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ABSTRACT

The Proceedings of PME-XVI has been published in three volumes because of the large number of papers presented at the conference. Volume 1 contains: (1) brief reports from each of the 11 standing Working Groups on their respective roles in organizing PME-XVI; (2) brief reports from 6 Discussion Groups; and (3) 35 research reports covering authors with last names beginning A-K. Volume II contains 42 research reports covering authors with last names beginning K-S. Volume III contains (1) 15 research reports (authors S-W); (2) 31 short oral presentations; (3) 40 poster presentations; (4) 9 Featured Discussion Groups reports; (5) 1 brief Plenary Panel report and 4 Plenary Address reports. In summary, the three volumes contain 95 full-scale research reports, 4 full-scale plenary reports, and 96 briefer reports. Conference subject content can be conveyed through a listing of Work Group topics, Discussion Group topics, and Plenary Panels/Addresses, as follows. Working Groups: Advanced Mathematical Thinking; Algebraic Processes and Structure; Classroom Research; Cultural Aspects in Mathematics Learning; Geometry; Psychology of Inservice Education of Mathematics Teachers; Ratio and Proportion; Representations; Research on the Psychology of Mathematics Teacher Development; Social Psychology of Mathematics Education; Teachers as Researchers in Mathematics Education. Discussion Groups: Dilemmas of Constructivist Mathematics Teaching; Meaningful Contexts for School Mathematics; Paradigms Lost - What Can Mathematics Education Learn From Research in Other Disciplines?; Philosophy of Mathematics Education; Research in the Teaching and Learning of Undergraduate Mathematics; Visualization in Problem Solving and Learning. Plenary Panels/Addresses: Visualization and Imagistic Thinking; "The Importance and Limits of Epistemological Work in Didactics" (M. Artigue); "Mathematics as a Foreign Language" (G. Ervynck); "On Developing a Unified Model for the Psychology of Mathematical Learning and Problem Solving" (G. Goldin); "Illuminations and Reflections--Teachers, Methodologies, and Mathematics" (C. Hoyles). (MKR)

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INTERNATIONAL GROUP FOR
THE PSYCHOLOGY OF
MATHEMATICS EDUCATION

# PROCEEDINGS
# of the
# SIXTEENTH
# PME CONFERENCE

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University of New Hampshire
Durham, NH (USA)
August 6 - 11, 1992

Volume I

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University of New Hampshire
Durham, NH (USA)
August 6 - 11, 1992

Volume I

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# PME XVI PROCEEDINGS

**Edited by William Geeslin and Karen Graham**
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## PREFACE

The first meeting of PME took place in Karlsruhe, Germany in 1976. Thereafter different countries (Netherlands, Germany, U.K., U.S.A., France, Belgium, Israel, Australia, Canada, Hungary, Mexico, Italy) hosted the conference. In 1992, the U.S.A. will again play host to PME. The conference will take place at the University of New Hampshire in Durham, NH. The University was founded in 1866 as the New Hampshire College of Agriculture and the Mechanic Arts. The state legislature granted it a new charter as the University of New Hampshire in 1923. The University now has about 800 faculty members and more than 10,000 students enrolled in 100 undergraduate and 75 graduate programs. The University's Mathematics Department has a strong history of commitment to research and service in mathematics education. We are pleased to be the host site for PME XVI.

The academic program of PME XVI includes:

- 92 research reports
- 4 plenary addresses
- 1 plenary panel
- 11 working groups
- 6 discussion groups
- 2 featured discussion groups
- 31 short oral presentations
- 40 poster presentations.

The short oral presentations represent a new format for sessions at PME.

### *The review process*

The Program Committee received a total of 181 research proposals that encompassed a wide variety of themes and approaches. Each proposal was submitted to three outside reviewers who were knowledgeable in the specific research area. In addition, one or more program committee members read each paper. Based on these reviews each paper was accepted, rejected, or accepted as a short oral presentation or poster. If a reviewer submitted written comments, they were forwarded to the author(s) along with the Program Committee's decision.

## *List of PME XVI Reviewers*

The Program Committee wishes to thank the following people for their help during the review process.

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# HISTORY AND AIMS OF THE P.M.E. GROUP

At the Third International Congress on Mathematical Education (ICME 3, Karlsruhe, 1976) Professor E. Fischbein of the Tel Aviv University, Israel, instituted a study group bringing together people working in the area of the psychology of mathematics education. PME is affiliated with the International Commission for Mathematical Instruction (ICMI). Its past presidents have been Prof. Efraim Fischbein, Prof. Richard R. Skemp of the University of Warwick, Dr. Gerard Vergnaud of the Centre National de la Recherche Scientifique (C.N.R.S.) in Paris, Prof. Kevin F. Collis of the University of Tasmania, Prof. Pearla Nesher of the University of Haifa, Dr. Nicolas Balacheff, C.N.R.S. - Lyon.

*The major goals of the Group are:*

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
- To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

*Membership*

Membership is open to people involved in active research consistent with the Group's aims, or professionally interested in the results of such research.

Membership is open on an annual basis and depends on payment of the subscription for the current year (January to December).

The subscription can be paid together with the conference fee.

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1 - 1

# Working Groups

## WORKING GROUP ON ADVANCED MATHEMATICAL THINKING (A.M.T.)

Organizers: Gontran Ervynck, David Tall

- **SESSION I: INTRODUCTION TO THE PROCESSES-OBJECTS THEME**
  Four initiators will present different approaches to what seems to be basically the same theory.
  - Michèle Artigue (France): *Tool and Object status of mathematical concepts; the case of complex numbers.*
  - Anna Sfard (Israel): *On Operational-structural Duality of Mathematical Conceptions.*
  - Ed Dubinsky (U.S.A.): *A Theoretical Perspective for Research in Learning Mathematics Concepts: Genetic Decomposition and Groups.*
  - David Tall (U.K.): *The Construction of Objects through Definition and Proof, with emphasis on Vector Spaces and Group Theory.*
- **SESSION II: DISCUSSION**
  Discussion of the contribution of the initiators. All discussion has to come from reflection on the content of the presentations. Disagreement is to be seen as a vehicle not for attempting to convince others of one's own view but of trying to find out the source of the disagreement.

- **SESSION III: CONTINUATION OF THE WORK ON RIGOR AND PROOF**
  The initiators are:
  - Dick Shumway (U.S.A.): *The Role of Proof and Definition in Concept Learning.* The intention is that a link should be established with the subjects discussed in Sessions I and II.
  - John Selden (U.S.A.): *Continuation of the work on rigor in mathematics* . Three aspects of rigor in undergraduate mathematics will be discussed: (1) Is it possible to construct objects through definition and proofs ? (2) What can be said about understanding the concept of proof itself ? (3) How can students learn to construct proofs and what kind of background knowledge is needed ?
- **SESSION IV: PREPARING THE FUTURE**
  Discussion of the work of the AMT Group at PME-17, Japan 1993.

## Working Group on Algebraic Processes and Structure

Coordinator: Rosamund Sutherland
Institute of Education University of London

During the Assisi meeting, the group aimed to characterize the multiple "jumps"/shifts that appear to be involved in developing an algebraic mode of thinking and to investigate the role of symbolizing in this development. Other concerns of the group are the role of meaning in algebraic processing; the potential of computer-based environments and implications for classroom practice. Key issues were discussed and worked on in small groups with the aim of producing a set of questions and working hypotheses for future collaboration.

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## Classroom Research Working Group

### Problems, standpoints and purposes of the working group

### A. *Problems*

1. In their research all the participants of this working group have encountered similar methodological problems arising from their developmental approach to classroom research. One of the problems is how to collect and to analyze the classroom data within the working group.

   An important aspect of our research is the defining of new variables for each new set of data. Different standpoints for 'good' research arise: doing classroom research for the data themselves, for the research methods involved, for enrichment mathematics education, for psychologic phenomena, for theoretical considerations.

2. Our research has raised many questions including:
   -ways of collecting data (video tapes)
   -repeatability
   -generalizability
   -falsification
   -objectivity
   -qualitative/quantitative aspects
   -reduction of collected data
   -developing new research methods and techniques

### B. *Standpoints*

The following perspectives are implicit in our research:

1. *Classroom Descriptors*

   As well as describing the data collected from various activities presented to and/or undertaken by the children, we record key classroom descriptors. In particular, details of actual instruction given by the teachers is noted. This is not common in research where any instruction involved is mostly described in global terms. We have found, however, in all our work that the type of instruction given by the classroom teacher can be a distinguishing feature in the data collection from the children.

2. *Mood conditions*
   One of the key issues to be considered by the group will involve searching for education conditions which produced a suitable mental climate for the children to work towards their own productions. They have *to bring the children into the mood to do so*. These conditions are mostly of a *social* character and help to legitimize the particular children's activities; the activities make *sense* to the children.

3. *Source*
   We consider the educational setting in the classroom (the manner of teaching, and so on) to be a *source of techniques and methods for the researcher*.

4. *Mutual nature of the research*
   Our focus is on *mutual research situations* in which the children can recognize themselves (e.g. as a writer, as an author, etc.). This is seen as important as it helps to justify the research objects (children).

## C. *Purposes of the working group Classroom Research*

1. To become aware of methods we use in the classroom, their possibilities and their constraints (watching video tapes)
2. To collect and develop methods and techniques which can be categorized under one of the standpoints mentioned above.
3. To collect and develop different mood conditions.
4. To criticize these methods, techniques and conditions and indicate their constraints.
5. To prepare a booklet to support researchers working in the area of classroom research and closely connected with the practice of teaching.

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E. ***Language***: English

*Working group on*

# CULTURAL ASPECTS IN MATHEMATICS LEARNING

Although the process of learning mathematics takes place in the school environment, this educational process cannot be isolated from the effects of the child's cultural context.
In other words, mathematical knowledge is a product of schooling filtered through culturally conditioned individual characteristics.

Our new Working Group has grown out of the Discussion Group *Learning Mathematics and cultural context*, which has been active since PME 13.
At PME 13, we explored the main interests of the participants in this area. The following themes were touched on: minorities in mathematics education, social pressures in mathematics education, the role of language in the acquisition of a given mathematical concept, learning and teaching in multicultural classrooms, teachers' conceptions of mathematics, problems of cooperative research in math education using a comparative approach for different cultures.
PME 14 and PME 15 represented our first attempt to focus on the following question: "What is the meaning of culture in the learning of mathematics?" Presentations of some research results in various areas led to a discussion about whether these cultural aspects are to be considered a starting point or an end-product.

At this point, we have identified and are working on several types of studies related to the cultural field:
1. Informal education and formal mathematical knowledge.
2. The effects of language and cultural environment on the mental representa-tions of students and teachers.
3. Cognitive processes in learning mathematics, using a comparative approach for different cultures.

The objectives of our working group consist of the following:
1. To exchange views on the impact of cultural context on the learning of mathematics.
2. To ensure that contact between conferences is maintained through the exchange of information about relevant research.
3. To contribute to the formulation of a methodological and theoretical framework by presenting original research at PME conferences. These contributions may be interdisciplinary in nature, possibly by making use of the fields of psychology, mathematics education, art education, and didactics of geography.
4. To identify the areas relevant to this approach where further research is desirable.

At PME 16, more specifically, we will invite participants to discuss the *problématique* of our working group as they are reflected in research situations presented by selected group members. At the request of last year's participants, a combined session of this working group and the *Social Psychology of Math Education* working group will be held at PME 16.

*Bernadette Denys*

# GEOMETRY WORKING GROUP

**Helen Mansfield**
**Curtin University of Technology, Western Australia**

At the Assisi meeting, the Geometry Working Group had the overall theme **Learning and Teaching Geometry: a Constructivist Point of View**. This theme was chosen because the committee of the Geometry Working Group believes that it is timely to examine constructivism as a theoretical framework for research into aspects of teaching and learning geometry.

Within this overall theme, two sub-themes were discussed. These were **What Constructivism has to say about Learning and How Teaching can Promote Learning in Geometry**; and **Helping Students to Construct Knowledge in the Geometry Classroom**.

In the first session of the Geometry Working Group, there were introductory presentations concerning the first subtheme followed by a discussion. The focus in the second session was on the role of computer environments in the learning and teaching of geometry.

The third session provided opportunities for group members to present brief papers on their current research. These papers did not have to report on work that was completed, but provided opportunities for the presenters to discuss work in progress, to seek feedback from other participants, and to discuss with colleagues collaborative research projects.

## PME XVI Working Group

# Psychology of Inservice-eduçation with Mathematics Teachers: a Research Perspective

Group Organizers: Sandy Dawson--Simon Fraser University, Canada
Terry Wood--Purdue University, USA
Barbara Dougherty--University of Hawaii, USA
Barbara Jaworski--University of Birmingham, UK

This is the fourth year the group has been studying the role of the teacher educator in doing inservice with mathematics teachers. Last year at PME XV in Italy, the group critically examined a proposed conceptualization of a framework for inservice education of mathematics teachers. These discussions gave rise to a revised draft of the framework. This draft was circulated and reactions to it were sought during the spring of 1992.

The organizers of the working group see the revised framework as a basis for the creation of a working manuscript (and thence a book) about INSET.

The preparation of a book is in line with the aim of the working group which is:

> to extend knowledge regarding the psychology of mathematics teacher inservice education, in order to broaden and deepen understanding of the interactions among teachers and teacher educators.

The group will meet for four sessions during PME XVI. The first two sessions will centre on collecting reactions to the revised framework, examining issues arising from this discussion, and to hearing accounts of how others on the international scene have attempted to handle the issues raised. The first half of the third session will be devoted to laying out the chapters for the manuscript. The latter portion of the third session will address the group's presentation at ICME7, the detailed planning for which will take place during the fourth session, a joint meeting with the other two Teacher Education working groups.

Though the work of the group is a carry over from discussions at the Italy meeting, new members are most welcome to join. The preparation of the manuscript will require input from participants representing a broad spectrum of the international mathematics education community served by PME. Hence, new participants are not only welcomed but are needed if the manuscript is to truly cover the spectrum of experiences lived by PME members.

When participants leave PME XVI they will have contributed to the preparation of an outline for a manscript on INSET, and will have made a commitment to write a chapter for the book based on their taken-as-shared experiences.

# Ratio and Proportion

**F.L. Lin, K.M. Hart and J.C. Bergeron**

Is the research topic 'ratio and proportion' dead or alive?

To tackle this question, this group have tried to define what is proportional reasoning abilities and found that what we have known is incomplete.

In the Mexico meeting, the group worked on some very fundamental questions, such as

(1) what is fractions? Is $\pi/2$ a fraction?

(2) what is ratio? Is $a:b = 3:4$ a ratio?

(3) what is the relation between ratio and fraction?

In the Assisi meeting, the group addressed these questions during one of their time slots.

The group also worked on advanced proportional reasoning. Discussion on some recent developments and problems for further investigation occurred.

In the remaining time slot, the group worked on questions such as,

(1) what is the origin of fraction/ratio concepts?

(2) how to develop a diagnostic teaching module on beginning fractions/ratio? ... etc.

Welcome to ratio and proportion group.

# The PME Working Group on Representations

Representations are key theoretical constructs in the psychology of mathematics education. For the purposes of our working group, the meaning of this term is quite broad. It includes:

- External, structured physical situations or sets of situations, that can be described mathematically or seen as embodying mathematical ideas. External physical representations range from peg-boards to microworlds.
- External, structured symbolic systems. These can include linguistic systems, formal mathematical notations and constructs, or symbolic aspects of computer environments.
- Internal representations and systems of representation. These include individual representations of mathematical ideas (fractions, proportionality, functions, etc.), as well as broader theories of cognitive representation that range from image schemata to heuristic planning.

Included in the scope of the Working Group are many kinds of issues. The following are just a few of the questions we have been addressing:

- What are appropriate philosophical and epistemological foundations of the concept of representation?
- How are internal representations constructed? How can we best describe the interaction between external and internal representations?
- What are the theoretical and practical consequences for mathematics education of the analysis of representations?
- How can the creation and manipulation of external representations foster more effective internal representations in students?
- What are the roles of visualization, kinesthetic encoding, metaphor, and other kinds of non-propositional reasoning in effectively representing mathematical ideas?
- How can linkages between representations be fully developed and exploited?
- Can individual differences be understood in relation to different kinds of internal representation?

Our group was guided during its first years by Frances Lowenthal (Mons, Belgium). I began coordinating it after the 1989 meeting in Paris, with the help of Claude Janvier (Montreal, Canada). In Mexico in 1990 we had 42 participants, and in Italy in 1991 we had 47. Some detailed notes from these two meetings will be available this year to participants; those who are not at the meeting are welcome to write for copies. We continue to aim toward publishing a special volume of the *Journal of Mathematical Behavior* devoted to the topic of "representation".

- Gerald A. Goldin, Center for Mathematics, Science, and Computer Education,
Rutgers University, New Brunswick, New Jersey 08903, USA

# RESEARCH ON THE PSYCHOLOGY OF MATHEMATICS TEACHER DEVELOPMENT

The Working Group *Research on the Psychology of Mathematics Teacher Development* was first convened as a Discussion Group at PME X in London in 1986, and continued in this format until the Working Group was formed in 1990. This year, at PME XVI, we hope to build on the foundation of shared understandings that have developed over the last five years.

## Aims of the Working Group

- The development, communication and examination of paradigms and frameworks for research in the psychology of mathematics teacher development.
- The collection, development, discussion and critiquing of tools and methodologies for conducting naturalistic and intervention research studies on the development of mathematics teachers' knowledge, beliefs, actions and thinking.
- The implementation of collaborative research projects.
- The fostering and development of communication between participants.
- The production of a joint publication on research frameworks and methodological issues within this research domain.

## Research Questions

At the Working Group sessions in 1991 it was decided that the focus for the 1992 Working Groups sessions would be the sharing of examples of the practice of mathematics teachers and teacher educators that inform our notions of what constitutes good pedagogy in general, and the role of assessment, in particular. The format of the sessions will include the presentation and discussion of brief, anecdotal vignettes. The following research questions may help to mould the thinking of researchers interested in contributing an anecdotal vignette to one of the sessions.

- Should professional development programs for teachers of mathematics be basically the same, the world over?
- Do we have examples of professional development programs that help practising teachers build confidence in their mathematical ability and in their ability to teach mathematics?
- Can the tension between constructivist ideas recommended in mathematics teacher development programs and assessment practices and pedagogy in the programs be reconciled?

## Proposed Outcomes of the Working Group at PME XV

*1. **Collaborative Research Projects:*** Members of the Working Group have overlapping research interests, and it is hoped that collaborative research projects can be mounted.

*2. **Publication of an Edited Collection of Research Papers:*** The Working Group plans to publish a collection of research papers.

*3. **Preparation of the Working Group's Presentation to ICME 7:*** A session at ICME 7, combined with the other two Working Groups involving teacher education, has been scheduled. At least one session at PME will be devoted to planning this session.

Nerida Ellerton, Convenor

## Working Group on "Social Psychology of Mathematics Education"

Alan J. Bishop, Organizer

All mathematics learning takes place in a social setting and particularly within the PME community, we need to be able to interpret, and theorize about, mathematics learning interpersonally as well as intrapersonally. Mathematics learning in its educational context cannot be fully interpreted as an intrapersonal phenomenon because of the social context in which it occurs. Equally, interpersonal or sociological constructs will be inadequate alone since it is always the individual learner who must make sense and meaning in the mathematics. Therefore, it is vitally important to research the ways this intra-interpersonal complementarity influences the kind of mathematical knowledge acquired by pupils in classrooms. In order to pursue this research it is therefore necessary to analyze and develop both theoretical constructs and methodological tools.

This is what the SPME working group has concentrated on. At PME 10, the first official meeting of the group, we tried out various small group tasks amongst ourselves and discussed their value as research 'sites' and also teaching situations. At PME 11, we moved to other social determinants of mathematics leaning, particularly thinking about influences of other pupils and of the teacher. At PME 12, we focused on the idea of "bringing society into the classroom" and the issues of justifying research which might conflict with what "society", considers education should be doing. At PME 13, we worked on two areas, firstly the ways in which the construct "mathematics" is socially mediated in the classroom, and secondly, the use of videos of classroom interactions, and their analyses. At PME 14, we considered the situation of bi-cultural learners, the social setting of the nursery-school, and the learning values of cooperative games. At PME 15, we considered the following: (1) bi-cultural learners - particularly ideas from the evidence of Guida de Abreu from Brazil, (2) the relationships between the social contexts of mathematics and the child development model, led by Leo Rogers from UK, (3) social issues of assessment, led by Luciana Bazzini and Lucia Grugnetti from Italy, and (4) aspects of cultural and social 'difference' which may be of significance in mathematics learning.

## Working Group: Teachers as Researchers in Mathematics Education

co-convenors: Steve Lerman and Judy Mousley

The group has been meeting annually since 1988 and as a working group since 1990. The aims of the group are to review the issues surrounding the theme of teachers as researchers in mathematics education, and to engage in collaborative research.

The stimulus for the notion that classroom teachers can and should carry out research whilst concerned with the practice of teaching mathematics comes from a number of sources, including: teachers as reflective practitioners; teaching as a continuous learning process; the nature of the theory/practice interface; the problems of dissemination of research when it is centred in colleges; research problems being generated in the classroom, and finding solutions within the context in which the questions arise. These themes are seen to be equally relevant to the teacher education situation, and provide a focus for the reflective activities of ourselves as teacher-educators.

Since the meeting in Assissi in 1991, we have established a network amongst members and circulated papers, ideas and questions. The programme in New Hampshire will centre around the issue that was raised in Assissi, namely what constitutes research in the context of teachers researching their/our own practice. We will also review the work of present and new members in this field and report on research carried out during the year.

# Discussion Groups

# DILEMMAS OF CONSTRUCTIVIST MATHEMATICS TEACHING: INSTANCES FROM CLASSROOM PRACTICE.

Richard Carter and John Richards[1]
Bolt, Beranek, and Newman 10 Moulton St.
Cambridge, MA 02138

The NCTM Curriculum and Professional Standards (NCTM 1989) and the California Framework (California Department of Education 1991) lay out a vision of how mathematics learning and teaching should happen. This vision is in strong contrast to what one finds in the vast majority of standard classrooms. This new vision is becoming common in the mathematics education community. Researchers have written about their own attempts to transform the classrooms they work in and the difficulties they have encountered (Lampert (1990), Ball (1990), Cobb (1991)) The NCTM's Professional Standards are full of classroom vignettes, and there are even some videotapes that show exemplary practices. Yet, there has been little written about what it means for regular classroom teachers to try to make the transition from traditional mathematics teaching to the vision of inquiry learning articulated in these documents. In this paper we present some of our own efforts to help classroom teachers make this transition and some of the enduring dilemmas these teachers have encountered.

# MEANINGFUL CONTEXTS FOR SCHOOL MATHEMATICS

**Luciana BAZZINI**, Dipartimento di Matematica, Università di Pavia, (Italy)
**Lucia GRUGNETTI**, Dipartimento di Matematica, Università di Cagliari, (Italy)

A Discussion Group explicitly devoted to the analysis of meaningful contexts for school mathematics had its first meeting at PME 15 in Assisi last year.

A primary reason leading to the establishment of this group was the growing interest in the role of contexts in mathematics education , as shown by recent research (for a basic bibliography, see the presentation of this Discussion Group in the Proceedings of PME 15, Vol. 1, pag. XXIX).

The two sessions of the Discussion Group took into account the role of context from a general point of view. It was noticed that the word "context" can have different meanings, related to socio- cultural or ethno-anthropological factors, or to the conditions in which teaching-learning processes take place. For our purposes, we defined context as the set of environmental conditions and experiences created to evoke thinking in the classroom in order to give meaning to mathematical constructions. Evidence suggests that the ability to control and organize cognitive skills is not an abstract context-free competence which may be easily transfered across diverse domains but consists rather of a cognitive activity which is specifically tied to context. This is not to say that cognitive activities are completely specific to the episode in which they were originally learned or applied. However, it is of vital importance to be able to generalize aspects of knowledge and skills to new situations. Attention to the role of context focuses on determining how generalization can be stimulated or blocked. A specific context ca represent a powerful opportunity for mathematical investigation but also a potential obstacle to abstraction.

In this perspective, we propose two main foci for the two sessions of this Discussion Group at PME 16. They are:

- analysis of how mathematical activity can be contextualized in experiences taken from children's extrascolastic knowledge;
- analysis of how school mathematics can be linked to other school disciplines, in view of a meaningful contextualization of mathematics itself.

In our opinion, special attention to the interaction of school mathematics and the world outside and of school mathematics and other domains gives rise to important questions related to the meaningfulness of a given context: meaningful for children, for mathematics or other.

Finally, questions related to how mathematical constructions can be contextualized and de-contextualized according to a spiral process can be discussed and investigated hereafter.

# PARADIGMS LOST: WHAT CAN MATHEMATICS EDUCATION LEARN FROM RESEARCH IN OTHER DISCIPLINES?

BRIAN A. DOIG The Australian Council for Educational Research

Many researchers appear to work in isolation from their brethren in related fields. Nowhere is this more true than in educational research. Language research has had little to say about the language of mathematics, yet the mathematics research literature is replete with references to 'the language of mathematics'. This 'language of mathematics' though, seems not connected to the notions of language generally used in language research. There are two questions here. First, why does this disconnection occur, and second are there indeed any benefits to be had from looking at other disciplines? I do not propose to enter the debate regarding to the former question, but do in regard to the latter.

Mathematics education research may benefit from looking at some related disciplines, but which? Let us look at one related discipline, namely science. Although science appears to be similar to mathematics to the uninitiated, and indeed historically was so, the end of the twentieth century sees two quite distinct research areas defined. Where previously researchers like Piaget investigated both science and mathematical concept development we now see separate studies. Scanning the relevant journals gives the impression that mathematics is about content and how it may be best taught, while a similar overview of science journals reveals an emphasis upon development of concepts. How this divergence has occurred is of no importance here, but rather that it exists.

Mathematics is supposedly about concept development, so can we use the science research as a guide to better research efforts in mathematics? I believe we can. An example of science research exploring children's conceptual development and providing information for teachers to better plan their students' further learning, is a study recently undertaken in Australia (Adams, Doig and Rosier, 1990). This survey of children's science beliefs used novel assessment instruments collectively entitled Tapping Students' Science Beliefs (TSSB) units. Children were asked to röle play, complete a short story or comment upon the activities of characters in a cartoon strip. By the use of modern psychometrics the data was collated and analyzed to produce continua describing the development of concepts over a number of scientific areas. Descriptions of students' likely scientific beliefs at various points along these continua make the planning of future experiences for these students much simpler and more likely to match the students' needs.

It is my contention that mathematics education can learn from these current efforts in science. For example, is it possible to construct assessment instruments that engage students and measure their underlying beliefs about mathematical concepts? The answer must be 'yes' if we are to attempt to create any sort of constructivist curriculum – one based upon the student's needs and perceptions, and not solely on the received wisdom of previous generations, which is apparently what we have. Ask yourself 'How different is my curriculum from that of my grandparents?'

## REFERENCE

Adams, R. J., Doig, B. A. and Rosier, M. (1991). Science Learning in Victorian Schools: 1990 Melbourne: The Australian Council for Educational Research.

### Discussion Group

# PHILOSOPHY OF MATHEMATICS EDUCATION

Paul Ernest

In mathematics education epistemological and philosophical issues are gaining in importance. Theories of learning are becoming much more epistemologically orientated, as in the case of constructivism. A number of areas of inquiry in the psychology of mathematics education, including problem solving, teacher beliefs, applications of the Perry Theory, and ethnomathematics, all relate directly to the philosophy of mathematics. Researchers in mathematics education are becoming increasingly aware of the epistemological assumptions and foundations of their inquiries. This is because any inquiry into the learning and teaching of mathematics depends upon the nature of mathematics, and teachers' and researchers' philosophical assumptions about it. Whilst many of these issues have been raised before at PME, none have been or can be resolved. This suggests that a continuing discussion would be useful and timely.

In fact, the most central of the philosophical issues, the philosophy of mathematics, has been insufficiently addressed at PME. Although reference has been made to it in a number of plenary and other presentations, there has not been sufficient recognition that it is undergoing a revolution, and the absolutist paradigm is being abandoned. Publications by Lakatos, Davis and Hersh, Kitcher and Tymoczko, for example, are pointing towards a new fallibilist paradigm. This has profound implications for the psychology of mathematics education. For if mathematics itself is no longer seen as a fixed, hierarchical body of objective knowledge, then what is the status of hierarchical theories of mathematical learning or of subjective knowledge of mathematics? One outcome is sure. They can no longer claim to be representing the logical structure of mathematics.

The aim of the group is to provide a forum for a discussion some of these issues, including:

1. Recent developments in the philosophy of mathematics.
2. Implications of such developments for the psychology of mathematics education.
3. The epistemological bases of research paradigms and methodologies in mathematics education.

This discussion group was first offered at PME-14 in Mexico. This meeting will continue the discussion begun there, and consider becoming a working group.

# RESEARCH IN THE TEACHING AND LEARNING OF UNDERGRADUATE MATHEMATICS: WHERE ARE WE? WHERE DO WE GO FROM HERE?

Organizers: Joan Ferrini-Mundy, University of New Hampshire, Ed Dubinsky, Purdue University, and Steve Monk, University of Washington

There is growing interest, particularly in the community of mathematicians, in questions about the teaching and learning of mathematics at the undergraduate level. Professional organizations such as the Mathematical Association of America and the American Mathematical Society have begun to encourage attention to this emerging research area within their conference and publication structures. This discussion session is organized by members of the Mathematical Association of America's Committee on Research in Undergraduate Mathematics, to promote a more sustained focus on this area of research. We will address the following questions:

Can we summarize major research areas and methodologies concerning the learning and teaching of undergraduate mathematics, and what are the most appropriate vehicles for sharing this work with a wider audience?

How can mathematicians and researchers in mathematics education collaborate to formulate and investigate significant questions about the teaching and learning of undergraduate mathematics?

How can we encourage more systematic and widespread interest in this area of research, while also maintaining high levels of quality for audiences of mathematicians, mathematics education researchers, and others?

What mechanisms can be developed for sharing work that has implications for practice, in terms of instruction and curriculum, with the community of college mathematics teachers?

Is it viable to propose a PME Working Group on the Teaching and Learning of Undergraduate Mathematics? What might be the relationship with the Advanced Mathematical Thinking Working Group?

A wide range of research has been undertaken concerning the teaching and learning of undergraduate mathematics. There are serious challenges in considering how this work might be summarized and organized so that it can be accessible and helpful to interested researchers and practitioners. Several working reference lists and bibliographies will be assembled for this discussion session, and participants are encouraged to supply additional material. Certainly the monograph produced by the PME Working Group on Advanced Mathematical Thinking provides a very useful organization. Additional compilations and formats might be helpful to various communities.

College and university teachers of mathematics often have serious and important questions concerning issues in student learning and in teaching.*Communicating Among Communities*, the final report of a fall, 1991 conference sponsored by the MAA, includes as one of its recommendations that "those faculty whose professional work is devoted to research in mathematics education, as well all those whose work centers on curriculum development or educational practice" should be appropriately rewarded. Issues in this area also will be raised.

Beyond the relatively well-developed body of work in advanced mathematical thinking, there certainly are other research directions and emphases in the area of undergraduate mathematics learning and teaching. These include various intervention-type studies to test curricular innovation or instructional strategies, studies of teaching processes, and studies about the mathematics preparation of preservice teachers. We hope to expand the discussion to determine the ways that these other lines of research, many of which have more profound implications for practice, may be extended and communicated.

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# VISUALIZATION IN PROBLEM SOLVING AND LEARNING

Maria Alessandra MARIOTTI, Dipartimento di Matematica, Università di Pisa, (Italy)
Angela PESCI, Dipartimento di Matematica, Università di Pavia, (Italy)

When we had the idea, last year, to start a discussion group on this theme we did not expect so large a presence. There were 45 participants from the following countries: Australia, Canada, Spain, Finland, Germany, Israel, Italy, México, Portugal, Sweden, UK and the USA.

Today many people are very interested in this topic and the related studies are multiplying. On this subject T. Dreyfus, last PME, gave a lecture "On the status of visual reasoning in mathematics and mathematics education". Our theme also intersects some aspects that are widly discussed by the Working Group on Representations, guided by G.A. Goldin. In the last three years the growing number of participants has made evident the growing interest in these problems. On the basis of last year's discussion, we think it opportune to direct our work along the following lines.

Since visualization, that is the action "to see" mentally, can be the result of different visual stimuli, among these we plan to deal in particular with graphical representations. By graphical representation we mean every graphical sign different from the written word: from a pictorial drawing to a schematic and symbolic one, up to the most specific mathematical signs. The graphical sign can be produced by a pupil, a teacher, a textbook, a computer and so on. During a lesson of mathematics, geometrical figures, symbols, schemes, tables, tree diagrams, arrows, ... are frequently used. Often they are not only didactic aids but visual messages which are crucial in building the "meaning" of a concept or in schematizing a problematic situation. In several instances these representations play a very important role: perhaps they are able to suggest mental images which are very effective and functional (for instance, in some memory tasks, in associations useful in producing cognitive acquisitions, in partially new resolution processes and so on). We consider very important to study the dialectics between graphical representations and internal cognitive processes and to discuss how this study can be faced.

Therefore we consider important that our group try to discuss the following problems:

a - To what extent and how are internal images influenced by external ones in arithmetic, algebra, geometry and analysis?

b - How are internal images used to generate external ones (diagrams, pictures, sketches,...) for example during problem solving processes?

c - Which graphical representations are particularly effective? In which conceptual contexts? For which ages? Which could be the reasons of their effectiveness?

d - Which are the most common misunderstandings in using external representations? How can we find a remedy for them? How can we be sure that the meaning of a graphical scheme is completely determined without ambiguity?

e - How to face the analysis suggested by the previous points? By which tools and methods? In which theoretical frames?

# Research Reports

# APPROACHES TO RESEARCH INTO CULTURAL CONFLICTS IN MATHEMATICS LEARNING

Guida de Abreu - Department of Education / Cambridge University, UK[*]
Alan J. Bishop - Department of Education / Cambridge University, UK
Geraldo Pompeu Jr. - Catholic University of Campinas, Brazil[*]

*With the aim of discussing alternatives approaches to research into cultural conflicts, some results from two research projects are presented. The first is concerned with clarifying the cultural conflict as experienced from the child's perspective, when her home mathematics is substantially different from the school mathematics. The second analyses changes in teachers' attitudes in the transition from a culture-free approach to mathematics teaching, to an approach that acknowledges the cultural-conflict.*

The recent constructivist framework, as exemplified by Saxe (1990), focuses on a level of mathematics learning where culture and cognition are constitutive of one another. Saxe developed his empirical studies in an out-of-school setting, candy selling on the street, and found evidence that the children gradually interweave their school mathematics with the mathematics generated by the participation in the out-of-school practices. A considerable amount of research describing the mathematical competence of people out-of-school, in contrast to in-school, is also available, e.g., (Carraher, 1988; Lave 1988), but little is known about the interactions occurring when children are confronted by the two sets of mathematics cultural practices, in a school setting. That seems a crucial area to clarify when developing new approaches to teaching in situations where the school mathematics culture is markedly different from that demonstrated outside school.

A second crucial area in such situations is that of the attitudes of teachers concerning the relevance of children's out-of-school knowledge for classroom teaching. There is a body of research on teachers' attitudes in mathematics teaching in general, but none which focuses on this specific aspect. This paper will be a report of ongoing research in both of these areas, illustrating as well the enormous research challenges facing mathematics educators working in cultural conflict situations, where 'cultural conflict' means the conflict the children experience in terms of contradictory understandings generated through their participation in two different mathematics cultures, one outside school, linked to their everyday practices, and the other at school.

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[*] - Guida de Abreu 's research is sponsored by CNPq / Brazil
Geraldo Pompeu's research is sponsored by Capes / Brazil

**Children's cultural conflicts**

The assumption that mathematical knowledge is cultural implies that its learning is associated with values, beliefs, rules about its use, etc. Therefore, the traditional belief that school mathematics is a culture-free subject is questionable, and there is a growing feeling that it should be treated as a specific school mathematics culture which is not taking into account the mathematics practiced in the out-of-school culture. That split between the two mathematics cultures is the source of conflict for children. By the nature of human cognition they should build their knowledge upon their previous understandings, but because of the cultural gap they are being faced with contradictions, which appear in different ways, such as: (a) beliefs ; (b) performance; (c) representations (d) self-concepts. To exemplify these aspects some results from an empirical investigation developed among Brazilian children, aged between 8 and 16, from primary schools in a sugar cane farming community, will be reported. This is a development of the research reported in Bishop and Abreu (1991).

(a) **Beliefs**: When investigating children's beliefs a great imbalance was found in terms of the value that they give to the outside mathematics, used in the predominant activity of the local economy, sugar cane farming and the in school mathematics (see Table 1).

**Table 1: Percentage of children's answers related to their beliefs about mathematics (n=26)**

| Children believe that: | % |
|---|---|
| . People working in an office **use mathematics** | 95 |
| . People working in sugar cane farming **do not use mathematics** | 72 |
| . The pupils, who **performed best** in school mathematics, work in offices | 81 |
| . The pupils, who **performed worst** in school mathematics, work in sugar cane farming | 73 |
| . People working in an office **are schooled** | 100 |
| . People working in sugar cane farming **are unschooled** | 77 |
| . Sugar cane workers **can work out sums** without being schooled | 73 |
| . Sugar cane workers **cannot do sums** without being schooled | 27 |

On the other hand they also acknowledged that sugar cane workers cope successfully with their everyday sums. This seems to be one area of cultural-conflict, which apparently they resolved in terms of contextualization, that is school mathematics is different from sugar cane mathematics. However, in practice they are coping with contradictions such as: their parents can do sums better than them, but they believe that people who are in sugar cane farming do not have proper knowledge; they need to rely on their parents' mathematics to get help to cope with difficulties in their school mathematics homework. Severina, 14 years old, a sugar cane worker's daughter, described by the teacher as an unsuccessful pupil, gives evidence of the contradictions:

*I (Interviewer) - Why doesn't that man [in a picture] on the tractor know mathematics?*
*S (Severina) - He doesn't know. He doesn't have a job. He works in sugar cane.*
*I - Is it possible that some people [in pictures] had never been to school?*
*S - [Among pictures with people in offices, markets, school and sugar cane she choose a man working in sugar cane.] Yes, this. I think that if he has been to school he would not be working in that place.*
*I - Any more?*
*S - These (again people in sugar cane).*
*I - Why?*
*S - It is the same. If they had studied they will not be working in that place. This is an example of those who had never been to school, like my father.*
*I - (...) You told me that your father doesn't know to write, but for oral sums he is the best. How does he help you in your mathematics homework?*
*S - I ask him, for example: how much is 3 times 7 or 8 and he answers. How much is 3 plus 12? He answers all.*
.....
*I - [On another interview] Can you tell what you think about the way your father did the sums, is it the same or different from the way you learned in school?*
*S - It is a different way, he does it in his head, mine is with the pen.*
*I - Which do you think is the proper way?*
*S - School.*
*I - Which do you think gives a correct result?*
*S - My father.*
*I - Why?*
*S - Because I just think so.*

**(b) Performance:** Analysing children's performance in group tasks, on which they were asked to imagine they were farmers, to allow them to bring their out-of-school knowledge to solve the task, no differences were found related to their school mathematical performance, but there seems to be a relation with gender (see Table 2).

**Table 2: Number of answers according to children's performance in school and gender**

| How the child understands the inverse relation | Number of answers given by pupils: | | | |
|---|---|---|---|---|
| | Successful at school | Unsuccessful at school | Boys | Girls |
| Only qualitatively | 11 | 15 | 1 | 25 |
| Both qualitatively and quantitatively | 25 | 25 | 31 | 19 |

The mathematics concept involved was the inverse proportional relation between the size of the unit of measurement and the total number of units needed in four tasks: halving and doubling the size of a stick used to measure length; halving and doubling the size of a square to measure area. Children show two types of understandings: (a) only qualitatively, e.g., if it is half the size of the unit, then I will need more units, but they show difficulties in the calculations, (b) qualitatively and quantitatively, e.g., this is half, then I need twice the number I have.

Again, these results brought more cultural conflicts into question. One is related to the contradiction in the child's performance in school and out, of which they seem to be aware, as for example, a 5th grade girl, unsuccessful in school who said: "*Sometimes my sister comes to my house, brings the money and I go shopping for her. They give me the note (account), I check and get it correct. But, in school there is no way, I cannot learn.*" Another is related to the contradiction between child-specific experiences and the task presented, e.g. this community has specific social roles for girls and boys, allowing boys to have jobs in agriculture or in the market, but which are improper for girls. Perhaps that difference between girls and boys could account for their engagement in different social practices outside which lead to specific mathematics understandings.

**(c) Representations:** When confronting the children with school-like tasks, the first thing that was obvious was the difficulty of the children of that community in coping with written language. This make sense since in the children's homes there is mainly an oral culture. They mention about their fathers that: only 37% can read, but 77% can do sums orally and 70% in writing. For the mothers, 57% can read, but only 47% can cope with sums orally or writing. Analysing the results of twenty pupils in four school tasks they succeed on 89% of the sums when solving them orally, but only 39% succeed in representing it in a written way acceptable in school (these findings agree with Carraher, Carraher and Schliemann, 1987, who described the oral strategies accounting for success). However, in the research presented here, the focus is on the process of producing a written representation for their oral solution. It seems as if they are **mixing their oral system of representation** with the written system, which is being taught in school. For example 50% of the third grade pupils in writing the sum of the sides of quadrilaterals, put more than one side in the same line, as in the following examples:

Example 1: After measuring a square with 5 (cm) each side, and answering orally that the total is 20, the child produced the following written representation

$$\begin{array}{r} \mathbf{5\ 5} \\ \underline{\mathbf{5\ 5}} \\ \mathbf{10\ 10} \end{array}$$

Reading the written answer: " **It is twenty. Because 5 plus 5 is 10, with 5 plus 5 is 10.**"

Example 2: Adding the sides of a parallelogram 3 by 4 (cm).

$$\begin{array}{r} 3\ 3 \\ \underline{4\ 4} \\ 7\ 7 \end{array}$$

Reading the written answer: " **It is seventy seven.**"

Both of these children wrote in the same way, but in the first example the child seems to follow the oral reasoning and give a correct result, while in the second the child reads the number following the rules of written numbers and giving an incorrect answer. This seems to be another way of experiencing the conflict, that is, children appear to be very confused when asked to choose which answer is correct. A child argued in one problem that the written sum is correct "*Because this one here (written) we did getting the numbers from here, working out, and checking*", while in fact the correct result was the one he did orally. On another problem the same child chose as correct the oral result.

**(d) Self-concepts**: Comparing children's self-judgements about their performances in mathematics with their teachers' judgements it was found that they do not agree in 55% of the cases. That high rate of disagreement between children's self-concepts and teachers seems to be another source of cultural conflict. The majority of the children who disagree with their teachers are the ones judged as low achievers by the teachers. The child's self-concept seems more coherent to their mathematics abilities in general, than the teacher's judgement based on the scores from school tests.

## Educational approaches to cultural-conflict

Cultural-conflict between in and out-of-school mathematics is being reflected in different kinds of contradictions that affect children mathematics learning, as exemplified by research results like those described above. There appear to be, from an educational perspective, two broad approaches to this conflict, one which ignores it and keeps the traditional mathematics teaching approach, and the other which acknowledges it. Following the second, different alternatives are followed in terms of the extent to which the home culture of the child will be taken into account in the school context. We refer to these as: assimilation; accommodation; amalgamation and appropriation. The way the school culture will interact with the child's home culture will vary according to the four alternatives, therefore raising different questions about children' learning; teacher's attitudes; school curriculum, etc. (see Table 3)

**Table 3: Different approaches to culture-conflict**

| **Approaches to culture conflict** | Assumptions | Curriculum | Teaching | Language |
|---|---|---|---|---|
| **Culture-free** Traditional view | No culture conflict | Traditional Canonical | No particular modification | Official |
| **Assimilation** | Child's culture should be useful as examples | Some child's cultural contexts included | Caring approach perhaps with some pupils in groups | Official, plus relevant contrasts and remediation for second language learners |
| **Accommodation** | Child's culture should influence education | Curriculum restructured due to child's culture | Teaching style modified as preferred by children | Child's home language accepted in class, plus official language support |
| **Amalgamation** | Culture's adults should share significantly in education | Curriculum jointly organised by teachers and community | Shared or team teaching | Bi-lingual, bi-cultural teaching |
| **Appropriation** | Culture's community should take over education | Curriculum organised wholly by community | Teaching entirely by community's adults | Teaching in home community's preferred language |

To illustrate one aspect of research into those approaches, some results from a study of teachers' attitudes are presented. This research, also in Brazil, investigated changes in attitudes occurring during the implementation of an 'accommodation' approach - specifically in the transition from traditional teaching, called 'canonical-structuralist', to one called 'ethnomathematical', which took into account the children's social and cultural knowledge and values. This work is the culmination of the research described in a preliminary form in Bishop and Pompeu (1991).

The research study was designed in three main phases: in the first, the theoretical background for the Ethnomathematical approach was introduced to the teachers; in the second phase, the teachers planned and developed six Teaching

Projects (TPs)' based on the Ethnomathematical approach; finally, in the third phase, the teachers applied the TPs with their pupils. Nineteen teachers were involved, in twelve ordinary state schools, teaching the six projects lasting between three and five weeks, to a total of 450 pupils. In order to monitor and assess the changes in teachers' attitudes, a questionnaire was applied, as an attitude 'thermometer', in three different points in the research study: at the beginning, after the theoretical phase, and after the application of the TPs (for details of procedure, see Bishop and Pompeu, 1991). The teachers also wrote about their attitudes and were interviewed. Three conclusions about the effects of the different phases of the implementation strategy upon the teachers' attitudes are relevant here:

From the general perspective of mathematics as a school subject, and in terms of the intended, implemented and attained levels of the mathematics curriculum, the theoretical phase significantly affected the teachers' attitudes towards the first two of these perspectives. That is, the introduction of the Ethnomathematical theoretical background to the teachers, substantially changed their attitudes towards the general perspective of mathematics as a school subject, and towards the intended mathematics curriculum (why mathematics occupies an important place into the school curriculum). According to the data, the emphasis on mathematics as a school subject most increased in relation to the 'particular' and the 'exploratory and explanatory' features of the subject, and most decreased in relation to its 'universal' and 'logical' features. Similarly, the emphasis on the reasons why mathematics occupies an important place into the school curriculum, most increased in relation to the 'social and cultural basis of the subject', and most decreased in relation to its 'informative' aspects. For example, one teacher wrote:

> *"What a big mistake it was to think initially that the 'cultural and social' basis of mathematics has so little importance. Mathematics is basically a product of the culture of each race. It grows from the needs of each society, and the experience of each one. These are the bases of its truth."*

The planning and development of the TPs, as well as their application with the pupils, most substantially changed the teachers' attitudes towards the attained level of the mathematics curriculum. In other words, the teachers' action as 'designers of curriculum (TPs in this case), and guides to learning', as suggested by Howson and Wilson (1986), most substantially changed their attitudes in relation to 'what abilities pupils should have after they have learnt mathematics'. According to the data, for example, after the application phase, the emphasis on pupils' ability to 'analyse problems' was the one which most increased its importance in the teachers' view.

Interestingly, the attitude questionnaire did not reveal any major change in teachers' attitudes towards 'how mathematics should be taught' (the implemented level of the mathematics curriculum). However, by the end of the study, the emphasis on a 'debatable' approach to the teaching of mathematics was the perspective which had most increased its importance from the teachers' point of view. In contrast, the emphasis on a 'one-way'and a 'reproductive' approach to the teaching of mathematics were the aspects which most decreased in their importance for these teachers. In addition, from some teachers comments, it was also possible to see other changes in teachers' attitudes at this level of analysis. One of these changes is related to the assessment procedure adopted in mathematics, and at the end of the research study, a teacher wrote about this:

> *"I was not expecting the kind of reaction which some of the pupils had (some pupils manifested disagreement about the final results of the assessment - researcher observation). On the other hand, (...) I learnt that the assessment procedure is too complex to be so little discussed. (...) After all, I believe that an assessment procedure should take into consideration the individual aspects of each pupil, demanding from each one of them a proportional response to his/her earlier experiences."*
>
> (More data will be presented at the conference)

## Conclusion

Cultural conflicts are increasingly being recognised as a source of mathematical conceptual, and attitudinal obstacles for pupils and teachers alike. The analysis, research approaches and findings reported in this paper indicate some promising directions which research in this area could take, and demonstrate the educational complexity which must be appreciated if progress is to be made.

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# RHETORICAL PROBLEMS AND MATHEMATICAL PROBLEM SOLVING: AN EXPLORATORY STUDY

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## *Abstract*

*This investigation examined ways in which a theory of knowledge telling and knowledge transforming from written composition might be relevant to mathematical problem solving. Rhetorical problems were identified in problem solving interviews as subjects attempted to understand the problem statement. These problems generally dealt with understanding language and were sometimes resolved as a result of expectations of text forms for mathematical problems. Revisions of text during mathematical problem solving occurred when the problem solver modified diagrams, charts, equations, etc. Revisions often occurred at critical times in the solution process.*

## Introduction

Although solving a mathematics problem and writing a composition are very different activities, from one perspective they have much in common. Like skilled writers, good problem solvers in mathematics must use and exert control over complex cognitive activities such as goal setting, planning, and memory search and evaluation. Some researchers on writing (Bereiter & Scardamalia, 1987; Carter, 1988; Flower & Hayes, 1977) consider writing to be a problem-solving activity. This view of writing has led researchers in mathematical problem solving and in writing to rely on some of the same sources in building theories in the two domains. Bereiter and Scardamalia developed a theory of written composition that involves two modes of mental processing called *knowledge telling* and *knowledge transforming*.

In this study, the investigator examined Bereiter's and Scardamalia's theory of written composition (1987) from the perspective of a mathematics educator interested in mathematical problem solving. The investigator, in effect, first stepped outside the domain of mathematics to acquire an understanding of the theory of knowledge telling and knowledge transforming as it applies to written composition. That understanding was then brought back into the domain of mathematics to ground the theory in data on mathematical problem solving. This paper reports on the component of the investigation of knowledge telling and knowledge transforming in mathematical problem solving (Adams, 1991) that identified rhetorical problems.

## Theoretical Background

The idea of two modes of mental processing has its roots in theories from cognitive psychology (Anderson, 1983). According to Anderson, one mode of cognitive functioning is "automatic" and "invoked directly by stimulus input." The other mode "requires conscious control . . . and is invoked in response to internal goals" (pp. 126-127). Bereiter and Scardamalia (1987) identified the characteristics of these modes of processing within the domain of written composition and labeled the first mode of processing *knowledge telling* and the second mode *knowledge transforming*.

Bereiter and Scardamalia (1987) proposed that, when a writer is engaged in knowledge transforming, the writer creates a rhetorical problem space and a content problem space. These problem spaces are not created if the writer is engaged in knowledge telling. The rhetorical problem space is tied to text production and contains mental representations of actual or intended text. One of its functions is to put thoughts into a linear sequence for output as written text. The content problem space is tied to idea production.

To make the transition from written composition to mathematical problem solving, the investigator viewed the representation of a mathematical problem created by the problem solver as "text." Examples of text forms in mathematical problem solving include charts, tables, and proofs. Knowledge of how to structure a mathematical proof is an example of knowledge of a way of forming text. Because mathematical problems are often presented in written form, the function of a rhetorical problem space in mathematical problem solving was assumed to include the interpretation of text as well as the creation of text. This assumption has a basis in literature on reading and writing: Birnbaum (1986) suggests that reflective thinking about language is a common thread between reading and writing. Readers and writers share the common goal of constructing meaning (Dougherty, 1986; Spivey, 1990).

The origins of the concept of problem spaces lie in theory related to computer simulations of human thought (see Newell, 1980; Newell & Simon, 1972; Simon & Newell, 1971). Simon and Newell (1971) describe a problem space as "the way a particular subject represents the task in order to work on it" (p. 151). Newell (1980) proposes that problem spaces are mental constructs "which humans have or develop when they engage in goal-oriented activity" (p. 696). Whether or not these mental constructs were created was of interest in this investigation as one means of distinguishing knowledge telling and knowledge transforming. For that purpose, it was useful to think of a problem space as setting boundaries on the knowledge structures used in finding a solution to the problem.

Although the concept of problem spaces is not new to research on mathematical problem solving (e.g., Goldin, 1979; Jensen, 1984; Kantowski, 1974/1975), the idea of a rhetorical problem space has not been proposed for mathematical problem solving. Researchers in mathematics education, however, have been interested in issues dealing with mathematics, language, and learning (e.g., Cocking & Mestre, 1988; Pimm, 1987). Pimm (1987) identified features of the mathematical writing system and the complexity of the syntax of written mathematical forms. He suggested that the same difficulties that children have with natural language are evident in learning mathematics.

## Research Perspective on Mathematical Problems

In this investigation, a problem solver was described as experiencing a problem if a direct route to a goal was blocked. Because a task that is a problem for one person may not be a problem for another person, a mathematical task cannot be labeled a problem until after the problem solver has worked on the task. Thus, in order to characterize the task as a problem, an observer must evaluate the mental activity of the problem solver by making inferences about that mental activity from what the problem solver says and does.

The mathematical tasks used in the investigation were written word problems. Problems were identified as *compositional problems* if the problem solver suspended attention to the problem goal identified in the problem statement in order to create his or her own goals for understanding and solving the problem. Mathematical tasks may be problems without being compositional problems.

As problem solvers work on mathematical tasks, they construct mental representations of the tasks. The mental representation may be thought of as a cognitive structure constructed on the basis of the problem solvers domain-related knowledge and the organization of that knowledge in memory (Yackel, 1984/1985). As the problem solver develops the mental representation, the problem solver's mathematical concepts and problem-solving processes may undergo change (see Lesh, 1985). Kintsch (1986) suggested that the problem solver builds a mental model of the text (problem statement) and a mental model of the situation described in the text. The mental model of the text "is built from propositions and expresses the semantic content of the text at both a local and a global level" (p. 88).

For this investigation, I assumed that the mental model of the text is part of the rhetorical problem space and the mental model of the problem situation is part of the content space, which I refer to as the main problem space. The contents of each problem space must be inferred from analysis of the problem solver's written representation of the problem and what the problem solver does and says while solving the problem.

## Design and Procedures

A theory developed in one domain cannot be expected to manifest itself in exactly the same ways in a second domain. Thus, the main purpose of the the study was to generate theory for mathematical problem solving and was not approached as a verification study. A general method of comparative analysis (Glaser & Strauss, 1967) was used throughout the investigation. In order to examine potentially different situations in which a grounded theory might manifest itself, purposeful sampling (Bogdan & Biklen, 1982) was used to select data from the mathematics education literature and to collect additional data. This procedure made it possible to compare "novices" and "experts" solving a variety of mathematical tasks.

A total of 21 interviews of subjects solving problems were coded and analyzed. Three levels of protocol analysis were used: (a) coding and categorizing of idea units in order to identify characteristics of the theory as it related to mathematical problem solving (b) identification of problem spaces created by the problem solver, and (c) characterization of the problem solving episodes as primarily knowledge telling or knowledge transforming. Consistency between the levels of analysis was monitored.

The identification of rhetorical problems proceeded in two ways: (a) Situations involving difficulties with language were examined. If the problem solver appeared to be pursuing a particular goal, explicitly or implicitly, and appeared to be purposefully working toward the goal, the activity was identified as taking place in a rhetorical problem space. (b) Situations in which text was modified were examined.

## Results

The first problematic situation for many problem solvers occurred as they attempted to understand the problem statement. For example an eighth-grade student read a problem and stated: "Okay . . . You have 19 coins worth . . . worth a dollar. How many of each type of coin can you have? . . . 19 coins worth a dollar. Well I . . . I don't know if all 19 coins are worth a dollar each or just one dollar." The student resolved the problem in this way: "but I guess if it was worth a dollar each they'd tell you." In this case, the student relied on her expectations of text forms for problem statements to make the problem situation meaningful.

Revisions of text were identified in the problem solving process and appeared to be undertaken for different reasons. For example, one subject, solving the coin problem above, modified a chart he was creating: "Um. I am going to put a column over here for the total so that I can keep a running total." Modifications of this type seemed replace creating a plan before beginning the problem solving process. One problem solver working a problem involving similar triangles modified a drawing in order to make the drawing more consistent with the problem solver's knowledge of similar triangles. The revision, in this case, seemed to be part of a process of reformulating the problem. Revisions also occurred after the problem solver observed an error in earlier work.

In summary, rhetorical problems were more easily identified during the process of understanding the problem statement than at other times during individual problem solving. Although revisions of text did not seem to indicate the same types of problematic situations that language presented in understanding the problem statement, revisions of text often occurred at critical points in the problem solving process.

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# ACTION RESEARCH AND THE THEORY-PRACTICE DIALECTIC: INSIGHTS FROM A SMALL POST GRADUATE PROJECT INSPIRED BY ACTIVITY THEORY

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*Action research is a context and a process by which practising mathematics teachers enrolled in post graduate study can explore and explain the relationship between theory and practice. From this starting point I develop the argument that action research is enhanced by teachers' prior engagement in theoretical debate on learning and teaching. In particular, key concepts in activity theory[1] provide teacher-researchers with useful tools to explore, change and reflect on their practice. These arguments are explored in this paper through the work of one particular mathematics teacher during his post graduate study. I will describe how his understandings and use of activity theory both shape, and are shaped by, his classroom practices. This view from the teacher-researcher as post graduate student is complemented by my own reflections as the supervisor of his work. Through the latter, a third thrust emerges as I become aware of the complexity of the questions provoked by the project, the limitations of activity theory and the constraints on such research within post graduate study.*

## ACTION RESEARCH, THEORY-PRACTICE AND THE B ED DEGREE.

As an educational method concerned to break the research-practice divide, action research has spawned many different interpretations and practices. Differences can be linked to the assumptions and interests underlying the projects (Grundy, 1987), and to whether projects focus inwards towards the classroom or also outwards towards the broader social structure (Liston and Zeichner, 1990). Common to all projects is a concern with **improvement** of classroom practice through the **involvement** of **teachers-as-researchers** in their own classrooms.

Action research is distinguished from good practice (what many teachers do anyway) in that it is systematic, deliberate and open to public scrutiny (McNiff, 1988; Davidoff and Van den Berg, 1990; Walker, 1991) and, while enhancing reflective teaching (Liston and Zeichner, 1990), is distinct from it in that reflective teaching, is not always conscious (Lerman and Scott-Hodges, 1991). Action research involves a continuous cycle of planning, acting, observing, reflecting and replanning instances of classroom practice. Through critical reflections, teachers not only develop their practice but also their theoretical understanding of that practice. **Action research thus provides for a constant interplay between theory, research and practice.**

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[1] *as developed in the Soviet school of thought.*

Recently I have encouraged mathematics teachers enrolled for post graduate study at the Bachelor of Education (B Ed) level at the University of the Witwatersrand (Wits) to turn the research requirements of their degree towards small scale action research projects. There is no particular innovation in this: in-service education, directed as it is towards curriculum change, has long drawn on the methods of action research. Currently, action research is being explored for its possibilities within pre-service teacher education (Liston and Zeichner, 1991) and M Sc study (Lerman and Scott-Hodges, 1991). Their motivation is not dissimilar to the two inter-related reasons for my advocating action research within the B ED: (i) The B Ed degree has a predominantly theoretical thrust. While students are generally excited by new ideas and ways of looking at education, these often have little impact on their educational practice. Action research can ground theory in practice. (ii) More broadly, practising teachers in South Africa, by and large, remain alienated from educational research and educational theory. They tend to perceive themselves as users and not producers of knowledge. In particular, mathematics teaching is characterised by a rather slavish adherence to a prescribed syllabus and its related prescribed text book (Adler, 1991). Action research offers possibilities for shifting such curriculum processes.

## ACTION RESEARCH AND ACTIVITY THEORY.

Embedded in action research is a view of learning as an active process. Teacher-researchers learn through action, and reflection on that action. Action research thus shares assumptions about learning with constructivist theories, be they naive, radical or 'socio-constructionist' (i.e. drawing on activity theory) (Bussi, 1991). All see knowledge, not as given, but as constructed through activity. It is thus not surprising that current research into mathematics teaching and learning from constructivist perspectives (eg Bussi, 1991; Jaworsky, 1991; Lerman and Scott-Hodges, 1991) are within an action research tradition . While radical constructivists and activity theorists share their rejection of knowledge as transmitted, they differ crucially in the importance they attach to the social mediation of learning. I do not wish to debate the pros and cons of these theories. In fact, like Confrey (1991), I would look to 'steering a course between Vygotsky and Piaget'. However, activity theory, with its emphasis on social mediation, provides useful concepts like the Zone of Proximal Development (ZPD) and the interpersonal becoming the intrapersonal with which mathematics teachers can examine social interaction and its impact on learning in their classrooms.

The focus of the rest of this paper is the work of one such teacher, Mark[2]. Mark's action research project(i) illuminates the **theory-practice dialectic** in the context of the B Ed course and (ii) provides particular insights into how key concepts in **activity theory** can be useful tools for a teacher attempting to reflect on, understand, and change his teaching practices.

## MARK'S RESEARCH - A BRIEF DESCRIPTION AND SOME ANALYSIS.

While enrolled in the B Ed, part-time, during 1990 and 1991, Mark was teaching mathematics in senior classes in a middle-class state (still segregated, whites-only) school, and bogged down yet again by 'word problems' in Std 9 (Grade 11). During his studies, he was inspired by the Piaget - Vygotsky debate on cognition. After engaging with some theoretical extrapolations from activity theory to mathematics teaching and learning, he wrote:

> Various authors, including Christiansen and Walther (C & W) (1985), and Mellin-Olsen (1986) have provided an elaboration of this theoretical framework (activity theory) into mathematical learning. The relationship between educational 'task and activity' is analysed in detail by C & W, whereas Mellin-Olsen locates activity within a broader socio-political context. Given that the focus of this project is specifically syllabus-related in terms of teaching word-problems with less emphasis on socio-political problems, I will draw primarily on the work of C & W.

There were thus two interacting starting points for Mark's project: (i) a problem identified in his own classroom, and, (ii) a desire to develop his interest in activity theory and knowledge as socially constructed. He acknowledged and then rejected socio-political concerns as outside the scope of his project. His selection from elements of activity theory is pragmatic and clearly shaped by his understandings of the scope of his project and the constraints of his practice. Right from the start, Mark's project is an interaction of theoretical and practical concerns.

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[2] *See Phillips, M (1991)*

C & W's product-process framework and their characterisation of 'drilling of problem-types' resonated with what Mark perceived as inadequacies in his (and others') practice. He translated C & W's contrast of typical and novel problems and their argument for internalisation through learning in two dimensions (action and reflection) into a series of tasks structured around the solving of 'word problems' related to quadratic equations. These were to be tackled by Std 9 pupils in a socially interactive setting so as to incorporate key activity theory concepts such as 'mediation in the ZPD' and the 'interpsychological becoming the intrapsychological'.

To facilitate both his own and others' critical reflection and interpretation of his strategies for changed practice, and because of his focus on mediation and interaction, Mark tape-recorded a group of learners, his interactions with them, and their interactions with each other. Mark's detailed self-critical reflections are not possible to reproduce in full here. Some of the most significant are captured in the following extracts and descriptions.

Despite his intent at establishing both pupil-pupil and teacher-pupil interaction Mark's transcript revealed that: 'I did most of the talking ... each pupil tended to interact predominantly with me'.

On the question of establishing inter-subjective meaning for the tasks, he noticed that

> '... I often asked the question "do you all agree?" without actually confirming whether they did ... '

As regards his mediation of activity, close scrutiny of the transcript revealed that:

> '...I provided a great deal of help, but my domination prevented pupils from collectively or independently solving the problems without my interference ...'

More specifically,

> '... I effectively mediated Grant's and Robert's activity... I failed to effectively draw others into the process of interaction ... I concentrated far more on the boys than on the girls ...'

And reflecting most critically on the last point he says:

> 'A hidden assumption that the boys are automatically more successful than girls at solving word problems seemed to prevail.'
>
> Page 4
> Anastasia: I don't understand
> Teacher: OK, Robert, see if you can explain to her
>
> Page 6
> Teacher: ... good Robert ... (and later)
> Teacher: Help Anastasia, Grant.
>
> Page 8
> Teacher: ...that's great. Look at Grant's attempt everybody.
> Anastasia: Gee wiz, kif hey.
> Teacher: ... show the others what you did.'

Examining who entered tasks, how, why, he observed that many different methods emerged, revealing a 'virtue' of his new approach in encouraging pupils to use their own methods rather than simply adopting a 'correct-method' mentality and that:

> '... Grant and Robert were the most actively involved on the tasks and came closest to solving all the problems... this seems to have a bearing on their acceptance of the tasks and their willingness to communicate in the group setting...The girls tended to be demotivated ... none of the girls were able to effectively solve any of the problems ... A reason for the girls' demotivation...might well be related to my ineffectual mediation ...Another, more subtle reason, which C & W overlook and which Mellin-Olsen addresses more adequately, is that girls' failure to solve the problems could be ... that they did not adequately accept the tasks as part of their own activity...'

In addition, he concludes that more time was needed for all to get at each task, and that in their construction, the tasks did not facilitate or provoke sufficient pupil-pupil interaction. He draws his insights into his conclusions:

> 'Although there were flaws in the tasks ... the approach adopted was by far an improvement ... Pupils were more positive ... and I was provided some interesting "revelations" about my own teaching of which I was not aware. My mediation was unwittingly sexist ... I dominated the activities ... With these insights ... I can further improve my teaching approach so that more pupils will benefit in the future ...'

and finally,

> 'By researching my own methods of teaching word problems in the context of 'task and activity', I have been able to provide an example of how research and practice can be integrated so teaching is enhanced ... the teaching process is truly a continual research one.'

## REFLECTIONS

There is no doubt that Mark gained tremendous insight into his practice. In his conscious attempt at preparing for socially interactive learning, he came to see that his practices were such that not only did he dominate classroom interaction, but his mediation was exclusive (focused on only two pupils) in general and gendered in particular. Without some systematic method of observing his mediating processes, he would still be unaware of how much he actually talks in class and to whom. This specific project, therefore, speaks volumes of the powerful impact of action research with an activity theory framework. However, the weakness of activity theory in relation to the 'process of internalisation of collective activity and the conditions of its functioning within the ZPD' (Bussi, 1991) is reflected in Mark's focus. His emphasis on the gender dimensions of the process of internalisation of collective activity fails to open up the difficulties attached to mediating a whole group all at once. What does the ZPD mean in whole class interaction? Commenting on this, Bussi (1991) notes that this is still an 'open problem in activity theory...(and) mathematics classrooms are suitable settings for further research.'

Mark's observations and reflections are explicitly theory-laden. The structure of his analysis fits the framework he established for his research. His discussion foregrounds tasks, mediation and interaction and, through this focus, Mark is able to develop and share detailed insight into with whom he interacted, how and to what effect. However, his failure to mediate all pupils, and in particular the demotivation of the girls in the class, required a reconsideration of his theoretical framework. C & W do discuss goal-directed activity, but (as Mark says above) this does not address the gendered outcome of his teaching. Gendered practices need to be interpreted in relation to wider social practices and Mark is pushed to reconsider the worth of Mellin-Olsen's location of activity in a socio-political context i.e. to alter and expand his initial theoretical frame. The interaction of theory and practice evident at the beginning is thus just as evident at the end of Mark's project.

An important question at this point, both for Mark and for teacher-educators, is how to sustain the 'symbiotic relationship between teacher as theory maker and teacher as developer of practice' (Jaworsky, 1991) outside of the supporting structure of the B Ed degree? The need for strategies such as support networks for past B Ed students becomes important if the gains made by Mark during his formal study are to be consolidated and developed. Within such networks, action research as the structured and rigorous activity described in my paper, can become a continuous part of a teacher's reflection on their practice.

The more serious challenge, however, lies in linking issues such as gender bias in the maths classroom to deeply rooted social practices. Once this link becomes clearer, as it did to Mark, the solution to the problem becomes less obvious. Mark will be able to draw from a large and growing body of literature on gender in mathematics education. The experience gained and the analytical and methodological tools developed in the research component of his B Ed should enable him to act, reflect and deal creatively with these issues. But whether and how this rational method of noticing, analysing and acting will get at deeply seated social practices, e.g. gender, remains a question.

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# STUDENTS' UNDERSTANDING OF THE SIGNIFICANCE LEVEL IN STATISTICAL TESTS

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## ABSTRACT

*In this paper the initial results of a theoretical- experimental study of university students' errors on the level of significance of statistical test are presented. The "a priori" analysis of the concept serves as the base to elaborate a questionnaire that has permitted the detection of faults in the understanding of the same in university students, and to categorize these errors, as a first step in determining the acts of understanding relative to this concept.*

## INTRODUCTION

One of the key aspects in the learning of the test of hypothesis, is the concept of the level of significance, which is defined as the "probability of rejecting a null hypothesis, when it is true". Falk (1986) points out the change of the conditional and the conditioned as a frequent error in this definition and the mistaken interpretation of the level of significance as "the probability that the null hypothesis is true, once the decision to reject it has been taken". Likewise, White (1980) describes several errors related to the belief of conservation of the significance level value $\alpha$, when successive tests of hypothesis are carried out on the same set of data, that is, relative to the so called "problem of the multiple comparisons".

In this paper the concept of level of significance in a test of hypothesis is analyzed, determining different aspects related to its correct understanding. The analysis of the components of the meaning of mathematical concepts and procedures should constitute a previous phase to the experimental study of students' difficulties and errors on the said objects. The study of the interconnections between the concepts enables us to know their degree of complexity and to determine the essential aspects that should be pointed out to achieve a relational learning and not merely an instrumental learning of the same (Skemp, 1976).

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Likewise, we describe the results of an exploratory study carried out on a sample of 35 students, that shows the existence of misconceptions related to each one of the aspects identified in the conceptual analysis. The errors of the students when faced with specific tasks indicate faults in the understanding of the concepts and procedures, and so, this analysis "should be considered a promising researching strategy for clarifying some fundamental questions of mathematics learning" (Radatz, 1980, p.16).

## CONCEPTUAL ANALYSIS OF THE LEVEL OF SIGNIFICANCE

In the classical theory (see for example, Zacks, (1981)) a parametric test of hypothesis is a statistical procedure of decision between one of the two complementary hypothesis $H_0$ and $H_1$, hypothesis that refer to the unknown value of a population parameter, starting from the observation of a sample. To carry this out, a statistic $\psi(x)$ whose distribution is known in terms of the value of the parameter, is used. The set of possible values of the statistic, supposing that the hypothesis $H_0$ is verified, is divided into two complementary regions, acceptance region A and critical region C, in such a way that having observed the particular value of the statistic in the sample we decide to accept the $H_0$ hypothesis if this value belongs to the region A and reject it if it belongs to C. We will only consider the case $H_0$: $\theta \varepsilon \theta_0$ of simple null hypothesis, to facilitate the discussion.

The application of a test can give rise to two different types of errors: to reject the hypothesis $H_0$ when it is true (type I error) and to accept it when it is false (type II error). Although we cannot know whether we have committed one of these errors in a particular case, we can determine the probability of type I error as a function of the value of the parameter, that is called the power function of the test:

$$\text{Power } (\theta) = P \text{ (Rejecting } H_0|\theta)$$

In the case of $\theta=\theta_0$ , we obtain the probability of rejecting $H_0$ with the chosen criteria, supposing that $H_0$ is true, the so- called probability of type I error, or level of significance $\alpha$ of the test:

$$\alpha = P \text{ (Rejecting } H_0 \; |\theta_0) = P \text{ (Rejecting } H_0|H_0 \text{ is true)}$$

The contrary event of rejecting the hypothesis $H_0$ consists of accepting it and its probability can also be expressed as a function of the parameter:

$$\beta(\vartheta) = P\ (\text{Accepting } H_0 | \vartheta)$$

In this case and whenever $\vartheta$ is different from the supposed value $\vartheta_0$, a type II error is being committed. As we can see, in the case of a simple null hypothesis, while the type I error has a constant probability, the probability of type II error is a function of the unknown parameter. Finally, and taking into account that the events to accept and to reject the null hypothesis are complementary, we see that the relationships between these probabilities are given by the following expression:

$$\alpha = 1-\beta(\vartheta_0)$$

In the understanding of the idea of level of significance, we can as a result of this, distinguish four differentiated aspects, that we have used in the elaboration of a questionnaire that enables us to identify and classify the misconceptions related to this understanding. This classification constitutes a first step in the categorization of the acts of understanding of synthesis of the said concept (Sierpinska, 1990), that would be added to the acts of identification, discrimination and generalization of the objects that intervene in its definition. These aspects are the following:

a) The test of hypothesis as aproblem of decision: between two excluding and complementary hypothesis, with the possible consequences of committing or not one of the types of error that are incompatible but not complementary events.

b) Probabilities of error and relation between them: understanding of the conditional probabilities that intervene in the definition of $\alpha$ and $\beta$, of the dependence of $\beta$ in terms of the unknown value $\vartheta$ of the parameter, and of the relation between $\alpha$ and $\beta$

c) Level of significance as the risk of the decision maker:

The values $\alpha$ and $\beta$ determine the risks that the decision maker is willing to assume in his decision and will serve, along with the hypothesis, for the adoption of the decision criteria.

d) Level of significance and distribution of the statistic; interpretation of a significant result:

The level of significance is the probability that the statistic chosen as a decision function takes a value in the critical region, in the case that the null hypothesis is true. Obtaining a significant result leads to the rejection of the null hypothesis, although this does not necessarily imply the practical relevance of this result.

## EXPERIMENTAL STUDY OF MISCONCEPTIONS

### Description of the sample

The study was carried out on a group of 35 students studying Statistics in their 2nd year of Civil Engineering in the University of Granada. Seventy five per cent of these students had not studied Statistics or Probability before and the rest had only studied it in some previous course. These students had studied Infinitesimal Calculus and Algebra in their first year of studies, so they can be considered to have an excellent previous mathematical base. The subject of Statistics, which includes the basis of descriptive statistics, probability theory and inference has been given three hours per week throughout a whole course, and the test having been carried out at the end of the same.

### Questionnaire used.

The questionnaire used consists of 20 questions, and was elaborated by the authors to study conceptual difficulties of the test of hypothesis. Due to the limitations of space we will only present the results obtained in four of the items, whose distracters have been chosen by trying to detect errors in the acts of understanding of synthesis referred to the level of significance. These items are the following.

ITEM 1:

The probability of committing both type I and type II errors in a test of hypothesis is:
A: 1; B: 0; C: $a \cdot b$; D: the product $ab$ , since the errors are independent

This item asks about the possibility that the two types of error can occur simultaneously. Since by carrying out a test of hypothesis we have a problem of decision, the null and alternative hypothesis are complementary like the events of accepting and rejecting the null hypothesis. However, the events of committing type I error or type II error are incompatible but not complementary.

ITEM 2:

A scientist always chooses to use 0.05 as the level of significance in his experiments. This means that in the long run:

A: 5 % of the times he will reject the null hypothesis.
B: 5 % of the times that he rejects the null hypothesis he will have made a mistake.
C: He will have mistakenly rejected the null hypothesis only 5 % of his experiments.
D: He will have accepted a false null hypothesis 95 % of the times.

In this item the definition of the level of significance appears as a conditional probability and the distracters refer to the incorrect interpretation of the same. In particular, in the classical inference, it is not possible to know the probability of having committed one of the types of error once the decision has been taken, although we can know the probabilities of type I or II error "a priori". That is, although we cannot perform an inductive inference about the probability of the hypothesis referring to the population, once the particular sample has been observed, we are able to make a deducive inference from the population of possible samples to the sample that is going to be obtained before having extracted it (Rivadulla, 1991).

ITEM 3:

When we change from a level of significance of 0.01 to one of 0.05 we have:
A: Less risk of type I error.
B: More risk of type I error.
C: Less risk of type II error.
D: Both B and C.

In this item we study the interpretation of the level of significance as a risk of error as well as the relationship between the probabilities $\alpha$ and $\beta$, which implies that it is not possible to simultaneously reduce the two risks, when the sample size has been fixed.

ITEM 4:

What can be concluded if the result in a test of hypothesis is significant?:
A: The result is very interesting from the practical point of view.
B: A mistake has been made.
C: The alternative hypothesis is probably correct.
D: The null hypothesis is probably correct.

The level of significance determines the critical and the acceptance regions of a test, together with the null and alternative hypothesis and the test statistic. The problem of carrying out a test of hypothesis has been transformed into that of

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dividing the population of possible samples in two complementary subsets: those who provide evidence in favour or against the null hypothesis. So, the level of significance is the probability that the statistic take a value in the critical region. One statistically significant result does not necessarily imply the significance (relevance) from a practical point of view.

## RESULTS AND DISCUSSION

The frequencies and percentages of responses to the different items are presented in Table 1. The relative difficulty of the same have been quite homogeneous, although somewhat higher in item 4 which refers to the interpretation of results and the difference between statistical and practical significance. From the analysis of the distracters that have been chosen by the students in the different items, we obtain a first information about the conceptual errors, that we classify in accordance with the previous conceptual analysis, in four sections:

**Table 1**

*Frequencies (and percentages) of responses in the items*

| | OPTIONS | | | | |
|---|---|---|---|---|---|
| Item | A(%) | B(%) | C(%) | D(%) | R. Correct |
| 1 | 5 (14.3) | •16 (45.7) | 5 (14.3) | 6 (17.1) | 16 (45.7) |
| 2 | 4 (11.4) | 9 (25.7) | •17 (48.6) | 4 (11.4) | 17 (48.6) |
| 3 | 4 (11.4) | 12 (34.3) | 1 ( 2.9) | •16 (45.7) | 16 (45.7) |
| 4 | 6 (17.1) | 2 ( 5.7) | •12 (34.3) | 14 (40.0) | 12 (34.3) |

• Correct option.

Misconceptions in the identification of a test of hypothesis as a problem of decision:

- Consideration of the type I and II errors as complementary events that are shown in the four responses to distracter D of item 2 and in the 5 responses to distracter A of item 1.

- Errors type I and II are not perceived as incompatible events. (5 responses to distracter C and 6 to D of item 1).

Misconceptions in the interpretation of the probabilities of error and their relations:

- Confusion of the two following conditional probabilities in the definition of the level of significance
$\alpha$ = P (reject $H_0 | H_0$ true) and $\alpha$ = P ($H_0$ true | $H_0$ has been rejected)
shown by the 9 responses to distracter B of item 2, that is the error mentioned in Falk's research (1986).

- Interpretation of $\alpha$ as P (reject $H_0$), that is to say , the suppression of the condition in the conditional probability, in the 4 responses to distracter A of item 2.

- Not to take into account the relationship between the probabilities of type I and II error (12 responses to distracter B of item 3).

Misconceptions in the interpretation of the level of significance as the risk of the decision maker:

- A higher level of significance gives less probability of type I error. (4 responses to distracter A of item 3).

- By changing the level of significance the risk of type I error does not change (1 case, in distracter C of item 3).

Misconceptions in the interpretation of a significant result:

- A statistical significant result is also significant from a practical point of view, (6 responses to distracter A of item 4).

- Since the level of significance is a very small value of a probability, it is associated with an incorrect result (2 responses to distracter B of item 4).

- Confusion of the significant result as one that corroborates the null hypothesis, this is confusion of the critical and acceptance regions (14 responses to distracter D of item 4).

## CONCLUSIONS

In the analysis of the responses to the questions put forward, the existence of a great diversity of misconceptions has been shown in the interpretation given by the students of the sample to the concept of the level of significance, thus completing the results of Falk (1986) and White's (1980) research. Although in an exploratory way, this study constitutes a first step towards the search for the structure of the components of the meaning of the test of hypothesis and the identification of obstacles in the learning (Brousseau (1983), that without doubt can contribute to an improvement in the teaching and application of statistical methods.

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# Students' Cognitive Construction of Squares and Rectangles in *Logo Geometry*[*]

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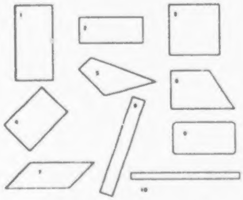
It has been argued that appropriate Logo activities can help students attain higher levels of geometric thought. The argument suggests that as students construct figures such as quadrilaterals in Logo, they will analyze the visual aspects of these figures and how their component parts are put together, encouraging the transition from thinking of figures as visual wholes to thinking of them in terms of their properties. Research has demonstrated that this theoretical prediction is sound; appropriate use of Logo helps students begin to make the transition from van Hiele's visual to the descriptive/analytic level of thought (Battista & Clements, 1988b; 1990; Clements & Battista, 1989; 1990; in press). The current report will extend the previous findings by giving a detailed account of how students' Logo explorations can encourage them to construct the concepts of squares and rectangles and the relationship between these two.

## The Instructional Setting

Students (n = 656) worked with activities from Logo Geometry (Battista & Clements, 1988a; Battista & Clements, 1991a; Clements & Battista, 1991), which was designed to help students construct geometric ideas out of their spatial intuitions. Control students (644) worked with their regular geometry curriculum. After introductory path activities (e.g., walking paths, creating Logo paths), students engaged in off- and on-computer activities exploring squares and rectangles, including identifying them in the environment, writing Logo procedures to draw them, and drawing figures with these procedures (Fig. 1).

Figure 1. "Rectangle: What can you draw?" Students are asked to determine if each figure could or could not be drawn with a Logo rectangle procedure with inputs and to explain their findings. They are permitted to turn the turtle before they draw a figure. From *Logo Geometry*.



Data came from two sources—case studies and relevant items from the Geometry Achievement Test that was administered to all students involved in the *Logo Geometry* project (Clements & Battista, 1991). None of the items from this test were related to Logo. The case studies were conducted by the authors, who observed and videotaped four pairs of students from grades K, 2, and 5 (two pairs) every day they worked on the materials.

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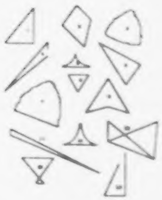
## Results

### Paper-and-Pencil Items



The first item, adapted from (Burger & Shaughnessy, 1986), asked students to identify rectangles. One point was given for each correct identification. Note that good performance on this item requires knowledge not only of the properties of rectangles, but of the fact that squares are rectangles.

*Directions:* Write the numbers of all the figures below that are rectangles.

There was a significant treatment by time interaction ($F(1, 1030) = 21.83, p < .001$). The Logo posttest score was higher than all other scores (Logo pretest, control pretest and posttest) and the control posttest scores were higher than both pretest scores ($p < .01$). It is noteworthy that Logo students showed dramatic growth between the pre- and post-tests for the squares (shapes 2 & 7). Control groups also showed growth on these items, but nowhere near as strong as did the Logo groups. For both of these shapes the most striking growth occurred in grades 4, 5, and 6. This may be due to students' increased knowledge of the properties of shapes or to the thinking engendered by the "Rectangles: What Can You Draw?" activity and class discussion. It is also relevant that the Logo group outperformed the control group on the parallelogram items; therefore, there was little indication that the students were simply overgeneralizing all quadrilaterals as rectangles. In a similar vein, students were asked on a separate item to identify all the squares in the same group of figures. Logo students significantly outperformed control students. There was no indication of an overgeneralization that "all rectangles are squares."

On another paper-and-pencil item, students were asked which geometric properties applied to squares and rectangles. Logo Geometry students improved more than control students. Thus, students' knowledge of properties was increased by work with Logo Geometry. According to the van Hiele theory, this lays groundwork for later hierarchical classification.

The increased attention that Logo students gave to properties, however, sometimes made it seem like their performance declined compared to control students. First, Logo more than control students claimed that rectangles have two long sides and two short sides. While not mathematically correct, this response indicates an increased attention to properties of figures—students consider it to be a property of rectangles. Logo students also claimed more often that rectangles had "four equal sides," possibly an overgeneralization from squares to rectangles or a misinterpretation of what the property states.

On the other hand, Logo students learned to apply the property "opposite sides equal" to the class of squares in much greater numbers than control students. Logo instruction may have helped students understand that the property "opposite sides equal" is not inconsistent with the property "all sides equal in l length." Most students could apply both properties to the class of squares, demonstrating flexible consideration of multiple properties that may help lay the groundwork for hierarchical classification.

In conclusion, Logo explorations helped students move toward van Hiele level 2 by focusing their attention on properties of figures. Explorations of relationships between shapes might have provided important precursors for hierarchical classification. Indeed, on the paper-and-pencil task in which students identified rectangles, the Logo Geometry group showed a strong increase in the frequency of identifying squares as rectangles, compared to the control group. This effect was particularly strong in the intermediate grades. No evidence of overgeneralization was found (e.g., that "all rectangles are squares").

There are several possible reasons why Logo Geometry instruction helped substantial numbers of students to identify figures consistent with the hierarchical relationship. First, when Logo Geometry students succeeded in identifying squares as rectangles on the rectangles item, they could have done so by asking themselves if a "Rectangle" procedure could have drawn each of the given shapes. Second, increased knowledge of properties of shapes and movement toward level 2 thinking may have enabled students to see squares as rectangles because squares have all the properties of rectangles. Class discussions of the classification issue may have suggested to students that squares should be classified as rectangles. Some students may have simply accepted this as a fact to be remembered. As we will see below, others made sense of this notion by using visual transformations.

### Case Studies

What is the basis for students' classifications of shapes?

Most of the students fell into either level 1 (visual), level 2 (property-based), or the transition between the two levels in the van Hiele hierarchy. The two student responses below illustrate these levels of thinking when classifying figures as squares. A second grader was examining her attempt to draw a tilted square in Logo. Although not really a square, she reasoned as follows:

Int: How do you know its a square for sure?
M: It's in a tilt. But it's a square because if you turned it this way it would be a square.

M does not refer to properties in making her decision; it is sufficient that it *looks like* a square.

Contrast this visual response with that of two fifth graders who had drawn a tilted square.

Int: Is it a square?
Ss: Yes, a sideways square. [Int: How do you know?] It has equal edges and equal turns.

So, what criteria do students use to judge whether a figure is a square? M was operating at the visual level; a figure is a square if it looked like or could be made to look like a square. The 5th graders required a figure to possess the properties of a square, demonstrating level 2 thinking in this instance.

Squares as rectangles

Students also dealt with the relationship between squares and rectangles in different ways. The first example of a kindergarten student illustrates an unsophisticated visual approach to judging the identity of shapes. Chris is using the *Logo Geometry* "Shape" command to draw figures of various sizes. He types S (for Shape), then types the first letter of the shape he wants (e.g., S for Square, from a menu), and finally receives a prompt to type a number for the length of each side of the shape. After first being puzzled that pressing R for rectangle required two numbers as inputs, Chris enters two 5s.

Int: Now what do the two 5s mean for the rectangle?
Chris: I don't know, now! Maybe I'll name this a square rectangle!
Int: That looks like a square.
Chris: It's both.
Int: How can it be both?
Chris: 'Cause 5 and 5 will make a square.
Int: But how do you know it is still a rectangle then?
Chris: 'Cause these look a little longer and these look a little shorter.
Int: Would this square [drawing a square with Logo] also be a rectangle, or not?
Chris: No.
Int: Even though I made it with the rectangle command?
Chris: It would be a rectangle square.

Even though Chris uses a terminology ("square rectangle") that suggests that he might be thinking of a square as a special kind of rectangle, his response of "No" indicates that he is not making a hierarchical classification. He also judges the figure to be a rectangle, not because it was made by the rectangle procedure, but because of the way the sides "look."

Int: So is a square a special kind of rectangle?
Chris: Yeah, if you pushed both numbers the same.
Int: How about 10 on two sides and 9 on the other two? Would that make a square? Or a rectangle? Or both?
Chris: It's both [a square and a rectangle].
Int: Is it a square?
Chris: Yes.
Int: How come it's a square?
Chris: 'Cause 9 is close to 10.

Again, we see the strength of visual thinking in Chris' judgments. He is willing to call the rectangle with side lengths of 9 and 10 a square, presumably because his visual thinking causes him to judge 9 close enough to 10 as side lengths. Contrast this with the second grader M's thinking about squares. It too was visual, but it was more sophisticated because of her use of visual transformations.

Robbie, another kindergartner, already indicated that he understood why two numbers must be input for a rectangle and only one for a square.

Int: What about this? What if I put in S R 5 5.
Robbie: That would be a rectangle for R.
Int: Right, and then I tell it 5 and 5.
Robbie: R draws on paper what he thinks it would be (a square) and calls it a square.
Int: How did that happen?
Robbie: Because if I goofed...and I think I put some number the same, I got a square, and I wanted a rectangle.
Int: Why is that?
Robbie: I can go wrong on the rectangles. Because the rectangle is like a square, except that squares aren't long.
Int: What else do you know about a rectangle? What does a shape need to be to be a rectangle?
Robbie: All of the sides aren't equal. These two [opposite] and these two [other opposite] sides have to be equal.
Int: How about 10 on two sides and 9 on the other two? Would that make a square?
Robbie: Kind of like a rectangle.

Int: Would it be a square too?
Robbie: [Pause.] I think may.... [Shaking head negatively.] It's not a square. 'Cause if you make a square, you wouldn't go 10 up, then you turn and it would be 9 this way, and turn and 10 this way. That's not a square.

Robbie too is not using any type of hierarchical classification. He thinks of squares and rectangles in terms of visual prototypes— "the rectangle is like a square, except that squares aren't long." And according to his past experiences, he, like most students, decides that rectangles have opposite sides equal, but not all sides equal.

In conclusion, the Logo microworlds proved to be evocative in generating thinking about squares and rectangles for these kindergartners. Their constructions were strongly visual in nature, and no logical classification per se, such as class inclusion processes, should be inferred. Squares were squares, and rectangles rectangles, unless—for some students—they formed a square with a Logo rectangle procedure or they intended to sketch a rectangle, in which case the figure might be described as a "square rectangle."

The comments of another second grader illustrate how visual thinking is used by some students to make sense of relationships between figures (Battista & Clements, 1991b). This student, who had previously discovered that she needed 90° turns to draw a square, used 90s on her first attempt at making a tilted rectangle, reasoning as follows:

C: Because a rectangle is just like a square but just longer, and all the sides are straight. Well, not straight, but not tilted like that (makes an acute angle with her hands). They're all like that (shows a right angle with her hands) and so are the squares.
Int: And that's 90 [showing hands put together at a 90°]?
C: Yes.

She then stated that a square is a rectangle.

Int: Does that make sense to you?
C: It wouldn't to my [4 year old] sister but it sort of does to me.
Int: How would you explain it to her?
C: We have these stretchy square bathroom things. And I'd tell her to stretch it out and it would be a rectangle.

It "sort of made sense" that a square is a rectangle because a square could be stretched into a rectangle. This response may be more sophisticated than one might initially think, for C had already demonstrated her knowledge that squares and rectangles are similar in having angles made by 90° turns. Thus, she may have understood at an intuitive level that all rectangles could be generated from one another by certain "legal" transformations, that is, ones that preserve 90° angles.

A fifth grader was working on the square in the "Rectangle: What Can You Draw?" activity.

Jon: This one [pointing to the square] is not a rectangle. It's a square. It has equal sides.
Int: Can you do it with your rectangle procedure?
Jon: No, because the sides are equal. So that would be a "no."
Int: So, no matter what you tried, you couldn't make it with your rectangle procedure?
Jon: You couldn't no, because the sides are equal.
Int: On your rectangle procedure, what does this first input stand for?
Jon: The 20? These sides.
Int: What does the 40 stand for?
Jon: Yea, you could do it. If you put like 40, 40, 40, and 40. [again, motions]
Int: Ok, try it.

Jon: So that would be a square?
Int: Can you draw a square with your rectangle procedure?
Jon: You could draw it, but it wouldn't be a rectangle.

Even with prompting, Jon is resistant to calling the square a rectangle. In his conceptualization, one can draw a square with the rectangle procedure, but that does not "make it" a rectangle.

Here is another fifth grader discussing the issue.

Teacher: Why do you think a square is not a rectangle, Jane?
Jane: Each side is equal to each other. But in a rectangle there are two longer sides that equal each other and the other two sides equal each other but they're short.

This response is typical. Jane has simply elaborated the essential visual characteristics of the set of figures she thinks of as rectangles. So, because almost all of the figures that she has seen labeled as rectangles have two long sides and two short sides, she includes this characteristic in her list of characteristics or properties. The teacher asked how she could make a square with the RECT procedure.

Jane: Because you put in two equal numbers. And that's the distance [length] and the width. If they are the same amount, then it will come out to be a square.
Teacher: So it did come out to be a square? That is a square you're telling me?
Jane: Yes, and a rectangle. But it's more a square, because we know it more as a square.

The second grader below tries to deal with the problem by inventing new language, much like one of the kindergartners that we discussed.

Int: Is everything that RECT draws a rectangle?
Bob: That's (points to square on the screen) not a rectangle.
Int: How come?
Bob: Because the sides are the same size?
Int: So ... this square [pointing to the square on the sheet] is not a rectangle?
Bob: I think it's a special kind of rectangle.
Int: So is this (pointing to the square on the screen) a rectangle?
Bob: It's a *special kind* of rectangle.

So Bob dealt with the conflict of a square being drawn by a rectangle procedure by inventing a language that allowed him to avoid the uncomfortable statement that a square is a rectangle by saying that a square is a *special kind* of rectangle but not a *rectangle*.

Other fifth graders trying to come to grips with the same question in a class discussion.

Lisa: I have a different question. Why can't we call squares equilateral rectangles?
Keith: A square classifies as a bunch of things. Equilateral rectangle doesn't classify as all the things that are square.
Teacher: Give me an example of a square that isn't an equilateral rectangle.
Keith: Well, like a diamond.
Teacher: [Draws one and has Keith clarify that he means a diamond with 90° turns. Keith still maintains that the drawing is not an equilateral rectangle.]
Lisa: All you have to do is turn it and it would be both a square and an equilateral rectangle in my definition.

Interestingly, and illustrating his lack of hierarchical classification, Keith does not think a square and an equilateral rectangle are the same. Lisa, who does, still uses visual thinking to support her argument. In the episode below, the teacher has asked the students whether a variable square procedure (SQUARE :X) can be used to make a variable rectangle procedure.

K: No. There are two longer lines on a rectangle. They are longer than a square. All the lines are not equal in a

rectangle; they are in a square. So if you think that, you can't draw a rectangle with a square procedure.
P: In the sense that the 10 or whatever you put down for the square represents all the sides, which wouldn't work because all the sides would be equal. So you'd have to make a new procedure for it.
J: You have mentioned that opposite sides are parallel and equal. It's the same way with a square except that all sides are equal. So that the two sides that are parallel are still equal. So a square in the sense that you're saying is a still a rectangle, but a rectangle is not a square.
Teacher: Can we build any rectangle with the square procedure?
J: Yes you can.
Teacher: Can I build a rectangle with sides of 20 and 40?
J: No, sorry. You can't build every single rectangle with the square procedure, but you can build one rectangle with the square procedure.

In pairs, students now move on to the Rectangle: What Can You Draw? activity. As they get to the square on the sheet, J says "It's a square." P illustrates his confusion over classification, saying "A square can be a rectangle, wait. A rectangle can be a square but a square can't be a rectangle." J starts to correct him "A square can be a rectangle. P interrupts, "Oh yeah [laughs]."

In this episode, all of these students see that the square procedure cannot be used to make rectangles. J, however, is the only student who seems capable of comprehending the mathematical perspective of classifying squares and rectangles. However, her comment "in the sense that you're saying" suggests, that she has not yet accepted this organization as her own. The episode below further illustrates that she has not yet adopted a mathematical organization in her classification of shapes.

Int: If I typed in RECT 50 51, what would it be (before hitting return)?
P: Probably about a square.
J: A rectangle but it wouldn't—
P: It would be a rectangle but sorta like—
J: It would be a rectangle, but it wouldn't be a perfect square. [They hit return.]
J: You see it's not a perfect square.
P: [Measures the top side (the longer) with his fingers.] It's only one step off.

Even though P and J say that the 50 51 rectangle is a rectangle and not a square, their language seems to indicate their belief in such a thing as an "imperfect square"—that is we presume, a figure that looks like a square but does not have all sides equal. They cling to an informal rather than logical classification system, one that still contains remnants of their visual thinking.

Finally, we examine the comments of a 6th grader during a class discussion of the square/rectangle issue raised by trying to draw the square with the rectangle procedure. Kelly asked "Why don't you call a rectangle a square with unequal sides?" After the teacher defined a rectangle as a shape that has four right turns and opposite sides parallel, however, Kelly stated "If you use your definition, then the square is a rectangle" (Lewellen, in press). Kelly's comments, like those of the 5th grader J, clearly indicate an ability to follow the logic in the mathematical classification of squares and rectangles. But neither student has yet made that logical network her own—each still clings to the personal network constructed from previous experiences. As van Hiele says, "Only if the usual [as taught in the classroom] network of relations of the third level has been accepted does the square have to be understood as belonging to the set of rhombuses. This acceptance must be voluntary; it is not possible to force a network of relations on someone" (van Hiele, 1986, p. 50). For Kelly or J to move to the next level requires them to reorganize their definitions of shapes in a way that permits a total classification scheme to be constructed. That is, the attainment of level 3 does not automatically result from the ability to follow and make logical deductions; the student must utilize this ability to reorganize her or his knowledge into a new network of relations. In this network, "One property can signal other properties, so definitions can be seen not merely as descriptions but as a way of logically organizing properties" (Clements & Battista, in press). Normally this entails making sense of and accepting the common definitions and resulting hierarchies given in the classroom.

## Conclusions

Logo environments can promote students' movement from the visual van Hiele level to the next level in which students think of shapes in terms of their properties. Logo explorations of relationships between shapes such as squares and rectangles differentially affect students at different levels of thinking. For some students such as second-grader C, such explorations cause their visual thinking to become more sophisticated, incorporating visual transformations that express their knowledge of these relationships. For several of the fifth graders, the explorations engendered analysis and refinement of their definitions for shapes in terms of properties, further promoting the attainment of level 2 thinking. And finally, for some, such explorations promoted the transition to level 3 thinking—first they understand a logical organization of properties, and finally they adopt it.

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# ARITHMETICAL AND ALGEBRAIC THINKING IN PROBLEM-SOLVING[1]

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*The fact that students have difficulty acquiring and developing algebraic procedures in problem-solving, considering the arithmetical experience that they have acquired over years, calls for a didactic reflection on the nature of the conceptual changes which mark the transition from one mode of treatment to the other. In this perspective, our study seeks to characterize the spontaneous problem-solving strategies used by Secondary III level students (14 -and -15-year-olds), who have already taken one algebra course, when solving different problems. The analysis of the problem-solving procedures developed by these students reveals the differences between the conceptual basis which underlie the two modes of thought.*

The difficulties experienced by students learning algebra have been the subject of many studies which have shown that certain conceptual changes are necessary to make the transition from arithmetic to algebra (Booth, 1984; Collis, 1974, Kieran, 1981; Filloy and Rojano, 1984; Hercovicz and Linchevski, 1991; Arzarello, 1991...). In the area of problem-solving, which is one of the important heuristic functions of algebra (Kieran, 1989) and which proves very difficult for students (Lochead, 1988; Kaput, 1983; Clément, 1982; Mayer 1982), the analyses examining the passage to an algebraic mode of thinking have either focused on a certain dialectic between procedural and relational thought (Kieran, 1991; Arzarello, 1991), or on the symbolism and/or the solving of equations. With regard to the latter, the history of mathematics shows that algebra began to develop well before symbols were used to represent unknown quantities. Rhetoric was an important stage in this development among the Arabs, for whom language was the natural means to represent the (known and unknown) quantities of a problem to be solved and to express the solution process. The studies carried out among students also show that most of them, from high school to university, solve algebraic problems in an "abridged" style (natural syncopated language) rather than in a symbolic style (Kieran, 1989; Harper, 1979). Few studies, however, have focused on the students' *reasoning* in solving the problems.

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[1] This study is part of a larger project undertaken by a group from CIRADE, subsidized by the FCAR (Quebec), which is researching the conditions for the construction of algebraic reasoning and representations, with regard to the situations which allow for their emergence and development.

In our didactic perspective, the main objective of our analysis was to gain a better understanding of the *conceptual basis* which underlie the arithmetical mode of thought on one hand, and the algebraic mode of thought on the other, as well as the possible articulation-conflicts which are possible in the transition from one mode of treatment to the other.

From a *didactic point of view*, because of the previous experience acquired by the students, problem-solving appears to be an interesting terrain for examining the two modes of thought and the conceptual changes which mark the passage from the arithmetical to the algebraic thinking. Moreover, from the *historical point of view*, the solving of problems played an important role in the development of algebra. It is at the heart of the algebra of Diophantus and of the Arabs, and is explicit in Vieta's objective of developing a method that could solve every problem. Thus, problem-solving is a doubly interesting terrain for the examination of the emergence of the algebraic mode of thought and its characteristics. This historical analysis is now being carried out, and is the object of investigation of some of the members of our team (Charbonneau, 1992; Lefebvre, 1992; Radford, 1992).

**Objective of the Study**

By examining the ways in which secondary school students (Sec. III, 14-and-15-year-olds who had already taken an algebra course) spontaneously solved different types of problems, this exploratory research project, carried out with a small group of students, aimed to analyse the solution processes of the students. In the characterization of the arithmetical and algebraic procedures used, the accent was placed not on the use of symbolism, but rather on the students' capacity to grasp the known and unknown quantities in the problem, and their way of solving it.

**Method**

In order to delineate, on an exploratory basis, the procedures used by students, and, through these, to better elucidate the differences between the *conceptual basis* which underlie the arithmetical and the algebraic thinking, 54 students from two regular classes in a Montreal area public high school (Secondary III, 14-and-15-year-olds) were given a paper-and-pencil test with five different written problems to solve[2]. The choice of the students' level (they had taken an introductory course in algebra) made it possible for us to show the conflicts that can arise for students at this stage when facing two possible modes of solving the problems.

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[2] The different problems presented, involving complex relations, could all be solved a priori by either arithmetical or algebraic reasoning, even if some of the methods appear more complicated than others.

### Analysis of the Results

Our analysis centered on one of the problems, which read as follows: "588 passengers must travel from one city to another. Two trains are available. One train consists only of 12-seat cars, and the other only of 16-seat cars. Supposing that the train with 16-seat cars will have eight cars more than the other train, how many cars must be attached to the locomotives of each train ?"

In this problem, different solution processes were possible. These took into account a certain implicit mental representation of the data and the relations which linked the elements involved, a representation which evolved during the solution process. How can we distinguish between the arithmetical and the algebraic procedures in the ways that this data and these relations were dealt with ?

A preliminary analysis of the above problem brought out the key elements around which the solution will be organized: "the total number of passengers: 588", the existence of "two trains", of "16-seat cars", "12-seat cars", and the "eight cars more" that one kind of train had in relation to the other.

However, the resolution of the problem required the use of other elements which made it possible to "build bridges" between the different data, elements which were not at all explicit in the problem: the number of 16-seat cars and 12-seat cars, the relation between the two types of quantities involved: the number of cars and the number of passengers, which must be built from the rates given in the problem, the number of passengers in each train... This a priori analysis brought to light important reference points which guided the subsequent analysis of the students' ways of solving the problem.

### SOME ARITHMETICAL PROCEDURES

It was easily observed that certain elements were retained by the students, and that these were used as a kind of point of entry , or engagement, in the organization of their solution procedures: a) the **two** trains; b) the **whole**: the 588 passengers; c) the **difference** between the number of cars of one type and those of other type; d) the data: "16-seat cars" and "12-seat cars".

In general, the arithmetical procedures were organized around these four **known elements**, in attempts to build bridges between them to be able to work with known data. The unknown quantity therefore appeared at the end of the process. Two types of entry points, or engagements, were distinguished. In the first case, the first two elements (a and b) frequently gave rise to a numerical strategy which we call **equitable partition**, which consisted in dividing the number of passengers by the number of trains (in this case, two) to obtain the number of passengers in each train (see Procedure 3). Another less frequent type of engagement was the **adjustment of the difference** between the two trains (c) at the beginning, to obtain two trains having the same number of cars (see Procedures 1 and 2).

**1. Procedure taking the difference into account at the beginning:**

| **Student:** | **Comments:** |
|---|---|
| 8 x 16 = 128 passengers | Numbers of passengers in the 8 extra cars |
| 588 - 128 = 460 passengers | By eliminating the extra cars, the number of cars in each train is equal |
| 460 ÷ 28 = 16.4 | 12 seats + 16 seats = 28 seats<br>(one 28-seat car train) |

**Answer:**
17, 12-seat cars and 25, 16-seat cars

This strategy clearly showed the modifications which occurred in the representation of the problem during the solution process: this representation was not at all static. The problem, and the relations linking the data had to be transformed by the students into a new configuration of the whole, which made it possible for the calculations to progress. The arithmetical procedure used here, which only dealt with the known elements, could not advance without those necessary modifications, because at the beginning there was no relation directly linking the known quantities provided in the problem.

**2. Another procedure taking the difference into account at the beginning, followed by partition:**

| **Student:** | **Comments:** |
|---|---|
| 16 x 8 = 128 | Numbers of passengers in the eight extra cars |
| 588-128 = 460 | Modification of the initial representation into a new configuration of equality of cars (see previous strategy) |
| 460 ÷ 2 = 230 | The equitable sharing strategy |
| (230 ÷ 12 = 19, 230 ÷ 16 = 14) | Number of cars of each type |
| 12-seat car train -> 19 cars | |
| 16-seat car train -> 14 + 8 = 22 cars | Re-utilization of the difference |

Just as in the first procedure, the representation of the problem was modified during the solving process. The change from the initial representation of inequality to one of equality authorized the use of equitable partition schema.

**3. Procedure with partition at the beginning:**

| **Student:** | **Comments:** |
|---|---|
| $588 \div 2 = 294$ | Division by two of the given total |
| $294 \div 16 = 19$ | Calculation of the number of 16-seat cars |
| $19 + 8 = 27$ | The use of the difference |
| $27 \times 16 = 432$ | Number of passengers in the 16-seat car train |
| $588 - 432 = 156$ | Calculation of the number of passengers in the 12-seat car train |
| $156 \div 12 = 13$ | Number of 12-seat cars |

The more frequent recourse to the equitable partition schema at the beginning suggested a less complicated representation than the preceding one, in which the inequality of the number of cars had to be taken into consideration. The students' errors in all of the arithmetical procedures occurred precisely in the coordination of the equitable partition schema and the inequality of the number of cars.

**PROCEDURES BETWEEN ALGEBRA AND ARITHMETIC** (revealing a process in formation)

In the following strategy (see Procedure 4), after undertaking an arithmetical method, the student subsequently abandoned it, and wrote an equation. The solution of the equation was used immediately afterward in a step which went back to an arithmetical procedure.

| **4. Student:** | **Comments:** |
|---|---|
| *Arithmetical trial:* $16 \times 8 = 128$<br>$588 - 128 = 460$ | previous procedure which took the difference into account at the beginning |

*Algebraic step:* $x + 8x = 588$; $9x = 588$; $x = 65$

*Arithmetical procedure:*

| | |
|---|---|
| $65 \div 2 = 32$ | equitable partition schema: the number of cars is divided by two |
| $32 - 8 = 24$ | use of the difference |
| $24 \times 12 = 288$ | passengers travelling in the 12-seat cars |

In this example, the student began by adjusting the number of passengers to arrive at two trains having an equal number of cars. In this arithmetical engagement, there is a semantic control of the situation and the relations which link the elements involved. When the student left this procedure in favour of an algebraic one, the continuation shows that there was no longer any control over the rates (which appear to be completely ignored) or the difference, although the algebraic treatment of the equation is correct. There was a complete loss of control over the situation. But as soon as the student returned to arithmetic, the control was regained. This and the following examples clearly show the distinctions effected by the student in the transition from one mode to the other. The passage to algebra requires the construction of a more global representation of the problem, which is in opposition to the sequence of dynamic representations which are the basis of the arithmetical reasoning.

**5. Student:**

*Arithmetical trial:* 16 x 8 =128; 588 - 128 = 460; 460 ÷ 2 = 230 (Control of the situation)

*Algebraic trial:*

$$\begin{aligned} 588 + 2 + 12 + 16 &= 16 + 8x \\ 578 &= 16 + 8x \\ 562 &= 8x \end{aligned}$$

Note that the order of the terms of the equation followed that of the presentation of the numbers in the text (loss of control - the student did not take into account the meaning of the quantities and of the problem).

**6. Arithmetico-algebraic strategy:**

**Student:**

8 x 16 = 128; 588 - 128 = 460 (difference taken into account)

Then the student switched to an algebraic mode, with the equation: $12x + 16x = 460$, and ended by solving the problem.

## ALGEBRAIC PROCEDURES

In contrast to the arithmetical procedures, in the algebraic procedures, the representation of the problem and the calculations do not generally undergo a parallel development. The solution process - which in arithmetic is based on a necessary transformation of the representation of the problem, in relation to meaning of the numbers obtained in successive calculations - needs at the beginning a representation of the relations between the data. It requires then for the student a global representation of the problem, from the start of the procedure, to infer an external symbolic representation modeling these relations, in the form here of an equation. Once the equation is expressed, the algebraic calculations often proceed independently of this representation of the situation. If the semantic control of the problem is re-established, it only happens at the end of the process. This type of engagement, totally different in its management of the data, is based on an element which is not present in arithmetic, that is, the introduction of precisely that quantity which is sought, the unknown quantity. We find there the analytical character of algebra so important to Vieta.

**Student:**

1st x 12
2nd (x+8)12 $588 = x.12 + (x+8)16$
$588 = 12x + 16x + 128$
$-12x - 16x = 128 - 588$
$-28x = -460$
$28x = 460$
$x = 16.42$
1st => $16.42 \times 12 = 197.04$ 2nd => $(16.42 + 8)\ 16 = 390.72$
1st 197.04 2nd 390.72

The equality $-28x = -460$, for example, cannot be interpreted in the context of the problem. This distance from the problem, necessary to proceed with the algebraic operations, makes it impossible, at this point, to verify if the results obtained concur with what is sought in the problem. A further effort must be expended to reinterpret the results from the symbolic operations.

The analysis of the students' errors in constructing their equations, throughout their procedures, showed that they did not take certain elements, such as rates, into account. Their symbolizations only retained certain aspects of what had to be represented.

## CONCLUSION

This analysis brought out differences between the *conceptual basis* that underlie the arithmetical and the algebraic modes of thought.

Arithmetical reasoning is based on representations which are particular to it, and involves a particular relational process. The successive calculations which work with known quantities are effectively based upon the necessary transformation of the relations which link the elements present, requiring a constant semantic control of the quantities involved and of the situation.

In algebraic reasoning on the contrary, the relations expressed in the problem are integrated from the beginning into a global "static" representation of the problem, nevertheless requiring specific necessary representations for this. This engagement, which is quite different in its management of the data, is based on the introduction of the unknown quantity at the very beginning of the process, and requires a detachment from the meaning of both the quantities and the problem to solve it.

Our results suggest that the difficulty experienced in the transition from arithmetic to algebra occurs precisely in the construction of the representation of the problem.

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# CONSULTANT AS CO-TEACHER: PERCEPTIONS OF AN INTERVENTION FOR IMPROVING MATHEMATICS INSTRUCTION

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**Abstract.** *A form of co-teaching was utilized as one mode of intervention in a project to improve mathematics instruction in Israeli secondary schools. Initial reactions of pupils, teachers, school principals, and co-teaching consultants suggest that, on the whole, this is a viable in-service approach for demonstrating instructional strategies to teachers and for increasing their involvement in reflection and planned instruction.*

## Introduction

For the past two years, a project for improving mathematics instruction has been on-going in six comprehensive secondary schools in the Northern Negev region of Israel. The project, which will continue for at least another year, was undertaken following a needs assessment survey which revealed that many teachers lacked proper teaching credentials and that few pupils were taking and passing national matriculation examinations in mathematics and the sciences at the end of Grade 12 (Ben-Chaim & Carmeli, 1990). The project is concerned in its entirety with improving mathematics and science teaching and learning in Grades 7 through 12. Attacking the problem from a holistic perspective, different forms of activity are being carried out at the various levels of instruction which include: 1) weekly workshops and individual consultation for Grades 7-9 teachers, 2) individual assistance for Grades 10-12 teachers, and 3) co-teaching of some Grades 10-11 classes. All modes of activity are explicitly geared towards helping teachers with average and above-average pupils, i.e. those with the ability to matriculate. Project consultants have extensive prior experience as teachers and as consultants in their subject area.

The co-teaching mode was undertaken primarily in mathematics classes. It has been selected as the focus of the present paper insofar as, compared to workshops and individual consultation which are somewhat common teacher in-service activities, co-teaching as an intervention mode is generally unknown. This form of activity was adopted in eight Grade 10 mathematics classes in 1990-91 and in nine Grades 10 and 11 classes in 1991-92. Four teachers have been involved in this activity for two years.

## Co-teaching

Co-teaching may be viewed as a form of team teaching in which two teachers are responsible for the educational advancement of a single class. As reported by Goodlad (1984), team teaching was extensively tried out in different schools in the United States during the 60's as one solution to the teacher shortage problem. Accordingly, qualified and experienced teachers were expected to work together with new and under-qualified teachers, thus ensuring both maximal use of personnel resources and the supervision of the less-qualified.

Co-teaching as a means for altering teaching behaviors in the context of a project is uncommon. However, the rationale behind such an approach is similar to that described by Goodlad. By pairing a project consultant with a particular teacher, expertise knowledge can be shared as both take responsibility for the instruction of a single class. In such manner, teachers are provided with an intensive, site-based, in-service experience: they are thus offered the opportunity to directly view expert teachers in action and to learn their strategies and approaches through joint-planning and coordination of lessons.

The co-teaching mode has been used in the project schools in the following manner. Throughout the course of the school year, on one set day every week, the consultant comes to give a regular classroom lesson in the co-teacher's class. The class teacher observes the lesson and often assists the consultant. Following the consultant's weekly lesson, co-teaching pairs meet to discuss the lesson and to plan the next week's teaching schedule. During these discussions, consultants endeavor to raise pedagogical and didactical issues relevant to the mathematical topic being taught. Since topics taught in the consultant's lessons are an integral part of the regular instructional curriculum, careful coordination must be made with the classroom teacher. Teachers and consultants try to plan their lessons and adjust their pace of instruction so that the consultant's lesson can be a natural continuation of the material taught by the teacher during the week. Accordingly, the consultants make suggestions to the teachers as to how to continue the teaching of the material and try to define for them what students need to accomplish in order to enable their own next planned lesson to be carried out smoothly.

By teaching actual classes, consultants are able to directly demonstrate different methods of instruction, to show how they cope with learning problems, and to demonstrate how to integrate material. They also become familiar with the needs of the specific class which enable them to give better advice to the teacher concerning appropriate materials, level of instruction, and pacing. Their intimate knowledge of the pupils and their demonstrated teaching skills are furthermore intended to enhance their credibility in the eyes of the teachers.

Classroom teachers are exposed to new ways of dealing with the curricular material and are able to view these methods in action in the natural environment of the class. In addition, they are given the opportunity to participate in collective efforts to plan instruction and to learn about teamwork. It should be noted that this mode of activity has involved only those teachers who are relatively new to teaching the grade level in question or who lack experience teaching it using up-dated materials.

On occasion, a change is made in this co-teaching schedule such that the regular teachers conduct the class on the day of the consultant's visit and the consultant observes the lesson. Observation enables the consultant to view the teacher in action, to diagnose the teacher's weaknesses and strengths in the classroom, and to concentrate activity with the teacher in the areas particularly requiring assistance.

Pupil progress in these classes is monitored through periodic examinations, some of which are specific to the class, prepared by the teacher and consultant together, and some of which are general, prepared by the Weizmann staff for all participating schools. Dates for the latter tests are set in advance which is intended as an external incentive to the co-teachers in their preparation of the pupils.

## Operational Difficulties

Intervention of this kind inevitably encounters numerous organizational problems along the way. One of the major difficulties is the problem of adjusting teachers' schedules to fit project activities. It means that each participating class must be studying mathematics on the day the consultant comes to teach and that their teachers have at least one free period for discussion and planning after viewing the consultant's lesson. Difficulties are also encountered regarding the coordination of teacher and consultant lessons. Classroom teachers are not always able to accomplish all that was planned for the week (often due to the cancellation of classes for school purposes) and the consultant is forced to change his/her own planned lesson accordingly. The co-teaching pair maintains telephone contact during the week so that the consultant is kept abreast of class progress and can make alterations as required.

The type of difficulties which particularly interested project directors were those whose source was psychological rather than organizational in nature. With regard to co-teaching, three questions were of special interest: 1) How do the teachers accept the consultant as a co-teacher? 2) How do the pupils respond towards having two teachers, one of whom is external to the school? and 3) How do the consultants themselves feel about their intensive involvement in someone else's classes? The central issue is whether or not the teacher's status in the classroom is undermined by the fact that an outside expert shares with him/her the teaching responsibilities for the class.

As the project progresses, information is being gathered on the reactions of the different parties to co-teaching as a form of intervention. This information is being collected through questionnaires to pupils, consultants, and teachers as well as by means of interviews with teachers, consultants, and school principals. Results from the first 1½ years of project operation are cited below; results from the full two years will be presented at the conference.

## Reactions to Co-teaching

Pupil reactions. At the start of the school year, it was carefully explained to pupils in the designated classes that both co-teachers would be responsible for their mathematics learning and that the teacher from the Weizmann Institute would be teaching them once a week on a regular basis. The general impression obtained from teachers and consultants was that pupils easily accepted this situation. Open-ended questionnaire responses from pupils in three classes indicated that, in two of the three, reactions were very positive and many pupils showed great enthusiasm, commenting that having two teachers was more interesting, made the material easier to understand, and reflected a more serious attitude in the school towards the importance of learning mathematics. In the third class, pupils also had positive comments to make but many of them complained that the pace of instruction was too quick for them and expressed a preference for their own teacher who they felt was sufficiently capable to teach them on his own (a comment supported by the consultant herself).

*Teacher reactions*. Teachers who were interviewed towards the end of the first year expressed satisfaction with the arrangement. They commented that working together increased creativity and resulted in better worksheets and examination forms. They felt that by working with a co-teaching consultant they had learned to better apportion instructional time among curricular topics.

Teachers felt that observing the consultant in the classroom was particularly useful. On a questionnaire administered to all project teachers, the observation of a lesson given by a consultant was the highest rated project activity, receiving an average rating of 4.22 out of 5 on usefulness. Teachers commented that observing the pupils from the side made them see the class differently and gave them greater insight into classroom dynamics and individual pupil difficulties. In addition, many of the teachers, after watching the consultant give a lesson, expressed amazement at seeing their pupils achieve higher levels of comprehension than they had previously thought them capable of reaching.

It is particularly significant that all teachers who were asked to participate for a second year raised no objections; rather they expressed satisfaction with the idea. One teacher, who co-taught with a consultant last year in Grade 10 and this year in Grade 11, has already requested to continue next year with a consultant in Grade 12. She feels that if she co-teaches once at each grade level, then she will be prepared to work on her own in these classes in the future.

*Consultant reactions*. On the whole, consultants felt comfortable with the co-teaching approach and were very satisfied with their close involvement in classroom instruction. Only one consultant expressed some discomfort insofar as she felt that the teachers she was helping were already good teachers and really did not require such intensive assistance. Talks with consultants revealed that they strongly believed in this form of intervention, noting that teachers with whom they had co-taught during the first year had made significant improvement which carried over into the following year. Changes occurred particularly with respect to lesson planning (greater thought given to goals, structure, and pacing) and the ability to design better worksheets.

Consultants felt that several factors made their entry into the classes as co-teachers acceptable to both teachers and pupils. First of all, explanations given to both groups at the start of the school year emphasized the joint responsibility of the co-teachers for the class. The pupils easily accepted this situation as natural and teachers did not feel that their self-esteem had been harmed. Secondly, most of these classes were plagued by severe discipline problems and the addition of another teacher was generally viewed with relief by most regular teachers who were only too happy to share their problems with someone else.

Although teachers were presented as equal, pupils however sometimes perceived the consultant as the more expert and saved up questions to be asked during the consultant's lesson. The consultants did not feel, however, that this was a problem for the teachers. They reported that since many of these teachers were relatively new to the profession or to teaching these grade levels with an up-dated curriculum, they tended to feel unsure of themselves and help from the consultants was welcomed.

As noted by Fullan (1982), Sarason (1982) and others concerned with educational change, resistance to change efforts is to be expected and perfectly natural in the transition to new modes of behavior. Moreover, teachers who are normally left alone in their classrooms do not usually take favorably to direct interference in the management of their "territory". Under the circumstances, it is almost surprising that the responses have been thus far so positive to the co-teaching form of intervention which entails intensive "meddling" in the teachers' territory.

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# THE REDUNDANCY EFFECT IN A SIMPLE ELEMENTARY-SCHOOL GEOMETRY TASK: AN EXTENSION OF COGNITIVE-LOAD THEORY AND IMPLICATIONS FOR TEACHING

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*The results of three experiments indicate the inadequacy of some conventionally formatted instructional material and emphasize the debilitating effect redundant material can have during initial instruction. Elementary-school children learning a simple paper-folding task learned more effectively from instructional material using diagrams alone than from material containing redundant verbal material, and self-explanatory diagrams with redundant material eliminated were superior both to instructions containing informationally equivalent text and to instructions consisting of redundant diagrams and text. This redundancy effect was evident not only when text was redundant to diagrams, but also when information was conveyed solely by means of diagrams. These findings extend the generality of the redundancy effect and have important implications for teaching.*

## Background

Printed instructional material in subject areas such as mathematics and physics typically use text and diagrams. Traditionally, especially in textbooks, the text and the illustrations are presented in a separated format, usually side-by-side. It has been shown in a number of subject-areas that integration of text and diagrams enhances learning. (Chandler and Sweller, 1991; Sweller, Chandler, Tierney and Cooper, 1990; Tarmizi and Sweller, 1988; Ward and Sweller, 1990). In terms of cognitive load theory, the act of splitting attention between and then mentally integrating textual and diagrammatical material presented in the traditional format imposes an unnecessary cognitive load and reduces cognitive resources available for learning. Comparison with modified instructional material incorporating physically integrated text and illustrations generates the **split-attention effect**.

The effect occurs only when the text and illustrations are unintelligible in isolation. Both text and diagram are necessary for the information to be understood. Sometimes, however, a procedure can be learned from diagrams alone, any accompanying text being irrelevant, or "redundant". In such cases, students tend to ignore the redundant text, and pay attention solely to the diagrams. Through integration of text with diagrams, however, students can be forced to pay attention to the redundant text even though diagrammatic material is sufficient by itself. It has been shown in several contexts (Chandler and Sweller, 1991) that in such cases learning is less efficient than when students are able to ignore the text. This is referred to as the **redundancy effect**.

**Procedure**

In the three experiments reported here, the effect on learning of both redundant text and redundant diagrams (each with respect to diagrams) was examined in the context of a paper-folding task. This task is found in the "space" strand of many elementary-school curricula and consists of folding a circular paper disk according to a sequence of instructions until a triangular shape is obtained. Each experiment consisted of two phases: an *acquisition phase* in which subjects learned the task by means of the instructional material provided, and a *testing phase* in which they carried out the task without aids. The only difference between the treatment groups was the format of the instructional material used in the acquisition phase.

In each experiment, children were treated singly. Each subject was asked to use the instructional material as an aid to learning the task. The time needed for this acquisition was recorded. The subject was then given a paper disk and asked to carry out the task without aids, the time taken to complete the task and the accuracy with which it was performed being recorded. In each phase, a time of ten minutes was recorded for subjects who over-ran this time.

**Experiment 1** Two sets of instructional materials were used in the acquisition phase: a sequence of diagrams intelligible by themselves ("diagrams-only" format), and the same sequence of diagrams accompanied by written instructions that referred to the diagrams ("redundant" format) [see Figure 1]. These written instructions were redundant to the diagrams but, unlike the diagrams, were unintelligible in isolation. Because LeFevre and Dixon (1986) have indicated that subjets are inclined to rely on example information (especially if it is in diagrammatic form) and to ignore written instructions, it was stressed to each subject in the "redundant" group that the written materials must be read. It was thought that children using integrated instructional material with extraneous information eliminated (*red't*) would learn more effectively than those using a format that includes redundant, but not self-sufficient, written material (*non-rd*).

The mean times in seconds, and the percentages of subjects correctly completing the task, are presented for both phases in the following table.

| | acquisition phase | | | | testing phase | | | |
|---|---|---|---|---|---|---|---|---|
| *N=15 for each group* | mean time | | % correct | | mean time | | % correct | |
| | non-rd | red'nt | non-rd | red'nt | non-rd | red'nt | non-rd | red'nt |
| | 369.9 | 463.7 | 66.7 | 33.3 | 277.8 | 457.6 | 60.0 | 26.7 |

The values of $t$ for the comparisons of mean times were $t=1.23$ (ns) for the acquisition phase and $t=1.90$ ($p<0.05$) for the testing phase. Comparison of percentages correct by means of Fisher's exact test with Overall's correction yielded a significant difference ($p=0.03$) for the acquisition phase, but not for the testing phase. The superiority of the non-redundant group provides evidence for the redundancy effect.

**Experiment** 2 In this experiment, the effect of self-sufficient diagrams alone and self-sufficient verbal instructions alone were compared. In addition, instructions for a "redundant" third group were constructed by presentation to students of both diagrammatic and textual material. Since the third step in Experiment 1 had proved difficult for many children, it was subdivided into three steps for Experiment 2, the accuracy of the representation being important if children are to construct an accurate mental image (Johnson-Laird, 1983). Thus, three experimental groups were used, each using a different set of instructions: diagrams-only format, text-only format, and diagrams-and-text format [see Figure 2], the latter being a "redundant format" because the parts were intelligible alone. Cognitive load theory postulates that the use of the redundant format will have a debilitating effect on learning in comparison with the use of a diagrams-only format. Both textual information and the redundant format are difficult to process for different reasons and no expectation could be stated with regard to their relative efficiency. It was expected that children using diagrams only (*diag*) would learn more effectively than those using informationally-equivalent written instructions (*text*) and those using a format in which the same diagrams are accompanied by redundant, informationally-equivalent written instructions (*red*), and that children using text only would learn better than those having redundant information.

The mean times in seconds, and the percentages of subjects correctly completing the task, are presented for both phases in the following table. There were fifteen subjects in each group. The data were analyzed by means of planned orthogonal contrast tests using *F*-tests in the case of the mean times and a test for homogeneity of binomial proportions (Marascuilo, 1975) in the case of the percentages correct. For the acquisition phase, the values of *F* were 3.07 (ns) for **the diag *vs* combined red+text** contrast and 0.03 (ns) for the **red *vs* text** contrast; for the testing phase, the respective *F*-values were 5.29 ($p<0.05$) and 0.11 (ns). The values of the test statistic for the same contrasts based on percentages correct were respectively 5.00 ($p<0.05$) and 0.00 (ns) for the acquisition phase, and respectively 6.10 ($p<0.05$) and 0.16 (ns) for the testing phase. These results indicate that the diagrams-only format was superior to the other formats.

| acquisition phase | | | | | | testing phase | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| mean time | | | % correct | | | mean time | | | % correct | | |
| diag | red | text | diag | red | text | diag | red | text | diag | red | text |
| 387.3 | 462.6 | 518.5 | 66.7 | 33.3 | 33.3 | 253.3 | 455.1 | 424.5 | 66.7 | 26.7 | 33.3 |

**Experiment 3** To date, investigations of the redundancy effect have concentrated on redundant text. Experiment 3 was designed to examine the effect of redundant diagrams. The diagrams-only instructional materials of Experiment 2 were compared with a modified version of these materials, in which "back views" were provided for some steps [see Figure 3]. Although it might be thought intuitively that this extra

information would enhance learning, it may be hypothesized from cognitive load theory that the inclusion of these extraneous, redundant diagrams imposes extra cognitive load and has a debilitating effect on learning. It was thought that children using diagrams only (*diag's*) would learn more effectively than those using a format in which the same diagrams are accompanied by redundant diagrams (*red'nt*).

The following table shows mean times in seconds, and percentages of subjects correctly completing the task for both phases.

| acquisition phase | | | | testing phase | | | |
|---|---|---|---|---|---|---|---|
| mean time | | % correct | | mean time | | % correct | |
| diag's | red'nt | diag's | red'nt | diag's | red'nt | diag's | red'nt |
| 244.6 | 396.0 | 90.0 | 60.0 | 108.9 | 314.8 | 90.0 | 60.0 |

The values of $t$ for comparisons of mean times were $t=1.92$ ($p<0.05$) for the acquisition and $t=2.12$ ($p<0.05$) for the testing phase. Thus, the diagrams-only format was superior to the format in which the same diagrams were accompanied by redundant diagrams. Comparison of percentages correct using Fisher's exact test with Overall's correction yielded no significant difference for either phase.

## Discussion

The results indicate the inadequacy of conventionally formatted instructional material and emphasize the debilitating effect redundant material can have during initial instruction. Experiment 1 demonstrated the advantage of presenting information with redundant material removed. The additional text provided for the group studying the redundant format had an inhibitory effect rather than assisting comprehension, as is the normal intention.

The findings of Experiment 2 suggest that self-explanatory diagrams with redundant material eliminated are superior to instructions containing informationally equivalent text and to instructions consisting of redundant diagrams and text.

Experiment 3 demonstrated that the redundancy effect is evident not only when text is redundant to diagrams, but also when information is conveyed solely by means of diagrams; this extends the generality of the redundancy effect. Providing additional diagrams detailing a perspective necessary for the successful completion of the task proved to inhibit rather that facilitate learning.

For teaching and learning, the implications of these findings bear on the manner in which initial instructional material is presented. Teachers should exercise extreme care when providing students with additional and seemingly useful information. If the processing of additional information (whether it be textual or diagrammatic) with essential information imposes an extraneous cognitive load, it may have a detrimental rather than beneficial effect on learning. For optimum effect, the usefulness of additional information must outweigh the consequences of processing it.

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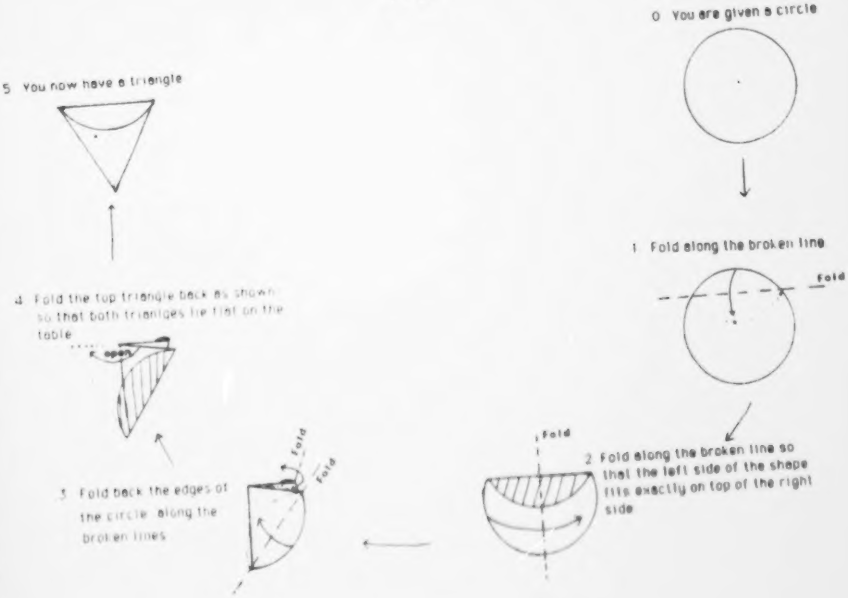

0 You are given a circle

1 Fold along the broken line

2 Fold along the broken line so that the left side of the shape fits exactly on top of the right side

3 Fold back the edges of the circle along the broken lines

4 Fold the top triangle back as shown so that both trianlges lie flat on the table

5 You now have a triangle

Figure 1: Redundant format instructional material for Experiment 1

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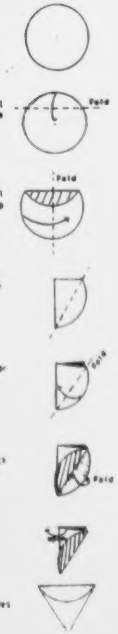

0 Take a circular piece of paper

1 Fold the top of the circle down so that its curved edge just touches the centre of the circle

2 Fold this shape in half so that the left side of the shape fits exactly on top of the right side

3 Imagine a line that runs from the exact bottom point of the figure to the top right-hand corner

4 Fold the upper curved flap back over this line

5 Fold the other curved flap back under this line in the same way

6 Open the centre fold

7 You now have a shape with three sides

Fold
Fold
Fold
Fold

Figure 2: Diagrams-and-text format material for Experiment 2

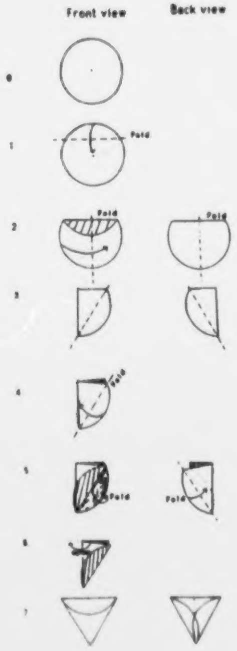
Front view
Back view
0
1
Fold
2
Fold
Fold
3
4
Fold
5
Fold
Fold
6
7

Figure 3: Redundant format instructional material for Experiment 3

# ON SOME FACTORS INFLUENCING STUDENTS' SOLUTIONS IN MULTIPLE OPERATIONS PROBLEMS: RESULTS AND INTERPRETATIONS

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*The report concerns the outcomes of 1023 pupils , aged 9-13 from two different instructional settings, on a problem involving several variables and solution strategies. Quantitative and qualitative analyses have been performed to assess the dependence of the strategies produced by pupils on age x instruction and numerical data. The findings from this study have led to some interesting interpretations regarding the students' transition to pre-algebraic strategies and the associated mental processes.*

## 1. Introduction

In recent years there has been an extensive array of studies which have investigated students' problem solving strategies in considerable depth (see Harel & C., 1991; Hershkovitz & Nesher, 1991; Lesh, 1985 ; Nesher & Hershkovitz, 1991; Reusser, 1990; Vergnaud, 1988)

In an attempt to add to this body of research , the study to be reported on has explored the mental processes underlying the strategies produced by students aged 9-13 when solving a non- standard contextually realistic problem involving multiple variables, operations and solution strategies ("trial and error" strategies, mental calculation strategies, "pre-algebraic strategies" - for a definition, see par.4-.... ).

The analysis of data focuses on quantitative and qualitative aspects of the evolution of strategies on the same problem, with respect to age and instruction, and the dependence of strategies on the numerical data.

A preliminary review of data suggests that pupils who are encouraged to perform a variety of strategies ("trial and error", hypothetical reasoning...) without rigid formalization and schematization requests reach the point of transition to "pre-algebraic" strategies earlier than those following more traditional instruction.

These results, along with some additional qualitative analyses of the protocols, bring to light some understanding of the roots of "pre-algebraic" strategies (with connections with research findings in the domain of pre-algebraic thinking: see par.5 ) .

## 2. The research problem

The purpose of this study was to better understand the mental processes ( i.e. planning activities, management of memory ...), underlying students' problem solving strategies in a "complex" situation. Towards this end the following problem was administered:

*"With T liras for stamps one may mail a letter weighing no more than M grams. Maria has an envelop weighing E grams: how many drawing sheets , weighing S grams each, may she put in the envelop in order not to surmount (with the envelop) the weight of M grams ?"*

Various numerical versions have been proposed to different classes:

| | money needed (C) | maximum admissible weight(M) | weight of the envelop(E) | weight of each sheet of paper (S) |
|---|---|---|---|---|
| (50,7,8) | 1500 | 50 | 7 | 8 |
| (100,14,16) | 2000 | 100 | 14 | 16 |
| (100,7,8) | 2000 | 100 | 7 | 8 |
| (250,14,16) | 3800 | 250 | 14 | 16 |

This problem was chosen because it represented a realistic situation for most of the students in this age range (interviews with a sample of Vth graders showed that over 75% of them think that the "cost" to send a letter must depend on its weight). In addition, it was possible to choose numerical values which kept into account the feasibility of mental calculations and the number of iterations needed to reach the result through progressive approximation from below. For example,in the transition from (50,7,8 ) to (100,7,8) there is an increase of iterations needed in an "approximation from below" strategy , but mental calculations are still easy .In the transition from (50,7,8) to (100,14,16 ) the mental calculations become more difficult, but the number of iterations needed remains the same.In the version with values (250,14,16) the mental calculations are yet more difficult and the number of iterations is increased. Finally, the problem format was similar to "evaluation problems" proposed in the Italian primary and comprehensive schools (multiple choices tests are not frequently utilized), but no such problem had ever been proposed to the students before.

### 3. Method

A pilot study was conducted at the end of 1990 with two classes of students in grades IV, V and VIII. The research problem with different numerical versions was administered and the findings were utilized to make subsequent choices, regarding the appropriatedness of the numerical versions with respect to grade level. In particular, the (250,14,16) case was excluded for the IVth-graders , due to the difficulties encountered by many subjects and the (50,7,8) case was not given to the VIII-graders, due to a great number of solutions written without any indications of the strategies performed .

For this study 63 IV,V,VI and VIII grade classes were chosen from schools in the north -west region of Italy, and this resultes in a total of 1023 participants. The study was conducted in October and early November 1991 (after about 4/6 weeks from the beginning of the school year). The classes were divided into two groups and were chosen in order to assure a similar sociocultural environment between them and suitable conditions for the experiment. In addition, in these classes age corresponded well with the grade level.

The classes in Group I, hereafter called the "Project" classes, included grades IV and V which are currently involved in a long term instructional innovation in the Genoa Group for primary school. The characteristics of this project which are relevant for this research, are presented in Boero(1989) and in Boero & Ferrari & Ferrero (1989) and are summarized below:

- the written calculation techniques are progressively costructed , under the guidance of the teacher, starting from the strategies spontaneously produced by pupils.This allows a great deal of "trial and error" numerical strategies to be performed by pupils, especially at grade II and grade III
- students,from the end of grade I, are required to provide verbal written representation of their strategies
- algebraic notation for an arithmetic operation is introduced only when the meaning of the particular arithmetic operation is mastered by majority of the class.No direct explicit pressure is exerced by teachers for the students to give formal representation of operations with algebraic signs (" words and numbers" resolutions are admitted up to the end of the primary school )
- in situations in which the students perform different strategies, comparisons of strategies (and formalizations) are organized and discussed (see Bondesan & Ferrari, 1991) ; and,
- problems involving more than one operation are proposed without intermediate questions.

The classes in the second group were composed of grades VI and VIII where the VIth graders come from "traditional" primary school classes . Here "traditional" instruction means:

- multiple operations problems guided by intermediate questions are widely proposed from the III to the V grade,
- "trial and error " or other case-by-case strategies are not encouraged,
- standard written calculations techniques are introduced early, and
- early formalization of arithmetic operations (with +, -, x, : signs ) is introduced and rapidly demanded as a standard code in problem solving (for single operations).

In Group II we have decided to select only VI and VIII grade classes with mathematics teachers affiliated to the Genoa group and working on a parallel, similar research project for the comprehensive school for the following reasons, emerging from our pilot study:

- difficulty to propose our problems at the beginning of the IV grade in "traditional" classes (because the "subtraction and division" problems are normally proposed, during the grade III, only as two-steps problems)
- difficulty to enter a "foreign" class and get completely verbally explicited resolutions (this is not frequent in Italy: it is requested only to indicate the most important calculations performed).
- for the comparison between VI graders' and VIII graders' performances, it was suitable to keep sociocultural variables unchanged
- the VIIIth graders of the chosen classes had not yet explicitely been involved with equations

The following table shows the distribution of the population involved in the study:

| | GROUP 1 | | GROUP 2 | |
|---|---|---|---|---|
| | n. of classes | n. of pupils | n. of classes | n. of pupils |
| grade IV | 9 | 145 | - | - |
| grade V | 24 | 396 | - | - |
| grade VI | - | - | 26 | 406 |
| grade VIII | - | - | 4 | 76 |

A sample analysis of the primary school copybooks of V grade and VI grade pupils belonging to the classes involved in this research showed these relevant differences

- the written calculation techniques were introduced 6-8 months in the primary school "Project" classes later than in parallel "traditional" classes
- algebraic notations were introduced in the "Project" classes 10-15 months later than in parallel "traditional "classes (for instance, the sign " - "was not introduced in the project classes before the second term of grade II ; the sign ":" was not introduced before the second term of grade III, and after at least one year of work on division problems)
- 20 to 25 "subtraction- division problems" were proposed in the "project" classes before the V grade (generally without an intermediate question ) while 40 to 52 subtraction/division problems were proposed in "traditional" classes from grade III to grade V ( 18 to 34 problems with the two step structure : first question asking for a substraction, second question asking for a division)

The problem was proposed by the teacher, with an "observer" present in the classroom. Up to the day of the study, the problem was unknown to the teacher . In order to avoid any difficulties in the classroom, if a pupil met with serious problems and insistently asked for help , he was helped by the teacher who provided written suggestions on the paper (this fact remained registered on the pupil's sheet of paper and these protocols were excluded from the following analyses). Pupils were asked to use only the sheet of paper on which the text of the problem was written. Some short interviews were performed (after the resolution of the problem) by the observer with pupils who had written very concise text, and with pupils who had adopted the "pre-algebraic" strategy in the cases (50,7,8) and (100,7,8), in order to understand their reasoning and motivation .

## 4. Results

The students' strategies resolutions have been analysed according to a classification scheme suggested by the data from pilot study, and corresponding to the aim of exploring mental processes underlying strategies.

Strategies were coded in the following manners:

"Pre-algebraic " strategies (PRE-ALG.). In this category the strategies involved taking the maximum admissible weight and subtract the weight of the envelop from it. The number of sheets is then found multiplying the weight of one sheet and comparing the product with the remaining weight, or dividing the remaining weight by the weight of a sheet of paper , or through mental estimates. If the problem would be represented in algebraic form, these strategies would correspond to transformations from :

$Sx + E \leq M$ to : $Sx \leq M - E$, up to : $x = (M - E)/S$

For the purposes of this research, we have adopted the denomination "pre-algebraic" in order to put into evidence two important , strictly connected aspects of algebraic reasoning, namely the transformation of the mathematical structure of the problem ("reducing" it to a problem of division by performing a prior subtraction) ; and the discharge of information from memory in order to simplify mental work . This point of view is connected to researches performed in recent years in the domain of pre-algebraic thinking (see par.5)

"Envelop and sheets" strategies (ENV&SH). This "situational" denomination was chosen by us because it best represented students' production of a solution where the weight of the envelop and the weight of the sheet are managed together . These strategies include "mental calculation strategies", in which the result is reached by immediate,simultaneous intuition of the maximum admissible number of sheets with respect to the added weight of the envelop; "trial and error" strategies in which the solution is reached by a succession of numerical trials , keeping into account the results of the preceding trials ( for instance , one works on the weight of some number of sheets and adds the weight of the envelope , checking for the compatibility with the maximum allowable weight ) ; "systematic approximation from below strategies", in which the result is reached progressively incrementing the number of sheets, adding the weight of the envelop and checking if the maximum allowable weight is reached or not; "hypothetical strategies", in which one keeps into account the fact that the weight of one sheet is near to the weight of the envelop, and thus hypothesizes that the maximum allowable weight is filled by sheets, and then decreases the number of sheets by one....... and so on

UNCLASSIFIED . While almost all of the strategies fit well in the above two categories, there were some which were difficult to interpret . For example, the transition from ENV&SH. to PRE-ALG. strategies during the resolution process (especially with the numerical versions (100,14,16) and (250,14,16 ) (example : "*16 + 14=30 , 30+16=46 . 100-14=86 , 86.16 =5 .......*" ; mixed strategies : "*16+14=30 ; 100-30=70 ; 70:16=4 4+1=5 sheets*" , etc. These "ambiguous" cases were thus coded as "unclassified"( the whole number of unclassified proofs was about 3% of the whole group).

**INVALID.** Another problem concerned the classification of incorrect resolutions in which solutions were lacking, or completely incorrect (for instance, strategies involving the amount of money ... divided by the weight of a sheet of paper) ; forgetting the weight of the envelope; or when numerical mistakes affected the final result in a relevant manner ( more or less than 10 times the correct result ).

In the case of numerical mistakes affecting only the final result in a "reasonable"way , or acritical presentations of the results (for instance, under the form : 43:8=5,375 sheets) protocols were classified (on the basis of the adopted strategy) .

The following tables represent a breakdown of the data:

TABLE 1 : (M,E,S) = ( 50, 7, 8 )

| | ENV&SH. | PRE-ALG. | UNCLASS. | INVALID |
|---|---|---|---|---|
| GROUP 1 / GR. IV | 71 (49%) | 26 (18%) | 6 (4%) | 42 (29%) |
| GROUP 1 / GR.V | 35 ( 38%) | 35 (38%) | 5 (5%) | 18 (19%) |
| GROUP 2 / GR.VI | 33 (34%) | 24 (25%) | 2 (2%) | 37 (39%) |

TABLE 2 : (M,E,S)=( 100,7,8 )

| | ENV&SH. | PRE-ALG. | UNCLASS. | INVALID |
|---|---|---|---|---|
| GROUP 1 / GR.V | 52 (39%) | 50 (38%) | 4 (3%) | 26 (20%) |
| GROUP 2 / GR.VI | 43 (34%) | 34 (27%) | 3 (2%) | 46 (37%) |

TABLE 3 : (M,E,S)=( 100,14,16 )

| | ENV.&SH. | PRE-ALG. | UNCLASS. | INVALID |
|---|---|---|---|---|
| GROUP 1 / GR.V | 24 (28%) | 36 (42%) | 4 (5%) | 21 (25%) |
| GROUP 2 / GR.VI | 25 (25%) | 32 (32%) | 2 (2%) | 42 (41%) |

TABLE 4 : (M,E,S)=( 250,14,16 )

| | ENV&SH. | PRE-ALG. | UNCLASS. | INVALID |
|---|---|---|---|---|
| GROUP 1 / GR.V | 13 (15%) | 42 (49%) | 6 (7%) | 25 (29%) |
| GROUP 1 / GR.VI | 11 (13%) | 34 (41%) | 1 (1%) | 37 (45%) |
| GROUP 2 / GR.VIII | 9 (12%) | 57 (75%) | 0 | 10 (13%) |

It is interesting to note from the analysis of the protocols that the students in the ENV&SH. strategies group who probably derived their solutions from a mental,global estimation of the situation represent (in grades V and VI) about 17% of the strategies performed in the (50,7,8) case, 13% in the (100,7,8 ) case ,while they are less than 3% in the (100,14,16) case .However,it is not easy to distinguish these strategies from the others (for instance in a protocol like this: "*5x8=40, 40+7=47* "), especially for the students in the GROUP 2, who might not have derived their solutions from mental evaluation , but from a succession of mental trials not reported on the sheet. Some interviews confirm the ambiguous character of these kinds of protocols . It is also interesting to observe that many usually successful problem solvers in each age group applied these strategies for (50,7,8 ) and (100,7,8) , while almost all of them were categorized as PRE-ALG. in the version with the numerical data (250,14,16).

A qualitative analysis of the data suggested the following:

- generally, the text of the PRE-ALG. strategies is linear, with subsequent declarations about the subtraction and the division and the result .For example :

"*I subtract the weight of the envelop : 250 - 14 = 236 , and I find the weight that may be filled with drawing sheets: then I divide : 236:16 = (calculations) = 14,75, and I get the number of sheets : 14* "

- frequently, the text of the ENV&SH. strategies is involved, especially in the (100,14,16) and (250,14,16) cases (where a global, mental estimation of the result is more difficult) .These are typical texts :

"*I multiply the weight of a sheet (16 grams) for a number chosen by chance, but not surmounting 10. and according to the result I multiply 16 for a lower or a greater number. When I get a number which works well I add the weight of the envelop (that is 14 grams). if the result exceeds 100 I make other trials. if the number does not exceed 100 I have solved the problem...* (+trials) ";

"*I multiply the weight of one sheet by a number of times such that their weight is contained in 250 g ,but I cannot exceed the weight of 250 g if I add the weight of one envelop to the weight of the sheets admissible with the 3800 liras fare . and so I must add a certain number of sheets to the weight of the envelop*(+trials) ";

- many ENV&SH. protocols from the students in the "Project" group reveal the students are in close proximity to a transition to a PRE-ALG. strategy .For example:

"*I must find the number of sheets which can be set in the envelop in order not to overcome 250 g , but with my sheets I must not arrive to 250 g, in order to be able to add the weight of the envelop and not to overcome 250. I must stop before.* (+trials up to 224)

The same kind of protocols is infrequent with students in the "traditional" classes

- there were 3 solutions in which students began with PRE-ALG and moved to ENV&SH. strategies , while transitions in the contrary direction were observed in 21 solutions

It should be noted, however,that it was not easy to evaluate and compare the protocols from different classes,because of the influences of different teaching styles , both past and present .This is especially true in the case of VIth graders,who had only for weeks of instruction with the same teacher.

## 5. Conclusions and discussion

A preliminary review of the results shows that there is a clear evolution with respect to age x instruction from ENV&SH. . strategies towards PRE-ALG.strategies within and between numerical versions (this is found in homogeneous groups of pupils: transition from IV grade to V grade; and from VI grade to VIII grade . both with classes of the same schools, and teachers working in analogous manne. Another interesting finding relates to the (50,7,8) case, where we find strong increase of PRE-ALG. strategies for students from grades IV to V in Group 1; this increase has occurred despite the fact that a PRE-ALG. resolution may be more expensive (in terms of calculations to perform) than some convenient ENV&SH. strategies.Some interviews performed with pupils in different classes (after the test) showed plausible reasons for their passage to PRE-ALG. strategies (also in cases asking for easy mental calculations), and these included they felt more secure, it helped avoid confusion,and it gave greater evidence: *"security", "avoiding confusion", "greater evidence"*..... are expressions frequently utilized by pupils to explain the motivation of their choices.

If we also look at the protocols of pupils who appear to be ready to make the transition towards a pre-algebraic strategy in a more difficult problem , for example the students who write :

*"I repeat 16 grams (which is the weight of a drawing sheet) till I reach 100 , and then I subtract 14 grams (the weight of the envelop) and so I must consider one sheet less" ;*

*"I reason : 16 x .... = about 86 because 86+14=100 ; I count: 16 x 4 =64 , too little; 16 x 6 = 96, too much; 16 x 5 =80 : let us try: 80+14= 94 " ;*

*"I subtract 16 grams from 100 many times, each time checking if it remains 14 grams for the envelop : 100 - 16 =84, yes; 84-16=68, yes; 68- 16=52, yes; 52-16= 36, yes; 36-16=20, yes - and I stop, because the envelop weighs 14 grams .... "*,

we see that the motivations and access to pre-algebraic strategies may be different; but in all of them there is a form of reasoning that may derive from a wide experience involving production of "anticipatory thinking" (see also Boero, 1990) . That is to say , under the need of economizing efforts, pupils plan operations which reduce the complexity of mental work . This interpretation provides a coherence amongst different results, concerning the evolution towards PRE-ALG. strategies with respect to age, as shown in the solutions produced in grade IV to grade V and in grade VI to grade VIII, as well as with respect to the results involving more difficult numerical data. Indeed, in the (100,14,16) and (250,14,16) cases we saw the difficulties encountered by students when attempting to manage the weights of the sheets and the envelop together (see par.4) .

All this may explain also why the large experience of subtraction/division problems presented as two steps problems (with an intermediate question) in "traditional" classes does not seem to produce all the desired effects: experiencing time separation of tasks, according to the suggestions contained in the text of the problem, may not effectively develop planning skills in the same direction.

Concerning research findings in the domain of pre-algebraic thinking, we may observe that there is some coherence between:

- our results , concerning an applied mathematical word problem ( Lesh,1985) , proposed to students prior to any experience of representation of a word problem by an equation and prior to any instruction in the domain of equations; and
- Herscovics & Linchewski's (1991) results , concerning numerical equations proposed to seventh graders prior to any instruction in the domain of equations. For instance, they find that an equation like $4n + 17 = 65$ is solved performing $4n= 65-17$ and then $n= 48 :4$ by 41% of seventh graders, while an equation like $13n + 196 = 391$ is solved in a similar way by 77% of seventh graders . This dependence of strategies on numerical values is similar to that shown in our tables (compare data concerning sixth graders in the cases (50,7,8) and (250,14,16)).

Filloy & Rojano define (for numerical equations) the "didactic cut" *"as the moment when the child faces for the first time linear equations with occurrence of the unknown on both sides of the equal sign"* ; for applied mathematical word problems, a "didactic cut" might be considered when the child faces for the first time a problem where a separation of tasks (through an inverse operation) must be performed in order to simplify mental work and avoid "trial & error" methods. Our study gives some indications about the consequences of two different long term instructional settings on students' efforts to overcome the obstacle represented by such a "didactic cut".

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# MULTIPLICATIVE STRUCTURES AT AGES SEVEN TO ELEVEN

## Studies of children's conceptual development and diagnostic teaching experiments

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*ABSTRACT: The conceptions of children aged 7-11 of various multiplicative problems were studied using interviews and written tests. In four-number porportion problems, changes from easy integer ratios to 3:2 and 5:2, caused only small falls in facility in a familiar context (price), but considerable losses in a less familiar context (currency exchange). Geometric enlargement problems gave rise to the wrong additive strategy. A diagnostic teaching experiment showed successful use of the method and materials by 10/16 teachers after 2 days' training.*

### *Introduction*

This study has two parts. One consists of an analysis of primary school children's conceptions of multiplicative word problems in different contexts. The second part was a study of the effectiveness of a diagnostic responsive teaching method (Bell et al, 1985) and associated teaching material developed from pilot studies.

The 16 teachers involved attended a two days in-service course. They were then free to choose how to implement the teaching activities, and which activities to pick from the teaching material. The period of teaching in each class was two weeks. At least two lessons of each class were observed by the researcher to assess the teaching style being used.

### *Results concerning childrens concepts and misconceptions*

This paper presents the results of one group of problems. For further details see Brekke (1991).

The *four-number problems* (see below) vary with respect to *numerical relationship* (3 or 4 to 1 and 1.5 or 2.5 to 1) *and structural context*, rate (price), currency exchange, measure conversion and geometrical enlargement:

AL2: 3 sweets are sold for 9 pence. How much for 12 sweets?

BL2: 4 sweets are sold for 6 pence. How much for 10 sweets?

AL5: German money is called Mark. John changed £3 and got 9 Marks. Sarah has £12 to change for Marks. How many Marks will she get?

AU6: John changed £4 and got 6 dollars. Sarah has £10 to change for dollars. How many will she get?

BL5: Jane measured her book using her short pencil. It was 3 pencils long. Ian used his rubber to measure the same book. It was 9 rubbers long. Jane measured the table with her pencil. It was 12 pencils long. Ian also measured the table with his rubber. How many rubbers long will the table be?

(This text was accompanied by an illustration).

AL12: A triangle is 3cm wide and 12cm high. A copy is made of this triangle, it should be 9cm wide. How high must the copy be to have exactly the same shape as the triangle? (This text was accompanied by an illustration).

**Table 1**
**Percentage of correct answers and wrong additive strategies for the four-number problems**

| | AL2 | BL2 | AL5 | | BL5 | AL12 |
|---|---|---|---|---|---|---|
| Correct | 26.6 | 19.2 | 30.8 | | 15.9 | 9.3 |
| Wrong additive strategy | 1.4 | 1.9 | 12.1 | | 32.2 | 10.7 |

| | AU2 | BU2 | | AU6 | BU6 | AU11 |
|---|---|---|---|---|---|---|
| Correct | 59.6 | 49.3 | | 19.3 | 44.2 | 33.9 |
| Wrong additive strategy | 1.3 | 9.6 | | 56.4 | 28.8 | 8.3 |

There are not big differences in facilities when ratios are changed from 3:1 or 4:1 to 1.5:1 or 2.5:1 for the most familiar context of price, but when this change is combined with a less familiar context of currency exchange there is a considerable drop in facility. Children regress to more primitive ideas, in this case the wrong additive strategy. (Compare Hart, 1981; Karplus et al, 1983). Note also the widespread use of the wrong additive strategy for the measure conversion problem, which maybe compared with Karplus' "Mr Short and Mr Tall".

The idea of geometrical enlargement was not well understood. When asked to make a larger copy of the same shape as a given triangle, the children drew triangles which were roughly the same shape but without calculating or measuring. The problem of making an enlargement involves more than the pure numerical relationship. Young children lack the experience of linking numerical relationships with geometrical objects.

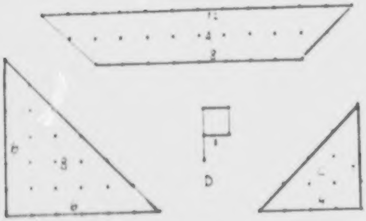
There are only small variations in use of a building up strategy across problem structure for the four-number problems for the younger children, and correct answers were equally distributed between multiplicative and additive answers. The exceptions were the problems with ratios 1.5 and 2.5 where building up strategy using the internal ratio was the most common correct method.

The context influences the choice of scalar or functional operator, with scalar procedures being dominant for rate and currency exchange problems, while the majority of correct answers to the measure and enlargement problems are obtained by a functional operator. Children tend also to start by considering the relationship between the two units used to measure the book, and applying this to the table. They are working within each object (book, table), while they in the previous items were working within the same measure (sweet, pence & £, $). Thus the dominant strategy of BU6 might be categorised as scalar. In AU11 the starting point is the relationship established between the snortest sides of the two triangles, and is thus a functional operator, though in this case it is also a scalar, since the unit of measurement (cm) is the same for the small and the large triangles (compare Bell et al, 1989). This preference for the functional relationship for geometrical enlargement problems is also reported by Friedlander, Fitzgerald and Lappan (1984). The dominance of the scalar operator is also reported in other studies (e.g Vergnaud 1983, Kurth 1988 and Karplus et al 1983). Around 85% of the wrong additive answers used the external difference as a constant for addition.

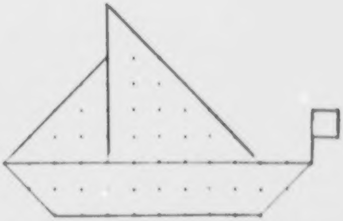
**New diagnostic teaching tasks**

The teaching unit is based on carefully chosen problems from different structural contexts.

WORKSHEET 5 U



These four pieces of a puzzle might be put together to make a boat.



The puzzle should be enlarged so that the sides of piece C will become [illegible] after they are enlarged.

a) Take one piece each to enlarge, use dotted paper.

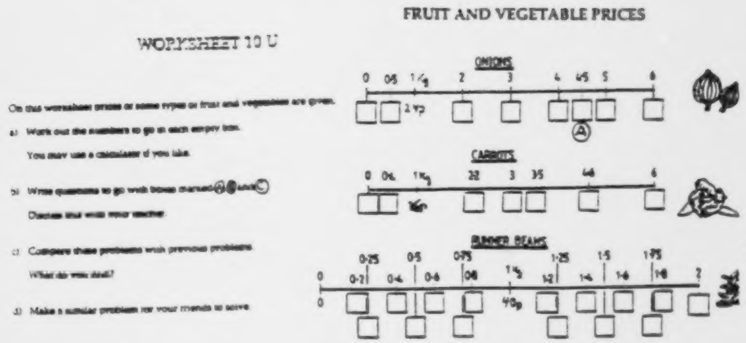
b) Cut out the pieces and bring together to make the boat.

c) Write down what was done and show this and the enlarged boat to your teacher.

d) If anything went wrong, discuss if there are other ways. Try again, write down what was done this time. Discuss with your teacher.

The main objective of the activities in Figure 1 is to focus on the inappropriateness of adding a constant difference in geometrical enlargements. The full set of activities also exemplifies the principle of starting with a difficult problem (5L) to bring out the expected misconception and following with an easier activity (6LZ) to give practice in using the correct strategy.

The purpose of the 'price-line' activity


FRUIT AND VEGETABLE PRICES

WORKSHEET 10 U

On this worksheet prices of some types of fruit and vegetables are given.

a) Work out the numbers to go in each empty box.
You may use a calculator if you like.

b) Write questions to go with boxes marked Ⓐ Ⓑ and Ⓒ.
Discuss this with your teacher.

c) Compare these problems with previous problems.
What do you find?

d) Make a similar problem for your friends to solve.

is to emphasise the need to interpret multiplication (or division) as a dimensionless *scale factor* along the double number line or as a *rate* across the number line.

*Success and communicability of the teaching method*

The classroom observations formed the basis for assigning the teachers to one of three categories according to the types of interventions leading to different levels of reflection on key aspects. These were: 1) the amount to which discussions of misconceptions were generated to bring key issues to the children's awareness, 2) the demand for explanations and justifications of statements, 3) the amount to which problem solving strategies were discussed, 4) the amount of discussion of problem structure, (classifying and making problems of the same structure) and 5) elements of generalisation. Teaching style A was described as highly concept intensive with a high level of reflection, where the elements described above were observed frequently. In a category B style these elements were observed sometimes and in category C scarcely. Of the teachers of the lower primary school classes, three were classed as style A, three as B and four as C; of the upper primary teachers, 3 were classed as A, 1 as B, 2 as C.

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Thus we may conclude that the 2 day training course was sufficient to enable 10 of the 16 teachers to acquire the method and to use it at A or B levels, and that these classes made clearly significant gains.

Further evidence of success of diagnostic teaching when done fairly well.

**Table 2 shows mean gains from pre to post test for the lower primary classes (max 22)**

| Class | 1 | 5 | 7 | 2 | 6 | 8 | 3 | 4 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Style | A | A | A | B | B | B | C | C | C | C |
| Pre Mean | 10.7 | 7.8 | 8.0 | 5.7 | 8.7 | 9.7 | 4.7 | 10.8 | 7.4 | 14.7 |
| Mean Gain | 2.3 | 5.5 | 3.1 | 1.9 | 3.7 | 3.6 | .9 | .6 | 2.7 | 1.3 |
| p value | * | * | * | .160 | * | * | .255 | .612 | * | .010 |

(* indicates a p value < 0.002)

A ONEWAY test applied to compare gains by teaching style showed significant differences between group C and the rest of the sample and no significant difference between group A and B. Significant improvements from pre to post test for every A and B class, and maintenance of scores through to the delayed post test, showed the long term effect of the teaching activities.

As expected the teaching activities have had various impact on problems from different categories.

Some findings are

1) The material has assisted a transition from employing the wrong additive strategy to delivering correct answers (few children used this strategy for the rate items). The shift was larger for style A and B classes than for style C.

2) The teaching activities have contributed to a change from several wrong categories to correct answers, but also from naive answers, which do not take into consideration the structural relationship between all the given numbers, to the wrong additive strategy, demonstrating that this strategy is a natural intermediate level of understanding of suchproblems.

3) The activities have not prevented children from employing the wrong additive strategy as a fall-back strategy when the number relationship has become more difficult combined with an unfamiliar context.

4) The errors children make in four number enlargement problems are clearly attributed to inexperience of geometrical enlargement, and secondly a failure in relating this to application of an appropriate number relationship. The activities in the material do not centre much around the first aspect.

The progress is considerable for all cartesian product items, showing that when children are helped to organise and represent the information in a systematic way, so that a repeated addition model can be employed, such problems are not particularly difficult.

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# MIDDLE GRADE STUDENTS' REPRESENTATIONS OF LINEAR UNITS

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*Students in Grades 5 and 7 were interviewed with a series of linear measurement tasks. The tasks were designed to investigate two aspects of students thinking about units of length: (a) the consistency with which they identify or construct units of length as line segment, and (b) the extent to which they reason appropriately about relationships between different sized units. It was found that many students applied a direct, point counting process to define units. However, the extent to which this occurred depended on the task situation. Point counting occurred with more students when numerals were juxtaposed with points- a phenomena which has instructional implications for the use of the number line for representing mathematical relationships.*

Children's conceptions of units initially develop in situations which involve enumerating, comparing and operating on sets of discrete objects. Regardless of variations in the physical attributes of the objects in a set, all objects represents equivalent units. Through years of counting experiences, children eventually learn to establish a one-to-one correspondence between their serial touching of or attention to each object and their simultaneous utterances of unique number names. If the one-to-one correspondence is not violated and the number names are uttered in a standard order, then the last number name uttered invariably determines the cardinal value of the set (Gelman & Gallistel, 1978). As such, children eventually construct a schema in which synchronous counting actions directly determine the number of units represented.

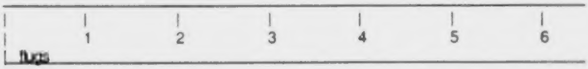
In linear measurement situations, the process of iterating linear units still conforms to the direct counting schema developed through the experiences of counting discrete units. However, when partitioning a line to represent linear units or when interpreting units represented as line segments, a direct relationship between synchronous counting actions and the number of units represented by the count is not invariant. Variations between the results of a counting process and the number of line segments implied by the count depend on a number of factors. These factors include whether one attends to line segments or points as the salient feature to be counted, and the plan of action followed. If line segments are the salient feature to which one attends then a direct relationship between the count and the number of units pertains. If points are the salient feature to which one attends, then the number of line segments are defined indirectly through the count of the points. For example, depending on whether one counts (1) all beginning and end-points, (2) only end-points, or (3) only internal points between line segments, 6 line segments would be represented by a count of 7, 6, or 5 points. It is necessary to develop a flexible counting schema in order to construct or interpret linear units adequately in such measurement situations. Students must incorporate notions of the geometric relationships between points and line segments, alternative plans of action implied by these relationships, and a means of evaluating the number of linear units implied by different counting procedures. Conceptions of counting developed in discrete unit situations are not sufficient. This paper explores the extent to which students in Grades 5 and 7 accommodate to these different measurement situations and construct alternative views of how units are determined. The results reported here are a small part of a larger study on students' representations of units and unit relationships in different mathematical domains (Cannon, 1991).

## Plan of the Study

Fifteen students from 2 schools participated in the study: 6 in Grade 5 and 9 in Grade 7. These students represented a range in mathematical achievement in each grade. A Measurement Concepts Test was administered, and later, each student was interviewed individually on a selection of the linear measurement tasks (See Figures 2). The tasks were designed to investigate two aspects of students thinking about units of length:

A. Ruler task (Interview & Test Task)

This ancient ruler measures lengths in "FLUGS." One "FLUG" is the same as two centimetres. Draw a line above the ruler that is 6 centimetres long.



B. Aggregate unit task (Interview & Test Task)

The line below is 4 units long. Draw a line that is 12 units long.

C. Partitioning tasks (Interview & Test Task)

This path is 5 units long.
a) Mark the 5 units on the path
b) Draw another path 3 units long.

Figure 2. Linear measurement tasks with explicit reference to units and number.

(a) the consistency with which they identify or construct units of length as line segment, and (b) the extent to which they reason appropriately about relationships between different sized units. Students were required to represent units in different problem situations.[1]

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[1] In addition, tasks were used in which students were required to compared the lengths of two "paths" which were made up of different sized line segments placed in irregular configurations. These comparison tasks were derived from Bailey (1974) and Babcock (1978). The nature of students reasoning strategies was investigated

Students' responses to the tasks reported in this paper were analyzed in terms of (a) the salient features to which they primarily attended when defining units, (b) how they interrelated units of different sizes. In the latter set of categories, with bi-relational thinking students accounted for the simple ratio between different sized line units, whereas with mono-relational thinking they treated different sized units as equivalent. Not all categories were applicable to all tasks. Distinctions in students' reasoning with multiple units (mono-relational versus bi-relational thinking) did not apply to the partitioning task. Only one unit was referenced in this task; all of the responses were necessarily mono-relational.

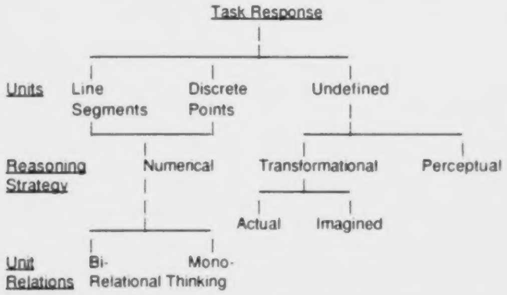

Figure 3. Analytical categories used to classify student responses to tasks.

## Results

Table 1 is designed to explore students' representations of units in response to the tasks in Figure 2 in several ways. First, it permits us to determine the general extent to which students constructed units which were either discrete points or line segments. Second, it allows us to compare each of the tasks with regard to the extent to which discrete points or line segments were constructed by students. And finally it reveals the extent to which their reasoning about the relationship between different units was mono or bi-relational. The students have been grouped in Table 1 according to the extent to which they represented units as discrete points or line segments: (a) predominantly discrete points, (b) predominantly line segments, and (c) consistently line segments.

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through these tasks (See Figure 3). Students' thinking about units and the comparisons of length in these situations are not reported here because of limits of space.

Table 1

Students' Representations of Units of Length and Relationships Between Units

| Response Groups | Ruler Task | | | | | | Aggregate Unit Task | | | | | | Partitioning Task | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Test | | | Interview | | | Test | | | Interview | | | Test | | | Interview | | |
| | D | Mi | L | D | Mi | L | D | Mi | L | D | Mi | L | D | Mi | L | D | Mi | L |
| Dominantly Discrete | | | | | | | | | | | | | | | | | | |
| James | M | . | . | B | . | . | B | . | . | . | . | B | M | . | . | . | M | . |
| Marlene | . | . | ? | . | . | ? | . | . | ? | . | . | M | M | . | . | . | M | . |
| Lolande | M | . | . | . | . | B | B | . | . | B | . | . | . | . | ? | M | . | . |
| Connie | NR | . | . | M | --> | B | . | . | ? | B | . | . | M | . | . | . | M | . |
| Fanya | B | . | . | . | . | B | . | . | B | . | B | . | . | . | ? | . | M | |
| Derek | ? | . | . | . | . | B | B | . | . | . | . | MB | M | . | . | M | . | . |
| Dominantly Line Segments | | | | | | | | | | | | | | | | | | |
| Edwin | B | . | . | B | . | . | . | . | B | . | . | B | . | . | M | . | . | M |
| Dahlia | B | . | . | . | . | B | . | . | B | . | . | MB | . | . | M | . | . | M |
| Tammy | B | . | . | . | . | B | . | . | B | . | . | B | . | . | M | . | . | M |
| Lara | B | . | . | . | . | B | . | . | B | . | . | B | . | . | M | . | . | M |
| Kasey | B | . | . | . | . | B | . | . | B | . | . | B | . | . | M | . | . | M |
| Line segment | | | | | | | | | | | | | | | | | | |
| Brock | . | . | B | . | . | B | . | . | B | . | . | B | . | . | ? | . | . | M |
| Kit | . | . | B | . | . | B | . | . | B | . | . | B | . | . | M | . | . | M |
| Coran | . | . | B | . | . | B | . | . | B | . | . | B | . | . | M | . | . | M |
| Pete | . | . | B | . | . | B | . | . | B | . | . | B | . | . | M | . | . | M |

Note The names of the students in Grade 5 are underlined

Units
D = discrete points
Mi = mixed, points & line segments
L = line segments

Unit relations
M = mono-relational
MB = mono- then bi-relational
B = bi-relational
? = no defined units
NR = no response

As can be seen in Table 1, most students represented units as discrete points in one or more tasks. However, none did so exclusively and a few never did so. There was a marked difference in the patterns of responses between different tasks. Students who represented units predominantly as discrete points, did so most consistently with the partitioning tasks. With the aggregate unit and ruler tasks, the form of their units was more variable. However, the ruler task was the situation in which the greatest number of students were inconsistent in the nature of their representation of units. This suggests a significant difference in the nature of students' thinking strategies in different linear measurement situations. A closer examination of how students responded to the partitioning and ruler tasks, in particular, reveals further differences in situational responses.

### Variations in Students' Representations of Units of Length: Partitioning and Ruler Tasks

In the partitioning tasks, some students appeared solely to attend to a direct relationship of counting points to determine the number of units as in Lolande's case (See Figure 4). Others incorporated attributes normally associated with linear units as in the case of Derek and James who both appeared to consider equal spaces between the points to be important, but varied in their attention to points and line segments as the salient feature for determining the number of units.

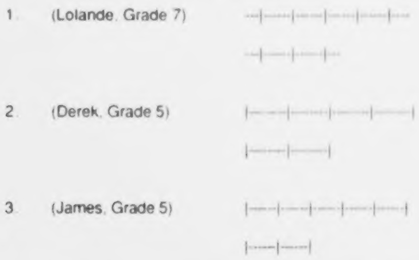

Figure 4. Examples of students' use of discrete points as units to partition a line into five units then drawing a line of three units.

When discrete units were used with the ruler task the reasoning behind these responses differed (See Figure 5). In the first example, the relationship between the size of the centimetre and flug was ignored. The points with each numeral determined the length of the line drawn. In the second example, the student attended to the 2:1 relationship between centimetres and flugs but counted the beginning and end points of the line segments as the units, beginning with the point associated with the 1 on the ruler. In the third example, the student converted centimetres to flugs using mental arithmetic and then represented the 3 flugs to correspond with the numerals on the ruler and not the number of line segments units.

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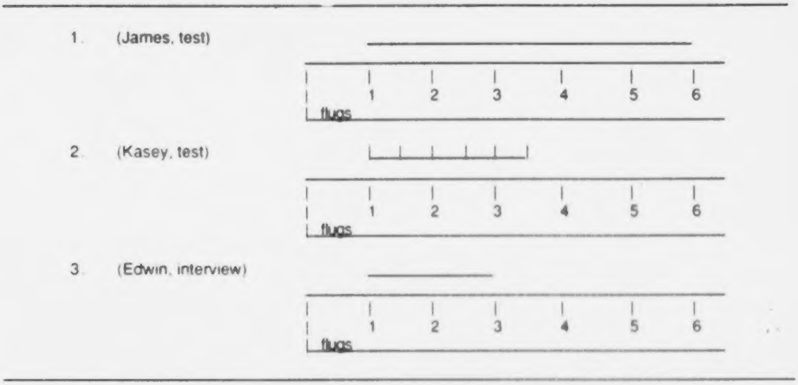

Figure 5 Students' use of discrete points as units with the ruler task.

### Point/Line Segment Conflict: Differences Among Tasks and Solution Strategies

Different procedures for constructing units influenced some students' attention to line segments or discrete points as units. Students who represented units predominantly as discrete points defined units as points when then used a partitioning process to resolve the aggregate units task. However, those in this group who solved the aggregate unit task by iterating line segments faced no ambiguity about how to determine the measure by the counting. The partitioning process led students to attend to the points rather than the line segments. With the ruler task there was the additional perceptual feature that points were juxtaposed with numerals. This juxtaposition further emphasized a counting relationship between points and the enumeration of units. All students who used discrete points as units and some who used line segments as units interpreted the "1" as the beginning marker of their representations of 6 centimetres, not as the end marker of the first "flug" unit. The structure of the ruler and the normal meaning of the numerals did not guide students' representation of 6 centimetres.

## Discussion

Differences in the representation of units of length often lay not in students' initial responses to the tasks, but in their reflection on the consequences of their first responses. For example, initial partitions of a line often were based on an assumption that the number of points determines the number of units. Upon reflection, many students revised their discrete counting plan and redefined their representation of units as line segments. However, the discrete counting schema predominated initially.

The representation of units as points or line segments appears to be influenced by perceptual and conceptual factors. The points are perceptually salient to the ruler and partitioning tasks. Attention necessarily is centred on points with the partitioning task and often centred on points during the ruler task. They are the component of the representation acted on synchronously with the verbal count, exactly the same actions as counting discrete units. However, it is indirectly through the points that line segments are defined as linear units and a conceptual understanding of this is necessary in the reflective process. One has to attend to the points, think about line segments, and keep track of the relationship between the count of points and the number of line segments. Even for students who in other situation focussed invariably on appropriate relationships between the count of points and the resultant number of line segments, the ruler situation generated additional attention to points. The common use of the ruler reinforces the notion that there is a direct relationship between the count of points and the number of units. Once a ruler is placed correctly, only the points and numerals have to be attended to "to read" the length. The discrete counting schema appears more likely to predominate regardless of a student's understanding of geometric relationships between the points and lines because the numerals and points are juxtaposed. The need to attend to other factors besides the points when representing or interpreting units is not recognized universally by the students.

It is insufficient to conclude, as Hirstein, Lamb, & Osborne, (1978) do, that a child who assumed that the count of points determines the number of linear units, "had no sense of a linear unit" (p.16). Students who attend only to points in one measurement situation did not necessarily do so in others. Eventually children must construct a conception of counting which admits to variable relationships between what is counted, how it is counted and the implication of such on the number units, a conception of counting that differs from their experiences in discrete situations.

## Implications

It has been reported extensively that the number line is a significantly difficult form of mathematical representation for students to interpret and use in a variety of instructional contexts (Behr, Lesh, Post, & Silver, 1983; Dufour-Janvier, Bednartz & Belanger, 1987; Ernest, 1985; Hart, 1981; Novillis-

Larson, 1980, 1987; Payne, 1975; Vergnaud, 1983). The tendency for the discrete counting schema to dominate when numerals are juxtaposed to points along a line may be one of the factors contributing to students alternative interpretations of mathematical relationships represented with a number line. This study was limited to students in the middle grades, but the author has observed the same application of a discrete counting schema when linear representation were constructed by pre-service teachers. It would appear that a fully flexible counting schema appropriate to the representation of linear units is long in developing.

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# CHOICE OF STRUCTURE AND INTERPRETATION OF RELATION IN MULTIPLICATIVE COMPARE PROBLEMS

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## ABSTRACT

*In this paper we present the results of a study about the processes used by 300 Primary School Children, in the 5th and 6th levels (11 and 12 years old), when they carry out compare problems. The analysis of written protocols has been made taking into account two qualitative solver variables: the choice of structure (additive or multiplicative) and the interpretation of the relation (direct o reverse) which pupils consider when they solve this kind of problems. The obtained results show that pupils' erroneous processes are related either to an inadequate choice of structure or to a reverse interpretation of the relation.*

Multiplicative word problems create many difficulties to children. There are several reasons for this but we are convinced that one of the most important is that pupils are normally confronted with a very scarce variety of standard multiplicative situations in their daily school work.

Multiplication and division concepts are usually taught having into account models which are deeply rooted in settled cultural elements. Multiplication is initially presented as a "repeated addition", even though this idea was temporally abandoned by the Cartesian product model. There is, however, no unanimity in the way to introduce the division concept.

Hendrickson (1986, p.26) stated that division is usually taught as a repeated subtraction, but this could be in American curriculum, because in Spain the main model is partitive division (cf. Castellanos, 1980, p.16).

Although these concepts are taught with the same basic model, pupils are expected to solve any kind of one step word problems of multiplicative structure, with the meaning Vergnaud has given to this term. Very often pupils cannot transfer what they have learnt from a specific standard word problem to another of the same conceptual field but different wording, and they will probably not be able to solve these problems by themselves, if there is not explicit teaching. It is useful therefore to establish the different models within the multiplicative conceptual field so that pupils could have specific instruction to overcome this gap. This is in line with the exhaustive investigation program Vergnaud proposed to study conceptual fields. Vergnaud (1990, pp.23-24) establishes six different points to carry out this empirical and theoretical work systematically. The two first are:

- *Analyse and classify the variety of situations in each conceptual field.*
- *Describe precisely the variety of behaviour, procedures, and reasoning that students exhibit in dealing with each class of situations.*

Vergnaud (1990; p.24) says that:

> *We have only bits and pieces of information on these complementary lines of inquiry.*

Both points have been widely treated in relation with additive structure conceptual field (cf. Carpenter, Moser, Romberg, 1982; De Corte & Verschaffel, 1987). This research has distinguished the very well-known semantic categories of problems.

Classification and analysis on multiplicative structure problems has been done from different points of view (Hart, 1981; Vergnaud, 1983, 1988; Schwartz, 1988; Nesher, 1988; Bell et all, 1988). Though in most of these studies multiplicative compare problems have not been considered, now the number of researchers considering this semantic category are increasingly growing (Hart, 1981; Nesher, 1988; Greer, in press).

**Semantic Factors in Compare Problems.**

In relation with addition and subtraction word problems, two semantic factors have been identified as the ones which may influence children's strategy when solving word problems: the static or dynamic character of the situation and the position of the unknown.

Multiplicative compare problems are static entities located at a time $T_0$. They can be mathematically described as a scale function between the referent set R and the compared set C

$$\begin{tikzcd} R \arrow[r, "f"] & C \\ x \arrow[r] & f(x) = \alpha x \end{tikzcd}$$

The scalar $\alpha$ may be used in a direct o reverse way, and so we have two possibilities:

$$\begin{tikzcd} R \arrow[r, "x\alpha"] & C & R \arrow[r, "/\alpha"] & C \\ x \arrow[r] & \alpha x & x \arrow[r] & x/\alpha \end{tikzcd}$$

If we work only with natural numbers, the scalar shows an increasing or decreasing comparison, respectively.

In Spanish increasing comparison are usually referred to with the following relational expressions: "x veces más que" (x times more than) and "tantas veces como" (times as many as).

Decreasing comparison, in turn, are referred to with relational expressions such as "x veces menos que" (x times less than) and "como una parte de" (as a part of). Similar expressions can also be found in the literature in other languages. For example, Harel, Post, Behr (1988: pp. 373-74) use the following relational expression:

> *"Ruth has 72 marbles.*
> *Ruth has 6 times as many marbles as Dan has.*
> *How many marbles does Dan have?"*

Hendrickson (1986, p. 29) uses this other one:

> *"There are 12 girls and 16 boys in a room.*
> *The number of girls is what part of the number of boys?"*

The relationship "n more than" can be interpreted in two different ways:

(1) as the additive relationship $A = n + B$ (i.e., $n = A - B$) or

(2) as the multiplicative relationship $A = n * B$ (i.e., $n = A/B$).

In the latter situation it is generally referred to as "n times as many". (Lesh, Post and Behr: 1988, p.101).

Vergnaud also considers relations such as "three times more" and "three times less" with a multiplicative meaning, expressing ratios (Vergnaud: 1988, p. 156). We have also found these relations in Spanish and French in old arithmetical text books:

> *"Se dice que dos cantidades variables son proporcionales cuando haciéndose una de ellas 2,3,4,... veces mayor ò menor, la otra se hace al mismo tiempo 2,3,4,... veces mayor o menor" (Sánchez y Casado: 1890, p. 85).*
>
> *"Deux quantitiés sont inversement proportionelles lorsque la première devenant 2,3,4... fois plus grande ou plus petite, la deuxième devient au contraire 2,3,4... fois plus petite o plus grande.*
> *Exemple:*
> *On a eu pour 100 francs 24 mètres d`étoffe; si on veut une étoffe 2,3,4... fois plus chère pour la même somme de 100 fr., on aura 2,3,4...fois moins de mètres" (Leyssenne: 1904, p. 240).*

Greer (in press) points out another important semantic factor on the compare multiplicative problems: the cultural dimension. Some performance differences with compare problems between English and Hebrew-speaking students can be explained because the simplicity of the Hebrew compare expression: P-3 instead of "3 times as many as".

All this makes us think that the relational expression used to build the comparison verbally is highly responsible for children's successful or unsuccessful performance when solving problems.

MacGregor (1991) has also pointed out the influence of this cultural and linguistic component and has analysed the misunderstanding between the relational expressions "times" and "times more".

## HYPOTHESIS

In our study we are going to use four different propositions to establish comparison in multiplicative compare problems: "veces más que", "veces menos que", "tantas veces como" and "como una parte de". Every one of these four expressions can be used in three different one-step multiplicative compare word problems. These three types differ in the unknown quantity (referent, scalar, or compared) of the comparative relation.

The relational term and the unknown quantity are both task variables, in the sense which Kilpatrick (1978) gives to them.

Having into account these two task variables, we establish twelve different multiplicative comparison word problems (see table 1). We claim the following hypothesis:

> *Error patterns need to be explained having into account not only the relational term or the unknown quantity but paying attention both to the two variables simultaneously and to their mutual influence.*

## METHOD

### Subjects

The subjects were 300 pupils from 4 groups of fifth-grade (11-years-old) and 4 groups of sixth-grade (12-years-old) in four Spanish schools at Granada. The project was done at the end of the academic year. According to the math curriculum, the notions of multiplication and division are introduced in the third grade.

**Tools and Procedure**

In this research we have worked on twelve multiplicative comparison word problems. These twelve problems arise from considering the task variable R (relational comparative proposition) with four values:

R1 = "times more than"

R2 = "times less than"

R3 = "times as many as"

R4 = "as one part of"

together with the task variable Q (unknown quantity on the relation) with three values:

Q1 = "compared unknown"

Q2 = "scalar unknown"

Q3 = "referent unknown"

We have controlled the following task variables: syntax, class of numbers and class of quantities. We used natural numbers and discrete quantities. To control learning effects, three homogeneous paper-and-pencil tests consisting of 4 one-step problems were prepared. In every test the items were problems that incorporated a different term of comparison in their statement. In all the problems we have used the static verb "to have". In the three tests the number size and the contexts used were controlled variables. The number triples used in the problems were (12, 6, 72), (18, 3, 54), (15, 5, 75), (16, 4, 64). Every pupil solved one 4-item test in a free-response form. All pupils completed the test in class. There was no time-limit to answer the test.

**Table 1**
*Six different types of the problems used in the study*

| | Increase comparison | Decrease comparison |
|---|---|---|
| Compared unknown $Q_1$ | Daniel has 12 marbles. Maria has 6 times as many marbles as Daniel has. How many marbles does Maria have?. | Maria has 72 marbles. Daniel has as many marbles as one of the 6 parts that Maria has. How many marbles does Daniel have?. |
| Scale unknown $Q_2$ | Maria has 72 marbles. Daniel has 12 marbles. How many times as many as Daniel does Maria have?. | Daniel has 12 marbles. Maria has 72 marbles. What part are Daniel's marbles in comparison to Maria's. |
| Referent unknown $Q_3$ | Maria has 72 marbles. Maria has 6 times as many marbles as Dan has. How many marbles does Daniel have?. | Daniel has 12 marbles. Daniel has as many marbles as one of the 4 parts that Maria has. How many marbles does Daniel have?. |

Note. Originally problems were in Spanish.

## RESULTS

We have classified pupils' answers to the twelve multiplicative compare word problems in three groups: right, wrong, and not answered. Answers are right when the pupil's process leads to the right solution, but we have not paid attention to small mistakes with operations. When the pupil has not given any solution, we have considered it a "not answered" reply.

We have received 1200 different answers to our problems; 694 were right answers (58%), 453 wrong (38%) and 53 "not answered" (4%). The right answers have been analysed in Castro et all (1991), and therefore we will only present here the analysis of wrong answers.

We present in table 2 the number of all the different processes which led to wrong answers found in everyone of twelve problems and in table 3 the number of the different processes leading to right answers. As we can see there is a great number of different wrong processes for every problem but only a few of the right ones.

**Table 2**
*Number of different processes leading to wrong answers*

| | $R_1$ | $R_2$ | $R_3$ | $R_4$ |
|---|---|---|---|---|
| $Q_1$ | 6 | 6 | 5 | 5 |
| $Q_2$ | 5 | 6 | 7 | 6 |
| $Q_3$ | 6 | 9 | 7 | 4 |

**Table 3**
*Number of different processes leading to right answers*

| | $R_1$ | $R_2$ | $R_3$ | $R_4$ |
|---|---|---|---|---|
| $Q_1$ | 2 | 1 | 2 | 2 |
| $Q_2$ | 3 | 3 | 4 | 5 |
| $Q_3$ | 1 | 1 | 1 | 1 |

If we add up both the wrong and right processes we can appreciate the variety of all the different processes which pupils have used to solve every problem (table 4).

**Table 4**
*Number of all the different processes for every problem*

| | R1 | R2 | R3 | R4 |
|---|---|---|---|---|
| Q1 | 8 | 7 | 7 | 7 |
| Q2 | 8 | 9 | 11 | 11 |
| Q3 | 7 | 10 | 8 | 5 |

## DISCUSSION

After the observation and analysis of pupils' written protocols we think that the most frequent mistakes can be explained under two basic error patterns:

1. *Change of structure*: the pupil understands the problem as if it had an additive structure (with the meaning Vergnaud gave this term).

For example, in the problem

María has 54 marbles.
Daniel has 18 marbles.
How many times as many as Daniel does María have?.

change of structure means that the pupil sets up as solution 54-18 or 54 + 18.

2. *Reversal of relation*: The pupil solves the problem using the reversal relation of the one which appears in the statement.

For example, in this problem:

María has 54 marbles.
María has 3 times as many marbles as Daniel has.
How many marbles does Daniel have?.

The reversal error means that the pupil proposes as solution 54x3 = 162.

In some problems the errors were mainly caused by only one of these patterns, but in others the errors were based on both patterns indistinctly. In table 5 we present the main error pattern for every one of the twelve problems; the wrong answers percentage over all the solutions (right, wrong, and not answered); and the most usual error pattern percentage over all the wrong answers.

We try to show that pupils' errors have been mainly produced by the same pattern, and although there are more different patterns, these appear with a very low percentage.

**Table 5**
*Most frequent error patterns and percentages over all the solutions and over all the wrong answers in every problem.*

| | $R_1$ | $R_2$ | $R_3$ | $R_4$ |
|---|---|---|---|---|
| $Q_1$ | Change of structure<br>9% (60%)<br>(*) (**) | Change of structure<br>20% (61%) | Ch. of str.<br>5% (50%)<br>Rev. of rel.<br>3% (30%) | Reversal of relation<br>11% (70%) |
| $Q_2$ | Change of structure<br>59% (92%) | Change of structure<br>60% (90%) | Change of structure<br>32% (80%) | Change of structure<br>32% (91%) |
| $Q_3$ | Ch. of str.<br>15% (46%)<br>Rev. of rel.<br>20% (56%) | Ch. of str.<br>23% (56%)<br>Rev. of rel.<br>18% (39%) | Reversal of relation<br>49% (84%) | Reversal of relation<br>30% (86%) |

(*) percentage over all the solutions.
(**) percentage over all the wrong answers.

The causes behind the error patterns detected in table 5 can be summarised in the following:

a) Errors in the four $Q_2$ problems are basically due to change of structure. This type of error is even bigger in $R_1$ and $R_2$ problems, where we find the words "more" and "less", respectively.

b) In the four $Q_3$ problems, errors are mainly of reversal relation, whereas in $R_1$ and $R_2$ variables we find both types of error patterns with a very similar percentage.

c) $Q_1$ problems has a very low percentage of errors.

d) $R_1$ and $R_2$ problems are mainly caused by a change of structure error pattern, but, as we have said in (b), in $Q_3$ problems we find reversal error pattern with a similar percentage.

## CONCLUSIONS

After the analysis of pupils' errors we can state the following conclusions:

1) The pupils in this study use two basic models to solve multiplicative comparison verbal problems. The two basic models have been: either they have used an addtitive structure pattern or they misplace the unknown in the compared ($Q_1$ problems). This is consistent with Fischbein theory of implicit models (Fischbein et all., 1985) and with the guiding frame model for understanding word problems, proposed by Lewis and Mayer (Lewis & Mayer, 1987).

2) Errors in Scalar unknown problems are mainly caused by a change of structure. Pupils usually identify this class with additive comparison problems, and so they give the a-b solutions instead off the a/b ones. This error pattern has also been detected with ratio problems by Piaget, Karplus and Hart (Hart, 1981).

3) Errors in "referent unknown" problems are based on the reversal pattern. Problems are solved as multiplicative structured but as if they were simple "compared unknown" model. Lewis and Mayer arrive at the same conclusion using the relation "times as many as" on consistent (compared unknown) and inconsistent (referent unknown) compare problems.

4) The two previous conclusions should be assessed considering the distracter effect produced by the relations "veces más que" (times more than) and "veces menos que" (times less than). Problem statements with these two terms lead to errors of change of structure. For this reason when we find these terms "scalar unknown" problems (which cause the same error pattern) the effect of both variables is reinforced, and this is why in both cases $R_1Q_2$ and $R_2Q_2$ we have the greatest percentage of error due to change of structure patterns. When these relational terms appear in "unknown referent" problems we find both types of error patterns indistinctly.

Our conclusions have to be understood in the controlled variables frame. That is, as Bell et all. have explained (1984, 1989), number size, class of numbers and the role of the numbers involved in a multiplicative relation could have influenced in the operations choice on multiplicative word problems. Our results could always have been affected by the change of these variables.

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# PERSONAL EXPERIENCE IN MATHEMATICS LEARNING AND PROBLEM SOLVING

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*This paper is based on a study involving three female first year junior college students, enrolled in a business mathematics course, to identify and understand the meaning of any uncharacteristic problem solving behaviour, from their perspective. The study is framed within a social perspective of mathematics. The paper discusses the students' use of a unique process of sharing "stories" of personal experiences which led to a "connection of knowledge" which was used to obtain a final solution to problems that had a context they could relate to their personal experiences. Based on this process, a conception of mathematics as experience is proposed.*

From my experience as a mathematics teacher, it became evident that many students engaged in problem solving processes or displayed learning behaviours that could not be meaningfully explained within a traditional conception of mathematics or the teaching of it. Such uncharacteristic behaviours seem to require educators/researchers to look beyond the "purely cognitive" (Cobb 1986) and pedagogical processes (Easley 1980) to understand the deeper meaning of their existence. This paper reports on a study in which I investigated the problem solving behaviour of three female first year junior college students, enrolled in a business mathematics course, to identify and understand the meaning of those aspects of their behaviour that seemed to fit this uncharacteristic label.

The study focused on word problems, specifically business math problems because of the students' circumstances. This focus became important because of my experience with students who often paid more attention to the social context instead of the mathematical context of these problems when trying to solve them. The literature also suggests that many students, particularly poor problem solvers, display the former behaviour. Traditional approaches to problem solving tend to favour the focus on mathematical context and discourage a focus on social context. Consequently, in this study, the latter behaviour was investigated to understand the implications of mathematics learning embodied in it.

The body of literature which provided a frame for this study falls in the category of a social perspective of mathematics where belief, personal meaning, culture, ... are important considerations(Fasheh 1982, Gordon 1978, Cobb 1986, Schoenfeld 1985, D'Ambrosio 1986). Within this framework, the social circumstances of the students play a significant role in their learning. In the case of females, some attempt has been made to explain the social implications of their treatment of problem context in terms of a preferred way of knowing rooted in relationships and connectedness (Belenky et al 1986, Buerk 1986). Thus, for example, they are likely to respond to problem context in terms of its humane qualities instead of its abstract ideas. However, by itself, this does not provide an understanding of the nature of the problem solving process these students engage in, from a social perspective. This study provides a way of beginning to fill this gap.

In this study data was collected through interviews, classroom observations, group and individual problem solving sessions, and journals. The analysis involved various levels of comparisons between the students' social biography, math biography and problem solving behaviour in terms of teacher-student relationships, peer interaction and problem solving strategies. Because of the limitation on the length of this paper, only the unique aspects of their problem solving strategy as a group and the conception of mathematics embodied in it will be presented.

## A DIFFERENT VIEW OF PROBLEM SOLVING

The unique problem solving behaviour of these students occurred when they were able to relate the context of the problems to their personal experiences. Their solutions of such problems presented a different way of viewing problem solving in math. The approach was not to focus on the mathematical context of the problem to understand it, but on the social context. It involved an exploration of the problem in a social or experiential mode instead of a cognitive mode. The process consisted of three stages: a sharing of "stories" of personal experiences, which led to a "connection of knowledge", which led to a final solution of the problem and a reflection on the consistency of their answer in the context of the connected knowledge. It seemed to be a different way of interpreting Polya's (1957) problem solving model. The following problem and excerpt of the group's solution of it will be used to discuss these aspects of the approach.

> Problem: A bookstore buys used books and sells those that are slightly damaged at 20% below the cost of a new book. If a customer paid $45 for a used book, how much did she save?
>
> Solution:
>
> (...)
> J: But it says you get 20% for slightly damaged books and the customer bought a used book, we don't know if it's slightly damaged. ... What if it's more than slightly damaged?
>
> L: That could be, which is why it's not working out. ... I don't usually buy used books, like school books, because I find they sell them for too much, even when they are more than slightly damaged....
>
> M: Well I always buy my books used ..., but it's from other people selling theirs and I don't take the price they give even if it's slightly damaged. ... No, but it's true. One time somebody was giving me a price and I just ..., I asked somebody else to give me a better price and I did. I got lots of sh-- from the first person but I still took it.
> J: But some people prefer to get the price they want than to take what they can get, which is so stupid if you don't need the book anyways. I know people like that.... Like this guy wanted to sell me a book for $30, he paid 35 for it. I said, "20". He said, "forget it." ...
>
> L: I know. ... But what you said makes sense because it could be more off if it is damaged a lot. But they don't give us that information. So ...

M: But we are dealing with a bookstore here. So they would want to sell it to make a profit too.

(...) [p.s.t.Nov.88]

In solving this problem, the students' first approach was not to look for a formula, recall a method illustrated by their teacher or use one from the text book, as they did with problems with abstract or "irrelevant" context. They started from scratch, depending more on their real life personal experiences to arrive at a solution, instead of an abstract connection to what they had learnt in class.

The "storying" stage started when their initial, individual attempts at a solution failed. To resolve the situation, they resorted to a special type of sharing. It wasn't a sharing of isolated opinions of what was wrong and how to fix it. It wasn't a cause-effect analysis of something that happened on their page. It was a different personal encounter, one involving a sharing of personal experiences directly and indirectly related to the problem context. "J" noted a "social" concern about the problem. She pointed out that the problem stated that books which were slightly damaged would be sold at a 20% discount, but it did not say if the book that a customer purchased was slightly damaged. It only said that the customer bought a used book. This generated a "discussion" of this "defect" in the context.

This "discussion" portrayed a different voice in relation to traditional problem solving processes; a voice that would likely be silenced in a traditional classroom because of its obvious deviation from the "norm". The discussion was not centered around "why didn't the math make sense", but "why didn't the context make sense". It also did not deal with the context in a general way or as a hypothetical situation. Instead, it personalized the context; integrating it into each participant's personal experiences.

The format of the discussion looked more like a narrative process as each person took turn at sharing a "story" of a personal experience related to the context. "M" and "J" shared an experience of buying used books while "L" talked about why she didn't buy used books. But these weren't just any "stories", they were biographical. They contained personal information that the others might not have known. For example, "M" talked about buying only used books, about not buying at "Eatons", about not allowing customers at work to take advantage of her, about not to give deals but to get them, about not to sell for a loss.... These were all reflections of her personal world as revealed in her "social biography". They were manifestations of her bargaining tendencies that defined the way she was and made sense of her world; a behaviour rooted in her childhood experiences. So she and the others seemed to be contributing something very personal in this sharing stage.

The outcome of this "story sharing" stage was a social construction of a reality of the problem as it was experienced by each of them. In particular, the "merging" of their personal experiences resulted in a "unique" type of knowledge used to solve the problems; a "personal connected knowledge".

I conceptualized "Personal connected knowledge" as the knowledge drawn out of the personal experiences of the individual members of the group by real images in the problem context and provides a concrete connection to the abstraction embodied in that context. It is an experiential reconstruction of the context of the problem. It is a reconstructed version of the original problem based on the "truths" extracted from their experience instead of those given in the problem. Thus it reflects what they care about; their meaning; their reality.

One way in which these qualities were manifested in the business problems they solved was through the group's concern about the negative implications of the context -- the unreasonableness of the business practices in the given situations, for example, the high cost of used books. Such concerns were usually "restoried" to create a context consistent with their perceived reality.

Once the "personal connected knowledge" was established, the students were able to get to a solution that was meaningful to them, but "correct" only when the "personal connected knowleg. ' did not conflict with the intended context of the problem. However, they seemed to be able to resolve such conflicts with appropriate teacher intervention.

## A DIFFERENT VIEW OF MATHEMATICS

The problem solving approach and other related mathematics learning behaviour of these students suggest a different conception of mathematics, a conception that is necessary to reflect its qualities when viewed from within a context of human experiences. The conception that emerged portrays mathematics, not as a dehumanized process or skill to be mastered, but a situation that is experienced in terms of human intention, fear, triumph, hope, .... One does not abstract the cognitive meanings from the human context, but deals with them holistically. These students' learning of mathematics was not an impersonal application of algorithms or problem solving strategies to some phenomena (real or fictitious) embodied in the problem, external to themselves. Instead, it was a sharing and connecting of personal experiences, a sharing and connecting of "self stories". This suggests that for them, the experience that is shared and connected is mathematics. In the business math problems they solved, mathematics became shopping, bargaining, selling ..., not the manipulation of the numbers abstracted from the experience. Consequently, from this perspective, not only the learning of mathematics, but mathematics itself is experience; an event in their life story.

This conception of mathematics provides a different way of viewing problem context and the way these students treated it. For them, it was not custom made clothing for some abstract concept, it was an event already "storied", or a "restorying" of one, in their life experiences. Thus "context" is viewed as the "storying" or "restorying" of a personal experience. Consequently, to talk about mathematics as embodied in the context is to talk about "self"; to talk about personal experiences.

This summary of this conclusion drawn from the study, does not do justice to the underlying conceptual considerations of the perspective being proposed. The goal, however, is to draw attention to an important dimension in considering what is mathematics, to understand some of the seemingly "bizarre" problem solving behaviour of many students and to deal with them meaningfully for the students.

## CONCLUSION

The outcome of this study is suggesting recognition of a dimension of mathematics that tends to be under-represented in the school curriculum thus denying students who understands mathematics in this mode the only opportunity to engage in a meaningful learning process. Given the current shifts in philosophy in mathematics education, this seem to be a timely outcome to be included in the broadened definition of mathematics in the school curriculum. Although the NCTM's Curriculum and Evaluation Standards (1989) have considered mathematics as problem solving, reasoning, connections, communications ... they do not seem to go far enough to include or explicitly recognize mathematics as experience in the context that emerged in this study. Similarly, the writing and cooperative learning movements in mathematics education have not gone far enough in terms of what is considered as personal experience in written and oral communication in the learning of maths. More attention is needed to the social autobiographical perspective in a narrative context to facilitate the way of knowing implied in this study.

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# INTERPRETATION AND CONSTRUCTION OF COMPUTER-MEDIATED GRAPHIC REPRESENTATIONS FOR THE DEVELOPMENT OF SPATIAL GEOMETRY SKILLS

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*In this paper we analyse the role that the computer may play in the development of the geometry skills that control representation by parallel perspective and orthogonal views. The study involved students of 12-13 years of age. Software especially designed by us was used to test problems of representation by orthogonal views of a poly-cube structure shown in parallel perspective, or vice versa. The a priori analysis of the teaching situations concerns the conceptual aspects inherent in the use of software with respect to problem-solving strategies that may be applied by the students. The discussion of the observations made concerns the role of computer mediation in the development of the students' strategies with respect to the involved knowledge.*

## 1. Introduction

Various studies carried out during the last few years stressed the difficulties that arise in the mastery of the projective system underlying parallel perspective and orthogonal projection drawings [1] [2] [3] [6] [8].
The difficulties that have been found concern the geometric conceptualization involved in the development of such notions as projection, change of point of view, and adoption of a reference system. Such notions are fundamental to permit an active control over the subject's perception with reference to the meaning-significant contents of the graphemes of perspective drawing or for the coordination of the points of view in the drawing of orthogonal projections. It is difficult to overcome these difficulties because in teaching practice, especially at comprehensive school level, there is a dearth of effective and tested paths towards development of skills of reading and of representations of the real physical space according to parallel perspective or orthogonal views.

The research about which we are discuassing in this paper concerns to what extent the computer may help the growth of this type of skills, encouraging the development of the concepts that the subject has about the projective system underlying these types of graphic representation [5]. The research that we carried out is circumscribed within the graphic space to tasks of reading and construction requiring the transition from a parallel perspective drawing to the corresponding orthogonal view, and vice versa.

Regarding the software specifications and the development of an a priori analysis of teaching situations, we availed ourselves of the collaboration of Claire Margolinas; the software implementation was developed by M.G. Martinelli for the thesis she wrote for her mathematics degree.

## 2. General Characteristics of the Software Implemented and Used

The software, written for the Autocad environment, exploits the potential of the Autolisp language. It permits to address the following two types of problems:

i) given a representation of a poly-cube structure in orthogonal views, construct the parallel perspective representation of the same structure;

ii) given a particular parallel perspective representation of a poly-cube structure, construct the corresponding orthogonal projections.

The second type of problem may be addressed in two different ways:

- constructing on each view the projections corresponding to any number of cubes before passing to the representation in another view (MOD 1);
- constructing on each view the representation corresponding to only one cube at a time (MOD 2);

The software was designed to structure the student's solution process in three distinct phases:

(1) the students make their observations on the statement of the problem and gather the information that they think may be useful for the implementation of their problem-solving strategy (solution anticipation phase);

(2) the students implement their problem-solving strategy exploiting the operational features of the software environment used (problem-solving strategy construction phase);

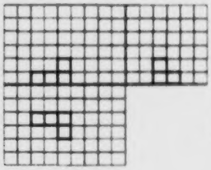
(3) the students have the possibility to carry out a validation of the problem-solving strategy that they have implemented, checking the correctness of the results obtained with respect to the proposed problem (stage of validation of the implemented strategy).

Each type of problem requires that the students work in both environments (orthogonal views and parallel perspective), using them respectively as starting and validation environment, and as working environment, and viceversa.

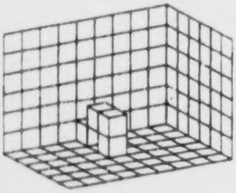
The basic geometrical elements manipulated by the used software are squares with unit side (working on orthogonal views), and cubes with unit edges (working on parallel perspective).

## 3. Learning Situations Proposed

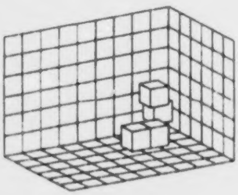
Here below we shall give the statements of the problems tackled by the students during the experiment. The three texts were introduced by a verbal question of the experiment leader, such as: "What you see on screen is a representation of the structure of an object. You must find what is the structure and represent it in this new environment".



**Prob A**

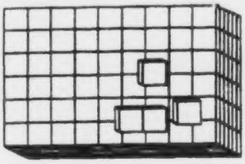


**Prob B (MOD 1)**

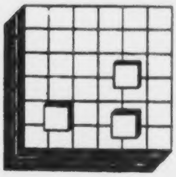


**Prob C (MOD 2)**

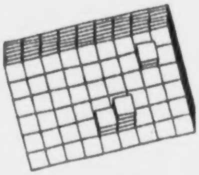
For problems B and C, specific commands are available with the visualization of the statement of the problem. These commands give to the student the possibility of exploring the poly-cube structure from three different points of view. Here below is an example relevant to problem C.



C1



C2



C3

The drawings illustrated here represent the object according to a non-transparent perspective view. Actually, by default the software represents the objects according to a wireframe (transparent) view, but a specific command permits to select also the non-transparent view of an object. We should also notice that in test B, the non-transparent representation does not permit to detect the presence of a fifth cube, that may be perceived in the transparent view, and is made more explicit by the reading and coordination of the other three available points of view.

Notice that in the views displayed on screen both the planes of the trihedron and those of the orthogonal views are grids and coloured so as to help the student see the correspondence of the parts.

The problem-solving strategy for test A is developed by inserting the cubes in the work-space defined by a tri-rectangular trihedron identical to the one represented in tests B and C. The cubes are inserted using the mouse to implicitly define on the horizontal plane the coordinates of a privileged vertex of the cube (the vertex whose projection on the horizontal plane is the nearest to the intersection of the three planes of the trihedron), and assigning to the cube an integer between 0 and 5 to define its height in relation to the horizontal plane, expressed in the units of measurements implicitly defined by the grids of the planes of the trihedral.

Underlying this way of working and representing is a conceptin of space that identifies the cube by means of the pair of coordinates of its projection on the horizontal plane, (seen as intersection of the row/column on the grid) and of the coordinate according to the axis orthogonal to it. It is also necessary to notice that the construction of each cube is possible only by specifying the coordinates of its projection only with respect to the horizontal plane; therefore, the three directions of the trihedron are not equivalent.

The construction of the solution strategy for tests B and C employs a mouse to insert the desired projections on the grids in the view planes. The use of software encourages a concept of space according to which each cube is identified by the pairs of coordinates of its projections on the three planes of a tri-rectangular Euclidean reference system, seen as intersection of row/column on the grids of the planes.

It should be noticed that in the solution strategy used for problem B (MOD 1), the demands made on the students to put into correspondence such pairs of coordinates are not pressing, as the construction of each view may be done independently from the others, even chronologically; on the other hand, in the solution of problem C (MOD 2) the coordination of such pairs of coordinates is necessary and should be performed with respect to the parallel perspective representation, in order to permit the unambiguous identification of each cube whose orthogonal view is desired.

By selecting a specific command, the students have the possibility of automatically accessing the orthogonal view or the parallel perspective representations corresponding, respectively, to the parallel perspective or orthogonal view constructed by them. This permits the validation of the employed solution strategies [7]. Finally, we should notice that the test of a problem, the students' solution and the representation automatically offered by the computer never can be simultaneously visualized on the screen. The students can pass from each one to another, selecting specific commands.

## 4. Experimentation Context and Methodology

Up till naw the experimentation was carried out on 5 pairs of students of 12-13 years of age, of the 2nd year of comprehensive school (Grade 7). Four pairs of students are in the same class. The teachers of these students belong to our research group; according to their opinions, these students belong to the upper half of their classes.

The problems were given to the students in the same sequence in which they are presented in this paper. To become familiar with the characteristic of the software, before attempting each of the three problems proposed, the students worked on three simple problems concerning the representation of only one cube. Also those problems constituted study situations.

Before starting the software activity, we asked each student to draw by her/himself on a blank sheet of paper the parallel perspective image of a cube and the three corresponding orthogonal views (top, front, and side).

No student had any difficulty in performing the task.

The students were instructed to use, during the tests, the blank sheets available to take notes. The entire work session of each pair of students was recorded by means of a tape recorder.

The authors of this paper, working respectively as experimenter and as observer, along with one of the students' mathematics teacher, assisted to the tests of each pair of students.

The analysis of the results is based on the transcription of the recordings made on tape, on the notes made by the students on the sheets of paper, on the printouts (made on a plotter) of the various validations performed by the students, and on the notes taken by the observer during the activity.

## 5. Our Hypotheses on the Role of the Computer

Our work endeavours to verify whether, and to what extent, the computer:

- may carry out an active mediation role in the students' learning process;
- may encourage the implementation of actions oriented towards the pursuit of a goal meant as anticipation of the future outcome of an action;
- may affect the inner mental processes of the subjects and the nature of the communication between them.

Within this framework, we formulate the following hypotheses:

a) the interaction with the computer may take the form of a social interaction in which action and communication may integrate dialectically;

b) the computer may have this important role in the learning process of the student only if the learning situations that the student tackles with the mediation of the computer permit the validation of the problem-solving strategies implemented by the pair of students;

c) a context permitting an interaction between "equals" affects the role that the computer may play in the learning process of the students involved.

Our hypotheses take in account the research findings according to which the use of computers may give to the concept of "proximal development zone", introduced by Vigotskij, a new perspective, i.e., "the child may do, with the aid of computer technology, things that he could not do alone or with the assistance of an adult" [7].

## 6. A Priori Analysis of the Learning Situations

The a priori analysis concerns the conceptual aspects of the use of the software in relation to the possible problem-solving strategies that the students may employ to solve the assigned problem situations, and also the conceptual aspects of the changes of strategy in the course of the computer activity.

The considered objects are abstract geometrical configurations (cubes and/or poly-cube structures), whose shape and location within the environment displayed on screen are not subject to any balance limitation. Since these are not objects characterized by a specific function, the shaping of a mental image of the represented object cannot be based on the identification of a known shape: the students have to construct it every time.

**6.1** - Problem A requires the interpretation of a system of views and the production of the corresponding parallel perspective representation.

The strategies implemented by the students may correspond to quite different levels of knowledge and of anticipation, and may be linked with the information that they establish before acting.

A priori, we identified four possible first approaches to the solution:

- proceeding without taking notes, trying to construct the representation trusting one's memory;
- reproducing the views on a sheet of paper, even if in different ways: sketch of the outlines alone, without reproducing the positions on the views' planes; position sketches, by means of coordinates; complete sketch of outlines and grids;
- sketching on a sheet of paper a parallel perspective representation of the object, with or without indication of a spatial location;
- drawing only one view (mainly from above) with a number placed over each square to indicate the number of cubes "present" at that position.

Generally speaking, we may expect that:

a) the behaviour of the students in implementing their solution strategy depends not only on the notes they take, but also on their knowledge of the system of the views. For instance, if the students never worked with orthogonal projections and do not know the geometrical rules that govern them, they may decide to draw three different objects corresponding to the three views as seen from above or three objects corresponding to the views as seen from the front.

b) at first the attention of the students is drawn mainly on the reconstruction of the object's shape; only later do they tackle the problem of the correct location of the object in the measured and oriented Euclidean space, especially if, in their anticipation, the coordinates of the object had not been stressed in any way.

**6.2** - To solve situations B and C, the students must understand what is the shape of the object and in which way it is placed within the trihedral space. They must gather the necessary information to be able to reproduce the views, considering that the single representation given in the statement of the problem is not sufficient.

A priori we identified four possible first approaches towards a solution:

- proceeding without taking notes, trying to construct the representation trusting one's memory;
- drawing the orthogonal projections directly after analysing the four parallel perspective representations offered by the software, recording or not the location of the object;
- reproducing the parallel perspective drawing on a sheet of paper to remember its shape, taking for granted the ability to obtain from it all the information necessary to the representation of the views;
- drawing the view from above, with a number placed over each square to indicate the number of cubes present at that position.

Generally speaking, we may expect that:

a) the initial strategies of the students are quite not different with regard to the construction modality to be used (MOD 1, MOD 2); they tackle problems B and C with the same spirit. Only later they may feel the need to conform their strategy to the operational characteristics of the available environment;

b) the solution method affects the setting up of an optimum strategy for the solution of the two types of task. When it is possible to insert all the projections on a view, and later on the other views (prob. B), the optimum strategy is based on the second approach, since there does not appear any strong necessity to link the object to the views . If it is necessary to insert the projections corresponding to one cube at a time (prob. C), the optimum strategy requires, beside the second approach, also the third or the fourth one; in fact it is necessary to remember the shape of the object, since for each projection it is necessary to recognize to which cube the projection corresponds, in order to be able to coordinate them coherently.

**6.3** - In all the types of problem tackled, the possibilities of learning offered by the software are linked to the students' development of strategies and knowledge in the course of the activity. During the process of construction of the solution strategy, the effect of every action performed by the students is visualized on the screen, yet without any indication whether it is adequate to the assigned task. The validation of one's own strategy or of a particular action, can be performed later, by comparing the test of the problem with the representation automatically produced by the computer in relation with the solution proposed by the students. The differences that students can possibly observe can induce them to go back to the performed actions, hence starting a dynamic process between anticipations and validations, that we believe meaningful for understanding the rules related to the representation by means of views and in parallel perspective.

This software, hence, offers a possibility of validation, but the decision whether , when, and how to use it belongs only to the students. Since we did not put any limitation on the number of validations that the students can perform during an exercise, we expect that this can influence the way they use it, hence that this can finally contribute to characterize the role of the computer in the students' learning process. We intend to evaluate a posteriori the students' behaviour in relation to the visual feedback offered by the computer.

## 7. Discussion of Some Results of this Study

The role of the computer as mediator in the students' learning process emerged from an analysis of the processes by which the students learned to overcome past errors and to construct and modify their solution strategy.

In our context we note that the computer-mediated activity allows an immediate actualization, by means of images, of the students' actions which can be judged based on the possibility of the available validation.

During our experimentation, we have observed that the visual feedback connected with the possibility of validation has been used in different ways by the students.

In same cases, overcoming past errors (as well as constructing a correct solution) has been based on a trial and error practice. The visual feedback connected with the validation allowed a correctness test of the involved actions. In these cases the students' actions depended on the way they perceived the visual feedback related to the past action.

For example, in problem A, all pairs of students have used a trial and error practice to develop a 3-D reference system conception suitable to the software operation.

We observe that the trial and error practice requires many validations and is characterized by the low level of the cognitive processes that students put in action.

Even though in some contexts, such as those linked with the discovery of the software operation, a trial and error practice very often can be the only useful approach for the students, we observe that in the task solution such approach is scarcely productive for the students' learning process.

We have observed that only a pair of students has been captive of the action - validation cycle that is typical of the trial and error practice; in the various tasks this pair has continued to perform actions based on successive approximations, relying every action upon the memory of the last visual feedback, without never carring out a global anticipation of strategy. We note that only this pair was unable to solve problem C, for which was necessary to work out a more articulate strategy than the trial and error one.

In all other cases the trial and error practice was used only at the very begining; subsequently we have observed that the actualizations through images offered by computer have allowed the students to observe "regularities" in the operation of the projection system underlying the use of the software, and to work out, in relation to them, behavioural schemes which have allowed the students to single out more articulated objectives for the problem solution.

For example, the observed "regularities" concern the correspondence between the position of a cube in the trihedral space and the localization of its projections onto the view planes, or the direction of the cube's edges with respect to the knots of the grid of the trihedral planes in the different possible points of view.

We observed that the elaboration of a behavioural scheme connected with regularities observed by the students is the result of a non formalized analysis produced by the dialectic between anticipation - action - actualization by means of images - validation, which is realized through the dialogue with the computer. A peculiar characteristic of the non formal analysis performed with the help of the computer is the binding that is established, while students of a pair communicate with each other, between the observed "regularity" and one or more "key words" taken from the vocabulary of both of them. Examining dialogues between students, results that, through the mediation of the computer, they have been able to give to these words particular meanings connected to the geometrical properties related with the observed regularities; such words tend to be implicitly transformed, during a work session, into "conceptual words" related to the geometrical knowledge which is requires by the problem solution. At the same time, also actions and controls, that the students have put in action in discovering a regularity, acquire a unitary meaning related to the naming process. This process leads to the construction of a behavioural scheme related to the observed regularities, and places at disposal tools that allow students to stop and reflect upon their own solution strategy, and possibly elaborate new anticipative hypotheses for the proposed problem.

Hence, we observed that, in these cases, the computer mediates the individual and pair activity of the students, influencing at the same time both the elaboration capabilities of each student and the quality of their communication.

The observations made during this experimentation raise two important problems.

The first problem is whether low level strategies, like trial and error, are intrinsic in any software with graphic feedback, hence in some cases avoidable only with teacher's assistence, or can be overcome by adding bindings to the dialogue between student and computer, suitably modifying the software characteristics.

The second problem concerns how to formalize the non formal analysis conducted by the students with the mediation of the computer, transforming what actually is an in-progress knowledge into a conscious learning of the geometric knowledge under consideration. We believe that this problem is closely correlated to the elucidation of the role that the teacher may play during and after the students' activity with the computer, that has not yet been studied in our work.

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# RESPONSES TO OPEN-ENDED TASKS IN MATHEMATICS: CHARACTERISTICS AND IMPLICATIONS

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## Summary

*The classification of the types of responses provided by primary and secondary schoolchildren to a particular form of open-ended mathematics task has enabled the investigation of the effects of factors such as collaborative work, age, instruction, question format, and culture or school system. The demonstrated reluctance of pupils to give multiple or general responses has been investigated through the distinction between the inclination to give such answers and the ability to do so. Comparison has been made with pupil responses to open-ended tasks in disciplines other than mathematics. Study has been made of pupils' accounts of their thinking while attempting such tasks and their justifications for their answers. Preliminary findings are reported of an on-going study into the learning outcomes of a teaching program based solely on the use of such open-ended mathematics tasks. The implications of this research are discussed with respect to the use of open-ended mathematics tasks for the purposes of instruction, assessment and as a research tool.*

## INTRODUCTION

This paper constitutes a progress report on an extensive and continuing program of research into the use of a particular form of open-ended mathematics task for the purposes of instruction and assessment. The task type has been given the title "Good Questions", and the characteristics of these Good Questions have been discussed elsewhere (Sullivan & Clarke, 1988; Clarke and Sullivan, 1990, Sullivan and Clarke, 1991a and b, Sullivan, Clarke and Wallbridge, 1991). Examples of 'good' questions are as follows:

*A number has been rounded off to 5.8. What might the number be?*
*Draw some triangles with an area of 6 sq cm.*
*Find two objects with the same mass but different volume.*
*Describe a box with a surface area of 94 sq. cm.*

The questions are different from conventional exercises in two major ways. First, these questions engage the students in constructive thinking by requiring them to consider the necessary relationships for themselves, and to devise their own strategies for responding to the questions. Second, the questions have more than one possible correct answer. Some students might give just one correct response, others might produce many correct answers, and there may be some who will make general statements. The openness of the tasks offers significant benefits to classroom teachers because of the potential for students at different stages of development to respond at their own level.

## THE STUDY

The following is a report of a five-stage project which sought to identify the way schoolchildren respond to Good Questions. The discussion of results is structured around the specific research questions addressed at each stage of the project.

### *General Method - Task administration*

In a typical administration, a set of four questions was given to participant classes of schoolchildren. The criterion for selection of classes was the willingness of their teachers to participate. Even though no teacher declined the invitation no claims are made about representativeness of the results for other schools.

In the first administration of tasks, the questions asked, the procedure for administration and the response coding system were as follows:

Subtraction

*Last night I did a subtraction task. I can remember some of the numbers.*

$$\begin{array}{r} 1\,4\,\_ \\ -\ \ 7\,\_ \\ \hline \_\ 4 \end{array}$$

*What might the missing numbers have been?*

Rounding

*A number has been rounded off to 5.8 What might the number have been?*

Area

*A rectangle has a perimeter has of 30 m. What might be the area?*

Fraction as operator

$\frac{2}{5}$ *of the pupils in a school play basketball. How many pupils might there be in the school and how many might play basketball?*

The format for administering the questions was the same in each class:

i) The question: "___ + ___ = 10 What might the missing number be?" was posed, and the responses suggested by the pupils were written on the chalkboard. The pupils in the class were asked to comment on what was different about this task from common mathematics questions. The response sought was that there are many possible answers.

ii) The first two questions were distributed (subtraction, rounding).

iii) The papers were collected and the answers reviewed. Again the possibility of multiple answers was discussed.

iv) The second two questions were distributed (area, fraction as operator)

The responses of the pupils to the tasks were coded. The coding was as follows:

0 meant no correct answers
1 meant only one correct answer
2 meant two or three correct answers
3 meant all or many correct answers
4 meant that a general statement was given

To illustrate the way that this code was applied to the rounding question, the following is the meaning of the codes. Individual correct answers were numbers such as 5.82 and 5.78. A code of "3" was given to a response like "5.75 5.76 5.77 5.78 5.79 5.80 5.81 5.82 5.83 5.84". Examples of responses which were considered to represent a general statement, "4", were "5.75 ... right up to 5.84999..." or "between 5.75 and 5.849".

## RESULTS

### Stage 1.

Stage 1 of the project addressed two questions:

> What types of responses do primary and secondary schoolchildren give to such open-ended tasks?
>
> Do the responses of the pupils vary depending on whether they work together or individually?

The purpose of this stage of the investigation was to ascertain the proportion of the pupils who gave each of the types of responses and to compare the responses of pupils in the different groups; individual, combined, and pair/ind. There were 39 pupils who completed individual responses, 49 students worked in pairs but submitted individual papers, and there were 39 pairs of students who gave combined answers. It was confirmed with the respective teachers that all of the concepts which are pre-requisite to these questions had been taught during the year prior to this study.

No clear differences between the groups emerged. The proportions who responded at each level did not appear to be influenced by whether they worked individually or with a partner. It had been anticipated that the pupils who had worked together would be more likely to give multiple or general answers. It was presumed that two minds might view each task differently and produce at least two responses, as well as alerting the pairs to the possibility of more than one correct answer. Whatever thinking or expectation is necessary to stimulate multiple or general answers appeared to be no more available to pairs than to individuals. Or conversely, whatever preconceptions limited the potential to give multiple answers affected both pairs and individuals alike.

## Stage 2

Stage 2 of the project addressed the questions:

> What effect does age or school experience have on the types of responses given to 'good' questions?
>
> Does the distribution of pupil response types differ according to culture or school system?

The same four questions were given to 99 year 10 students at two outer suburban high schools in Melbourne, to 97 year 10 students at a specialist mathematics/science school in Penang State, Malaysia, and to 86 year 10 students in two high schools in the USA. The questions were translated into Bahasa Malaysia for the Malaysian students. Subsequent independent re-translation of the Malaysian questions into English verified the accuracy of the translation. The protocol for the delivery of the questions was the same as in stage 1, and was followed in each case. The year 10 students worked individually.

The year 10 pupils were able to give better responses to each of the questions than the year 6 pupils. Fewer year 10 pupils made errors, and there were more who gave multiple and general responses. While noting that the pre-requisite content is prescribed at the primary level in curriculum documents, it was pleasing that most year 10 students were able to give satisfactory answers to the questions.

The profile of the responses of the year 10 students from Penang was marginally different from the year 10 Australian students. In each question a higher proportion of the Penang students were technically accurate in their responses to the four questions than the Australian students, but fewer gave general responses. The responses of the American students resembled the Malaysian sample on the first two tasks, the Australian sample on the third, and was quite distinctive with regard to the basketball question.

Overall, while there were more students at year 10 level than at year 6 who gave multiple and general responses, there was still a significant number of year 10 students who gave a single response even though it seemed to the researchers that a request for multiple answers had been implied by the wording of the question.

## Stage 3

Stage 3 addressed the following research questions:

> Does instruction increase the number of pupils who give multiple or general responses?
>
> Does question format affect the number of pupils who give multiple or general responses?

This third stage was an attempt to investigate the cueing inherent in the questions. On one hand, it was hypothesised that the pupils may have had too much experience at the single-answer type of question, and would need more experience than that provided in the protocol of stage 1. On the other hand, it was possible that the word "might" in the question may not, in a mathematical context, imply that more than one possible answer was required.

Although the two research questions for this stage are to some extent separate they were investigated concurrently. Three of the grade 6 classes who had participated in stage one were given instruction and another three of the grade 6 classes were asked the questions in a different format.

The lesson taught to the three classes aimed to broaden the pupils' view of what such questions are seeking. The first step in the lesson was to focus on the word "might". Questions relating to both everyday and mathematical situations were used to discuss both multiple and general answers. These classes are called the "instruction" group.

The other three classes were not taught a lesson but were given the same mathematical tasks in a slightly different format. Instead of phrasing the question like "What might the answer be?", words similar to "Give as many possible answers as you can" were used. This was intended to alert the pupils directly to both the possibility of multiple answers and to the requirement to give as many answers as they could. These classes are called the "question format" group.

There was a clear trend that the classes which responded to the questions with the revised format gave more multiple responses than the classes which had had instruction. It appears likely that there are pupils who are able to give multiple responses but who do not consider that the "might" questions invite such replies.

There was some indication that pupils' responses to particular questions were disproportionately influenced by the specific concept or concepts invoked. Because of the extended nature of open-ended tasks, the form of the individual question assumes greater significance than is the case with conventional closed tasks.

### Stage 4.

Stage 4 involved the implementation of a teaching experiment employing Good Questions and addressed the question:

> What are the learning outcomes of a teaching program based solely on the use of such open-ended mathematics tasks?

Stage 4 was conducted in a suburban Catholic primary school. The school had a high proportion of students from non-English speaking backgrounds and served a predominantly lower socio-economic community.

The experimental group was taught a unit on length, perimeter and area over seven one-hour lessons. The control group was taught the same topic. The teacher of the control class was instructed to follow the program presented in the most commonly used text. This was to simulate a standard approach to the topic. Data collected included attitudinal data, achievement data and observation data regarding classroom practices.

There were two interesting results. First, the experimental class were able to respond to the skill items as well as the control group, even though there had been no teaching or practice of skills in that class. Second, even though there more students in the experimental group who could give one correct response to the Good Questions than in the control, no students in either group attempted multiple or general answers. The meaning of this is unclear. During the program, many students in the experimental group were willing and able to give multiple and general responses to 'good' questions. It is not clear why they did not give such responses to the test items.

## Stage 5

Stage 5 sought to address the questions:

> Is the reluctance of pupils to give multiple or general responses to open-ended mathematics tasks replicated with open-ended tasks in other disciplines?
>
> Does the use of open-ended tasks for assessment purposes disadvantage any identifiable groups of students?

For stage 5 of the project, students at years 7 and 10 from three schools (one single-sex boys, one single-sex girls, and one co-educational) were asked to respond to open-ended items from a variety of academic contexts.

For instance, one question was:

> In a Victorian country town, the population fell by 50% over a period of 5 years.
> Why might this have happened?

The protocol guiding the administration took two basic forms derived from that for stage one. The administration varied task order and whether or not the requirement of multiple answers was made explicit in the task format.

Analysis disregarded the correctness of student response in applying the coding of stage one. As a consequence, the discussion which follows documents student intended response levels.

Order of task administration did not affect student response types.

Inferences which might be drawn from these results included:

- that the inclination to give single responses (or the reluctance to give multiple responses) is a product of schooling, and not peculiar to mathematics. Both year 7 and year 10 pupils were similarly recluctant to given multiple answers in all four academic contexts;
- that the explicit request of multiple responses produces a significant increase in the response level in other academic contexts, but not necessarily mathematics;
- that the ability to give multiple responses increases significantly with age, except in the context of literature.

## CONCLUSIONS

Mathematical power has been identified with the capacity to solve non-routine problems (NCTM, 1989), and open-ended tasks are seen as an appropriate vehicle for instruction and assessment of students' learning in this regard. Further justification for the use of open-ended questions for instructional purposes can be found in the work of Sweller and his associates (see, for instance, Sweller, 1989), where the use of goal-free tasks was associated with effective schema acquisition.

Are students' responses to open-ended tasks constrained unduly by their preconceptions about the nature of an acceptable response? It would appear that students in both year 7 and year 10 possessed a comparable reluctance to provide multiple responses. However, when multiple responses were explicitly requested, there was a significant increase in the proportion of multiple responses offered by pupils in Literature, Science and Social Science, but only to some items in Mathematics. This finding may be task-specific and warrants further investigation.

Are students' responses to open-ended tasks necessarily indicative of either mathematical understanding or capability? Certainly it appears that young children find it substantially more difficult than older children to provide multiple answers to mathematics tasks. This suggests that it may be inappropriate to ask primary school age children to give multiple responses. The legitimacy of relating student responses to non-routine and open-ended tasks to curricular content currently being studied continues to be the subject of research. Given current curriculum initiatives which employ open-ended tasks for assessment purposes (for example, CAP, 1989; VCAB, 1990), the results of this research assume some significance. Substantial additional research is required if we are to understand the significance of the meanings constructed by students in responding to open-ended tasks. Such research must address those student conceptions of legitimate mathematical activity on which their response inclinations are predicated (Clarke, Wallbridge & Fraser, 1992, and this research), and issues of cognitive load or working memory capacity and related developmental theories of learning outcomes which determine student response capability (for example, Collis, 1991; Sweller, 1989). These associated matters of inclination and capability must be understood if we are to employ such tasks with success in mathematics classrooms for the purposes of either instruction or assessment.

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# OVER-EMPHASISING PROCESS SKILLS IN SCHOOL MATHEMATICS: NEWMAN ERROR ANALYSIS DATA FROM FIVE COUNTRIES

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*The Newman procedure for analysing errors on written mathematical tasks is summarised, and data from studies carried out in Australia, India, Malaysia, Papua New Guinea, and Thailand are reported. These data show that, in each country, the initial breakdown point for a large percentage (more than 50% in four of the five countries) of the errors analysed occurred at the Reading, Comprehension or Transformation stages - that is to say, before the students had applied process skills such as the four operations. Additional data showing that Indian primary school students perform significantly better than Australian students of a similar age on straightforward computational exercises, but significantly worse on arithmetic word problems, raises the question whether there is an over-emphasis on process skills with insufficient attention being given to the role of language factors in mathematics learning.*

## The Newman Hierarchy of Error Causes for Written Mathematical Tasks

Since 1977, when Newman (1977a,b) first published data based on a system she had developed for analysing errors made on written tasks, there has been a steady stream of research papers reporting studies, carried out in many countries, in which her data collection and data analysis methods have been used (see, for example, Casey, 1978; Clarkson, 1980, 1983, 1991; Clements, 1980, 1982; Marinas & Clements, 1990; Watson, 1980).

The findings of these studies have been sufficiently different from those produced by other error analysis procedures (for example, Hollander, 1978; Lankford, 1974; Radatz, 1979), to attract considerable attention from both the international body of mathematics education researchers (see, for example, Dickson, Brown and Gibson, 1984; Mellin-Olsen, 1987; Zepp, 1989) and teachers of mathematics. In particular, analyses of data based on the Newman procedure have drawn special attention to (a) the influence of language factors on mathematics learning; and (b) the inappropriateness of many "remedial" mathematics programs in schools in which there is an over-emphasis on the revision of standard algorithms (Clarke, 1989).

### The Newman Procedure

According to Newman (1977a,b; 1983), a person wishing to obtain a correct solution to an arithmetic word problem such as "The marked price of a book was $20. However, at a sale, 20% discount was given. How much discount was this?", must ultimately proceed according to the following hierarchy:

1. Read the problem;
2. Comprehend what is read;
3. Carry out a mental transformation from the words of the question to the selection of an appropriate mathematical strategy;
4. Apply the process skills demanded by the selected strategy; and
5. Encode the answer in an acceptable written form.

Newman used the word "hierarchy" because she reasoned that failure at any level of the above sequence prevents problem solvers from obtaining satisfactory solutions (unless by chance they arrive at correct solutions by faulty reasoning).

Of course, as Casey (1978) pointed out, problem solvers often return to lower stages of the hierarchy when attempting to solve problems, especially those of a multi-step variety. (For example, in the middle of a complicated calculation someone might decide to reread the question to check whether all relevant information has been taken into account.) However, even if some of the steps are revisited during the problem-solving process, the Newman hierarchy provides a fundamental framework for the sequencing of essential steps.

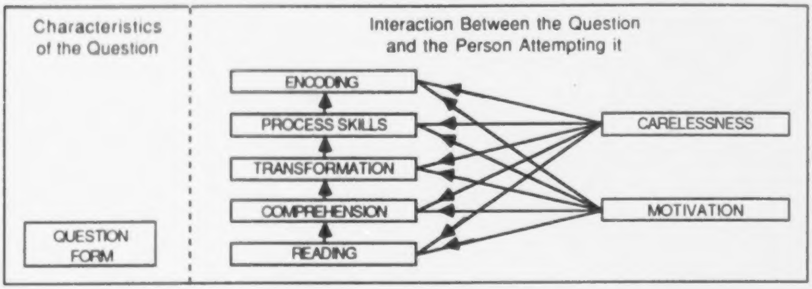
Characteristics of the Question
Interaction Between the Question and the Person Attempting it
ENCODING
PROCESS SKILLS
TRANSFORMATION
COMPREHENSION
READING
QUESTION FORM
CARELESSNESS
MOTIVATION

*Figure 1.* The Newman hierarchy of error causes (from Clements, 1980, p. 4).

Clements (1980) illustrated the Newman technique with the diagram shown in Figure 1. According to Clements (1980, p. 4), errors due to the form of the question are essentially different from those in the other categories shown in Figure 1 because the source of difficulty resides fundamentally in the question itself rather than in the interaction between the problem solver and the question. This distinction is represented in Figure 1 by the category labelled "Question Form" being placed beside the five-stage hierarchy. Two other categories, "Carelessness" and "Motivation," have also been shown as separate from the hierarchy although, as indicated, these types of errors can occur at any stage of the problem-solving process. A careless error, for example, could be a reading error, a comprehension error, and so on. Similarly, someone who had read, comprehended and worked out an appropriate strategy for solving a problem might decline to proceed further in the hierarchy because of a lack of motivation. (For example, a problem-solver might exclaim: "What a trivial problem. It's not worth going any further.")

Newman (1983, p. 11) recommended that the following "questions" or requests be used in interviews that are carried out in order to classify students' errors on written mathematical tasks:

1. Please read the question to me. *(Reading)*
2. Tell me what the question is asking you to do. *(Comprehension)*
3. Tell me a method you can use to find and answer to the question. *(Transformation)*

4. Show me how you worked out the answer to the question. Explain to me what you are doing as you do it. *(Process Skills)*

5. Now write down your answer to the question. *(Encoding)*

If pupils who originally gave an incorrect answer to a word problem gave a correct answer when asked by an interviewer to do it once again, the interviewer should still make the five requests in order to investigate whether the original error was due to carelessness or motivational factors.

### Example of a Newman Interview

Mellin-Olsen (1987, p. 150) suggested that although the Newman hierarchy was helpful for the teacher, it could conflict with an educator's aspiration "that the learner ought to experience her own capability by developing her own methods and ways." We would maintain that there is no conflict as the Newman hierarchy is not a learning hierarchy in the strict Gagné (1967) sense of that expression. Newman's framework for the analysis of errors was not put forward as a rigid information processing model of problem solving. The framework was meant to complement rather than to challenge descriptions of problem-solving processes such as those offered by Polya (1973). With the Newman approach the researcher is attempting to stand back and observe an individual's problem-solving efforts from a coordinated perspective; Polya (1973) on the other hand, was most interested in elaborating the richness of what Newman termed *Comprehension* and *Transformation*.

The versatility of the Newman procedure can be seen in the following interview reported by Ferrer (1991). The student interviewed was an 11-year-old Malaysian primary school girl who had given the response "All" to the question "My brother and I ate a pizza today. I ate only one quarter of the pizza, but my brother ate two-thirds. How much of the pizza did we eat?" After the student had read the question correctly to the interviewer, the following dialogue took place. (In the transcript, "I" stands for Interviewer, and "S" for Student.)

I: What is the question asking you to do?

S: Uhmm . . . It's asking you how many . . . how much of the pizza we ate in total?

I: Alright. How did you work that out?

S: By drawing a pizza out ... and by drawing a quarter of it and then make a two-thirds.

I: What sort of sum is it?

S A problem sum!

I: Is it adding or subtracting or multiplying or dividing?

S: Adding.

I: Could you show me how you worked it out? You said you did a diagram. Could you show me how you did it and what the diagram was?

S: (Draws the diagram in Figure 1A.) I ate one-quarter of the pizza (draws a quarter*).

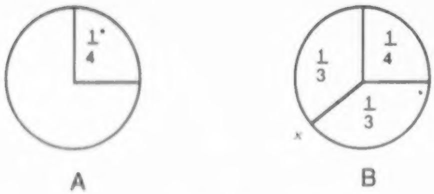

Figure 1. Diagrammatic representations of the pizza problem.

I: Which is the quarter?

S: This one. (Points to the appropriate region and labels it 1/4.)

I: How do you know that's a quarter?

S: Because it's one-fourth of the pizza. Then I drew up two-thirds, which my brother ate. (Draws line *x* - see Figure 1B - and labels each part 1/3)

I: And that's 1/3 and that's 1/3. How do you know it's 1/3.

S: Because it's a third of a pizza.

(From Ferrer, 1991, p. 2)

The interview continued beyond this point, but it was clear from what had been said that the original error should be classified as a *Transformation* error - the student comprehended the question, but did not succeed in developing an appropriate strategy. Although the interview was conducted according to the Newman procedure, the interviewer was able to identify some of the student's difficulties without forcing her along a solution path she had not chosen.

## Summary of Findings of Early Australian Newman Studies

In her initial study, Newman (1977a) found that *Reading*, *Comprehension*, and *Transformation* errors made by 124 low-achieving Grade 6 pupils accounted for 13%, 22% and 12% respectively of all errors made. Thus, almost half the errors made occurred before the application of process skills. Studies carried out with primary and junior secondary school children in Melbourne, Australia, by Casey (1978), Clements (1980), Watson (1980), and Clarkson (1980) obtained similar results, with about 50% of errors first occurring at the *Reading*, *Comprehension* or *Transformation* stages. Casey's study involved 116 Grade 7 students, Clements's sample included over 700 children in Grades 5 to 7, Watson's study was confined to a preparatory grade, and Clarkson's sample contained 13 low-achieving Grade 7 students. In each study all students were individually interviewed and with the exception of Casey, who helped interviewees over early break-down points to see if they were then able to proceed towards satisfactory solutions, error classification was based on the first break-down point on the Newman hierarchy.

The consistency of the findings of these Melbourne studies involving primary and junior secondary students contrasted with another finding, also from Melbourne data, by Clarkson (1980) that only about 15% of initial errors made by 10th and 11th Grade students occurred at any one of the *Reading*, *Comprehension* or *Transformation* stages. This contrast raised the question of whether the application of the Newman procedure at different grade levels, and in different cultural contexts, would produce different error profiles.

## Some Recent Asian and Papua New Guinea Newman Data

Since the early 1980s the Newman approach to error analysis has increasingly been used outside Australia. Clements (1982) and Clarkson (1983) applied Newman techniques in error analysis research carried out in Papua New Guinea, and more recently the methods have been applied to mathematics and science education research studies in Brunei (Mohidin, 1991), India (Kaushil, Sajjin Singh & Clements, 1985), Indonesia (Ora, 1992), Malaysia (Kim, 1991; Kownan, 1992; Marinas & Clements, 1990), Papua New Guinea (Clarkson, 1991), the Philippines (Jimenez, 1992), and Thailand (Singhatat, 1991; Sobhachit, 1991).

Rather than attempt to summarise the data from all of these Asian studies, the results of four studies which focused on errors made by children on written mathematical tasks will be given special attention here. The four studies, which have been selected as typical of Newman studies conducted outside Australia, are those by Clarkson (1983), Kaushil et al. (1985), Marinas and Clements (1990), and Singhatat (1991). Pertinent features of these studies, conducted in Papua New Guinea (PNG), India, Malaysia, and Thailand, respectively, have been summarised in Table 1.

Table 1
*Background Details of the Asian and PNG Studies*

| Study | Country | Grade level | Sample size | Number of errors analysed | Language of test & of Newman interview | Was the interview in student's language of instruction? |
|---|---|---|---|---|---|---|
| Clarkson (1983) | PNG | 6 | 95 | 1851 | English | Yes |
| Kaushil et al. (1985) | India | 5 | 23 | 327 | English | Yes |
| Marinas & Clements (1990) | Malaysia | 7 | 18 | 382 | Bahasa Malaysia | Yes |
| Singhatat (1991) | Thailand | 9 | 72 | 220* | Thai | Yes |

* Note that the 38 errors attributed by Singhatat to "lack of motivation" have not been taken into account for the purposes of this Table.

The percentage of errors classified in each of the major Newman categories in these four studies is shown in Table 2. The last column of this Table shows the percentage of errors in the categories when the data from the four studies are combined.

From Table 2 it can be seen that, in each of the studies, over 50% of the initial errors made were in one of the *Reading*, *Comprehension*, and *Transformation* categories. The right-hand column of Table 2 shows that 60% of students' initial breakdown points in the four studies were in one of the *Reading*, *Comprehension*, and *Transformation* categories. This means that, for most errors, students had either not been able to understand the word problems or, even when understanding was present, they had not worked out appropriate strategies for solving the given problems.

Table 2
*Percentage of Initial Errors in Different Newman Categories in the Four Studies*

| Study<br>Error Type | Clarkson (1983) (n = 1851 errors) % | Kaushil et al. (1985) (n = 329 errors) % | Marinas & Clements (1990) (n = 382 errors) % | Singhatat (1991) (n = 220 errors) % | Overall % |
|---|---|---|---|---|---|
| *Reading* | 12 | 0 | 0 | 0 | 8 |
| *Comprehension* | 21 | 24 | 45 | 60 | 28 |
| *Transformation* | 23 | 35 | 26 | 8 | 24 |
| *Process Skills* | 31 | 16 | 8 | 15 | 25 |
| *Encoding* | 1 | 6 | 0 | 0 | 1 |
| *Careless* | 12 | 18 | 21 | 16 | 14 |

## Discussion

The high proportion of *Comprehension* and *Transformation* errors in Table 2 suggests that many Asian and Papua New Guinea children have considerable difficulty in understanding and developing appropriate representations of word problems. This raises the question of whether *too much* emphasis is placed in their schools on basic arithmetic skills, and not enough on the peculiarities of the language of mathematics.

Further evidence for a possible over-emphasis on algorithmic skills was obtained in the Indian study (Kaushil et al., 1985) when the performances of the Delhi Grade 5 sample on a range of mathematical problems where compared with those of Australian fifth-grade children on the same problems. It was found that the Indian children consistently and significantly outperformed a large sample of Australian children on tasks requiring straightforward applications of algorithms for the four arithmetic operations (for example, 940 - 586 = ☐). However, on word problems, the Australian children invariably performed significantly better (see Table 3). Clements and Lean (1981) reported similar patterns when the performances of Papua New Guinea and Australian primary school students were compared on tasks similar to those shown in Table 3.

Interestingly, Faulkner (1992), who used Newman techniques in research investigating the errors made by nurses undergoing a calculation audit, also found that the majority of errors the nurses made were of the *Comprehension* or *Transformation* type.

Table 3
*Percentage of Indian and Australian Grade 5 Children Correct on Selected Problems (from Koushil et al., 1985).*

| Question | % Indian sample correct | % Australian sample correct |
|---|---|---|
| 940 - 586 = ❑ | 96 | 75 |
| 273 ÷ 7 = ❑ | 76 | 55 |
| A shop is open from 1 pm to 4 pm. For how many hours is it open? | 44 | 87 |
| It is now 5 o'clock. What time was it 3 hours ago? | 47 | 88 |
| Suniti has 3 less shells than Aarthi. If Suniti has 5 shells, how many shells does Aarthi have? | 42 | 73 |

The high percentage of *Comprehension* and *Transformation* errors found in studies using the Newman procedure in the widely differing contexts in which the above studies took place has provided strong evidence for the importance of language factors in the development of mathematical concepts. However, the research raises the difficult issue of what educators can do to improve a learner's comprehension of mathematical text or ability to transform, that is to say, to identify an appropriate way to assist learners to construct sequences of operations that will solve a given word problem. At present, little progress has been made on this issue, and it should be an important focus of the mathematics education research agenda during the 1990s.

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# REVISED ACCOUNTS OF THE FUNCTION CONCEPT USING MULTI-REPRESENTATIONAL SOFTWARE, CONTEXTUAL PROBLEMS AND STUDENT PATHS

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In this paper we argue that while the function concept should play a central role in the secondary curriculum, current curriculum based on formal definitions of function restricts both the conceptual and operational understandings students need to develop. We argue that building an understanding of functions through multiple representations and contextual problems provides an alternative epistemological approach to functions which suggests that experience working in functional situations, in *doing* functions, is more important than learning static definitions which mask its basis in human activity.

**Introduction.** The importance of the function concept in the secondary curriculum is virtually undisputed. Calls for the reform of mathematics place it at the center of the curriculum as an integrating concept and locate its importance in its modeling capacity. Moreover, with the increased power of computers and graphing calculators to display multiple and dynamic representations, one might expect the treatment of the function concept in the curriculum to be modified; however, there is little evidence that a theoretical framework for these changes has been carefully specified. In this paper, we outline the conventional educational view of functions and then suggest a three-part framework based on the use of: 1) dynamic multi-representational software, 2) contextual problems and 3) student interviews.

**The Conventional View.** In the conventional treatment of functions, the definition typically given is "a function is a relation such that for each element of the domain, there is exactly one element of the range." Such a definition does not necessarily preclude a multiplicity of approaches to functions; however, in textbooks and on assessment measures, one sees that a restricted view of functions emerges in which the algebraic presentation dominates the underlying assumptions about functions. These restrictions, which are overlapping, include an undue emphasis on :

1) the algebraic presentation with graphs as secondary and with tables a distant third.

2) a correspondence model which requires one to treat a function as a relationship between x and a corresponding y, rather than, for example, a covariation model where one can describe how the y values change in relation to each other, for given x changes (or vice-versa);

3) a functional format of "y=" so the equation is solved for y, and

4) a directionality of the relation, so that one can predict a y value for a given x.

Although there is nothing formally erroneous about such a concept of function, we wish to demonstrate that it is insufficient for approaches to functions which emphasize the use of multiple representations and contextual problems. We also wish to suggest that it makes it difficult to discuss, describe and evaluate the complexity and richness of a student's goal-directed investigation of functional situations.

**I. An Epistemology of Multiple Representations.** When one works in an environment that allows students to explore multiple representations of functions, one must learn to legitimize the use of a variety of forms of representation to describe functions. In Function Probe© (Confrey, 1991), a multi-representational software using graphs, tables, calculator buttons and algebraic approaches to functions, the student must coordinate a variety of representational forms. Meanings of the concept of function vary across these forms. To illustrate, we will work with the exponential function and demonstrate characteristics of the function which are more easily visible in different representations.

The Graph. A problem we have used with students involves setting up a time-line for a list of events which occurred over the earth's geologic history. We prepared a list of 'events' with dates. These events turned out to have a fairly uniform distribution when plotted on a log scale. One student tried plotting the number of years ago on the y-axis and the log of that number on the x-axis. The points lie on the graph of $y=10^x$. An idealized simulation of her graph(with differing y-scales) is shown below:

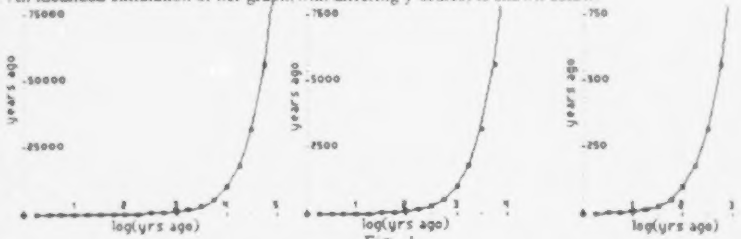
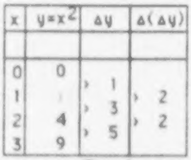


Fig 1

While scaling the y-axis, she noticed two things: First, no matter what value she used as the high y-value the graphs had the same shape. Geometrically, they were congruent. Second, the points were always bunched together near the origin, spread out near the top, and nicely spaced around the 'curve'.

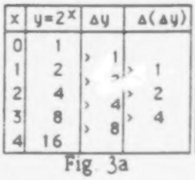
Thus she suggested that with appropriate scaling, she could have 'nicely-spaced' points for whatever period she wanted. Seeing multiplicative self-similarity as characteristic of the exponential graph can help students form a number of insights, including the equivalence of vertical stretches and horizontal translations, or the equivalency of half-life or doubling time. This quality of similarity is also easily recognizable in other visualizations of the exponential function and is a powerful way to recognize when an exponential function will prove an appropriate modeling device (sunflowers, nautilus shells, rams horns etc.).



| x | $y=x^2$ | Δy | Δ(Δy) |
|---|---|---|---|
| | | | |
| 0 | 0 | | |
| | | › 1 | |
| 1 | 1 | | › 2 |
| | | › 3 | |
| 2 | 4 | | › 2 |
| | | › 5 | |
| 3 | 9 | | |

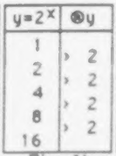
Fig 2

The Table. When students encounter a table for the exponential for the first time, they will often apply the strategies they have learned about polynomial functions, for example looking at the differences in the y's as the x values change at a constant rate. For polynomials looking at differences and differences of differences, etc. eventually leads to a column of constant differences and this is often what students expect (figure 2). When they do this with the exponential function, they may also see a repeating pattern emerging. However, if they are seeking to find a difference column that becomes constant over repeated application, they quickly learn of its impossibility, for all difference columns maintain the original constant ratio between terms (figure 3a).



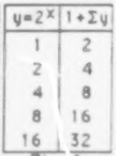
| x | $y=2^x$ | Δy | Δ(Δy) |
|---|---|---|---|
| 0 | 1 | | |
| | | › 1 | |
| 1 | 2 | | › 1 |
| | | › 2 | |
| 2 | 4 | | › 2 |
| | | › 4 | |
| 3 | 8 | | › 4 |
| | | › 8 | |
| 4 | 16 | | |

Fig 3a



| $y=2^x$ | ⊛y |
|---|---|
| 1 | |
| | › 2 |
| 2 | |
| | › 2 |
| 4 | |
| | › 2 |
| 8 | |
| | › 2 |
| 16 | |

Fig 3b



| $y=2^x$ | 1+Σy |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 4 | 8 |
| 8 | 16 |
| 16 | 32 |

Fig 3c

In Function Probe, we added a new resource, the ability to take ratios as well as differences and created a notation for doing so. Students learn to see a constant ratio as indicative of an exponential relationship (figure 3b). A verbal description, that for a constant change in x, the y values change by a constant ratio, is another signal for recognizing exponential situations. A second resource of Function Probe is the accumulation command. When one accumulates the exponential function, an exponential is produced(figure 3c). Between the accumulate and difference command, we can anticipate why the derivative and the integral of the exponential produce the exponential.

The Calculator. Logarithms often provide difficulty for students. We would attribute this both to their formal treatment in the curriculum and to the limitations of tables and most calculators which restrict students to either base 10 or base e. Thus they must invoke the somewhat mysterious change-of-base formula whenever they build an exponential or logarithmic model where the base (i.e. constant ratio) is not 10 or e. For example, in a situation where \$100.00 is invested at 9% interest, students will work out the annual multiplier (constant ratio) of 1.09 and eventually model the situation as an exponential function, $P = 100(1.09)^t$. If they have made a table of graph for the problem, they learn quickly that they can 'inverse' the function by changing the order of the table columns or reflecting the graph. However when trying to inverse the equation, they can easily write it as $P/100 = (1.09)^t$, but then are faced with the 'log problem'. Taking the log of both sides is a procedure that somehow works but has no apparent connection to their original way of making the equation. Because of this particular concern, we built two features into the calculator of Function Probe, a way to save procedures as user-defined buttons, and a button that takes any base as the input for either an exponential or a log function. Figure 4a represents a string of calculator keystrokes which calculates the amount accumulated after five years. By placing the 'variable' over the five in figure 4b, the user has designated this as a button, j1, that can now take any value for time as an input. If the student now wants to calculate what input creates the output of say 325, she can imagine undoing the button from right to left. Thus she might enter the set of keystrokes shown in figure 5a:

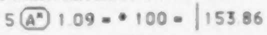


Fig. 4a

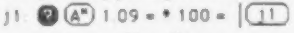


Fig. 4b

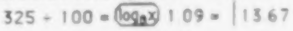


Fig. 5a

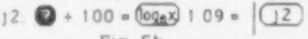


Fig. 5b

After dividing 325 by 100, they have to decide how to undo $1.09^t$. The "$\log_a x$" key becomes a key that undoes the action of an exponential. This ability to build and unbuild procedures and to 'undo' an exponential allows the log to play a stronger and more intuitive role in student's problem-solving. In addition the process of building algebraic equations from the linear procedures represented by calculator keystroke records and also building keystroke records from algebraic equations can provide strong assistance in helping student come to better understand the operational basis of algebra.

Integration of the Representations. In developing this example, we are not arguing that the traits of the exponential that are displayed cannot be seen across representations. In fact, searching for how to see them and learning to recognize them independent of the other representations is a valuable learning experience on the software.

**II. The Impact of Placing Functions in Contextual Situations.** The treatment of functions within multiple representations such as those described above ignores the question of how the function is generated, and/or where its application is witnessed or warranted. Others have argued for the value of contextual problems on the grounds that they are more socially relevant, realistic, open-ended, data driven, and inviting to students (Monk, 1989; Treffers, 1987; deLange, 1987) In addition to these important qualities, we see the value of placing functions into contextual situations as a challenge to the belief that abstraction requires one to decontextualize the concept from its experiential roots. Instead we see abstraction as the integration, reconciliation, juxtaposition of multiple schemes of action for a given concept.

In our work, we have chosen the contextual problems to highlight aspects of the function that are grounded in human actions. Piagetian research stressed the importance of the evolution of human schemes through the actions and operations one carries out on those actions. Reflective abstraction is the process by which the practical usefulness of those actions is acknowledged and the actions and operations become part of our mental repertoire in the form of schemes.

Accordingly, we stress the development of an operational schemes for understanding functions. For instance, to recognize contexts in which the exponential scheme is useful, we have postulated an underlying scheme called "splitting". Splitting, we suspect, has its roots in early childhood in sharing and congruence, primarily the binary split, and forms a basis for division (and multiplication) that is not well described by repeated subtraction (or addition). Doubling and halving are the simplest instantiations of it, and contrary to repeated addition views of multiplication, the split is a primitive multiplicative action which is often embedded in a repeated division and multiplication structure. The invariance in the operation is the constant ratio of 1:2 or 2:1. We witness students solving problems such as 426/18 by going 18, 36 (2 tallies), 72 (4 tallies), 144 (8 tallies), 288 (16 tallies), 476 (32 tallies) and then adjusting to get the exact result (Confrey, 1992a). The children understand that they can "reunitize" (Confrey, 1992b) from the unit 18, 36,. . . to the unit 288, to reach their goal more quickly. We have also argued for the coordination of the splitting structure with similarity as an underlying basis for the exponential function.

Such an approach to the concept of function locates a function within a family of functions and examines how that prototypic function is fit to the existing data or situation (Confrey and Smith, 1991). Thus, in a compound interest function, the principle is multiplied times the constant rate of growth factor multiplied by itself n times. In such an approach, a prototypic function might operate metaphorically. If the initial situation were of bunnies reproducing, then the principle functions as the initial number of bunnies and the reproductive rate is cast as the interest rate. It is not so much the specifics of the situations that remain invariant as the characters required in various contextual roles and the actions they each carry out in relation to each other.

The impact of such a view on the multiple representations is that one is encouraged to seek out how the actions, operations and roles are carried out and made visible (more or less) in the different representations.

**III. Functioning as a Human Activity.** Many mathematics educators prefer to speak of "mathematizing" to emphasize the role of the student(s) and/or teachers in doing mathematics. The reasons for this switch in language is that the process of doing mathematics is emphasized rather than the acquisition and display of traditionally accepted responses. We too find this shift to aid us in the understanding of mathematical ideas, for when we try to answer the question, 'what is a function?', our answers vary dramatically from when we seek to explain, 'what is the experience of understanding functions like?'

An Illustration. To illustrate this, consider one student's path[1] through the following problem: The tuition of Cornell University is $ 11,700. For the last five years, the average tuition hike has been 11.3%. What can you expect to pay when your children wish to attend the University? When will the tuition exceed 1 million dollars?

---

[1] This is a manufactured example, but is representative of the kinds of approaches we have witnessed repeatedly by students working on this problem.

The student, Ann, inputs 11,700 into the table. She calculates .113 x 11,700 and adds this to 11,700. Finding this tedious, she builds a button to carry out the actions. Her keystrokes for the button look like this: Cornell tuition: ● * 113 + ● = |J1. To figure out how many times to hit it, she figures, she'll have children in ten years, and then in eighteen years they will go to the University. So, she hits the button 28 times. To her astonishment, she sees the value over $234,000. To answer the second question, she wants a table, so she opens that window. She types in 11,700 in a column she then names, "cost" informally and c in the formal label. She uses the fill command and types in fill from 11,700 to 1,000,000 and chooses multiplication by .113. The computer gives her a warning that she has filled 50 entries and asks if she wants to continue. Her values have gone down and only the first entry is what she wanted it to be. After answering no, she goes over to the calculator and types 11,700 hits the button, J1, and sees 13022.10. She thinks she has figured out the problem and then goes to her table and types in n (for new) = c + 11,700. She realizes this produces only the correct first value, and feels frustrated at still not getting the other values, but persists long enough to create a column which has the values she gets from the calculator button listed next to the column labeled n.

Analysis. The description shown above describes the richness and complexity of the evolving function concept. Some characteristics of the functioning experience for Ann are: 1) it is embedded in a goal-directed activity of predicting cost as a function of time for t=28; 2) a covariation approach is used describing how cost changes as time increments by years; 3) an entry through numeric calculations is easily accomplished and she uses the repetition of the operation to create a button; 4) the results of the first question surprise her and give her firsthand experience with the rapid growth of the exponential; 5) she seeks out the table to create a record of her interim values and to be able to seek out the $1 million figure; 6) she thinks her method of filling by multiplication of .113 is the same as her calculator actions; 7) she recalculates the first value to set herself a specific goal; 8) she diagnoses her problem as needing to add the value $11,700; 9) she achieves her local goal but not her longer term goal; and 10) she cannot find a path immediately, and sets an interim goal of writing down her desired values.

The concept of function which emerges allows the specification and inclusion of:

1. epistemological obstacles such her experiences of failing to curtail 11,700*.113+ 11,700 into a single expression of (11,700)*1.113 to allow the use of the fill command and see the repeated addition.

2. affect into one's epistemological investigations, such as the surprise at the rapid growth rate, or her sense of ownership of the problem;
3. goal-directed behaviors, goals and subgoals, such as matching previous values or creating records;
4. inquiry skills where strategies for finding are expressed along with basic assumptions; and
5. contrasting, conflicting and supportive uses of multiple representations demonstrating the sequence and purpose of each representation.

The result of a revision of the function concept to incorporate such data would be to include in the function concept the idea of it representing a set of coherent stories to capture the evolutionary paths of student investigations.

**Conclusions**. In this paper, we suggest that the formal definitional approaches to the descriptions of the function concept fail to present a rich and complex enough framework for guiding the development of instructional methods. We explain how that framework must be revised to include the use of multi-representational approaches, to allow for the action-based schemes and conceptual roles that can result from placement in contexts and to describe the "functioning" experience as a personal or social experience. We suggest that such an approach is akin to the idea of a "concept image" expressed by Vinner (1983), and that a richer description of mathematical concepts is necessary to create the knowledge base for more effective forms of assessment.

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# APPLYING THEORY IN TEACHER EDUCATION: CHANGING PRACTICE IN MATHEMATICS EDUCATION.

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*Earlier research has supported the relationship between the setting in which learning occurs and the cognitive processes used as students approach tasks, and the quality of the resulting learning. Early school experience (often traditional) forms the basis of student-teachers' ideas about teaching and Mathematics and has a powerful influence on their initial classroom behaviour. The results of an educational intervention aimed at providing experiences as a basis for an alternative rationale are reported. Initial results suggest that many pre-service student teachers are able to develop a rationale for teaching practice based on their knowledge of how learning occurs and apply their developing rationale in practice.*

## Introduction

Earlier research on children's learning (see Crawford 1983,1984,1986,) supported the ideas of Luria (1973, 1982), Vygotsky (1978) and Leont'ev (1981) of the relationship between the social context in which learning occurs and the qualities of the cognitive processes used as students approach tasks and the resulting learning outcomes. In particular, the results indicated that the major cognitive demands of traditional teacher-centred instructional settings were for cognitive processes associated with memorization of declaritive knowledge and imitation of teacher demonstrations. In contrast, mathematical problem solving and enquiry made demands on students' metacognitive processes and higher order intellectual processes (simultaneous processing in Luria's model) associated with the formation of abstract concepts. All undergraduate student-teachers have many years of experience of instruction in school. In many schools in Australia, educational practice in mathematics is "little different from what it was 20 years ago" (Speedy, 1989:16).

Experience as a student in school forms the basis of student-teachers' ideas about being a teacher, about how learning occurs and even about Mathematics. Research (Crawford,1982;Ball,1987) suggests that these early experiences powerfully influence the classroom behaviour of beginning teachers. In particular, there is evidence (Crawford 1982, Speedy 1989) to suggest that traditional forms of teacher education have been largely ineffective in changing teaching practice or even student teachers' beliefs about teaching and learning.

Pressures for changes in teaching practice have never been so great. Cobb (1988) has described the present tensions between theories of learning and modes of instruction. In Mathematics, the advent of information technologies has significantly changed the role of Mathematics in societies and the roles of humans as mathematicians. The mechanical routines that have played such a large part in traditional mathematics curricula are now largely the function of electronic machines. As Speedy suggests:

> To be skilled in mechanics is no longer sufficient. To skilled in applying mathematical knowledge across the whole of real life situations is imperative.(Ibid)

The report below describes an attempt to apply theories about learning to the education of teachers.

## A Theoretical Description of the Problem.

According to Leont'ev's (1981) activity theory, cognitive development occurs as the result of conscious intellectual activity in a social context. The actual thinking that occurs during activity depends on the perceived needs and goals of an individual or group and the resulting ways in which they approach the tasks at hand. Leont'ev distinguishes between activities and operations. An activity involves conscious reasoning that is subordinated to a goal, operations are largely automated, unavailable for review and usually used unconsciously as a means to an activity. The quality of the learning outcomes reflects the quality of activity involved. The ways in which these factors effect learners is also described by Lave (1988:25) when she writes about a "setting" as a dynamic relation between the person acting and the arena in which they act. Engestrom (1989) extends the idea of Activity as proposed by Leont'ev with his notion of an "activity system". That is, a group of related people working together each bring with them their own needs and goals. In addition, any institutionalised system, such as a school, has an established set of cultural expectations about the relationships between the people who act within it. Like Lave (1988), Engestrom focusses in his system on the dynamic **relationships** between the actors and the context in which they act. Thus a school, or more specifically a Mathematics classroom, may be thought of as a very complex activity system or "setting" in which a large number of people interact according to largely subjective perceptions of expectations, needs and goals.

Student-teachers have long experience of teaching and learning in school. The Activity in the classrooms that they have experienced was largely directed by the teacher. The teacher did most of the higher order intellectual activity...the posing of questions, the planning, the interpretation and the evaluation. As students they have learned ABOUT Mathematics from the teacher and "how to" carry out selected techniques. What they know about being a teacher is the result of their experiences. Many of them have chosen to be teachers on the basis of such knowledge. They are attached to it and, understandably, many resist reviewing and modifying their beliefs in the course of their pre-service education.

At universities, a similar "setting" often persists in relation to their teacher education. Lecturers tell them about educational theories...they select the theories, set the assignments and evaluate them. Successful students learn to talk (and write) ABOUT teaching and learning in the appropriate ways. In Mathematics they generally learn the mechanics of using existing techniques to solve problems. Because teaching practice in Mathematics has changed little, student-teachers' practical experience in schools tends to support their initial perceptions about the role of the teacher.

In order to break the cycle, it seemed essential to provide student-teachers with a wider range of experiences both as learners and teachers. It also seemed important to construct a "setting" for their learning which facilitated a shift of attention away from preconceived notions of teaching towards an examination of the learning of the children in their care. The "setting" should also encourage intellectual activity directed towards the development of a practical rationale for teaching practice that is centred on a working theory of how learning occurs. To this end a teaching experiment was conducted with final year students at the university.

### The teaching experiment and outcomes

All final year students (n=45) of a Primary Bachelor of Education course were provided with information in the form of readings and tutorials about recent theories of learning. In the beginning, tutorial sessions focussed student discussions on the possible implications of research on learning for classroom practice in Mathematics. Thus a beginning was made in mapping knowledge about theory onto a known practical situation. Then students were then required to work in groups of three or four to plan, implement and evaluate a mathematical learning environment in which autonomous learning behaviour was encouraged and pupils were involved in investigation and enquiry for a large part of each session. Students worked in a local inner-city school one morning a week for seven weeks. The morning routine involved a half hour planning session involving all students, an hour in charge of a class (working in groups of three), and a review session of approximately 45 minutes. Tertiary staff were available for consultation and advice and observed student work in the classroom. Student-teachers met in working groups for an hour between sessions to plan and discuss. School staff agreed to negotiate with each group of students about the content and scope of activity in each class and to thereafter take a low profile and allow the students to take responsibility for the implementation of their planned projects. Students were advised to take turns at being "teacher", facilitator and observer. They were later required to conduct a similar project alone as part of their practicum experience in another school.

Initially all student-teachers were enthusiastic about the prospect of allowing children a more active role in Mathematics learning. Many had negative memories of their own mathematics education and expressed a commitment to ensuring that pupils in their care did not have the same kind of experience. All had written at length about recent research based learning theories in other parts of their course. All except one had studied Mathematics at matriculation level with some success. Despite this positive beginning, the process of applying theory in practice was fraught with tensions and inconsistencies. Some of these are listed in point form below:

1. Without exception student-teachers were confronted by deeply held beliefs about the need to always tell pupils about Mathematics before allowing them to begin an investigation. This belief dominated their behaviour for several sessions in spite of their awareness of the inconsistency of their behaviour in terms of their stated aims.
2. Despite a specific focus in tutorials on the use of open ended questions and instructions as stimulus material and for evaluation purposes, all began with an expressed need for all children to complete the set tasks in the "correct" way.
3. Most initially failed to distinguish between their needs, expectations and goals and those of the children.
4. All initially found it difficult to shift their attention away from the intentions of the teacher to the responses of the children.
5. All found working collaboratively in a group for a common goal difficult. They empathised with the similar difficulties experienced by children doing group projects at the school.

By the end of the first three weeks most were very frustrated. They were still taking charge of activity in the classroom and the children were colluding. School staff also believed that the children needed to be "told what to do". Many children appeared to lack the social skills to work effectively in groups. The discontinuity between their behaviour in the classroom and the facilitative role that was implied by learning theory was troubling most student-teachers.

Allowing the children ownership of the activity became a major focus of review sessions. Student-teachers experimented with gains and role-play as a means to help children take more assertive and socially collaborative roles. Gradually, for all groups of student-teachers there was a change in the dynamics of the classroom. One spoke of the initial experience as follows: "It felt like coming through a gate into a place that I didn't know existed." As the children became confident that their ideas were respected and a more active role in the Mathematics projects was appropriate, most responded enthusiastically. Both student-teachers and school staff "were amazed" at the knowledge of the children. The student-teachers,paradoxically, also became much more confident about choosing to take a directive role when it was perceived to be advantageous. Many also began to recognise the scope and **limitations** of modelling materials as aids to learning.

A survey was carried out at the end of the course. All students responded. Some of the results are summarized below:

97% indicated that they intended to use group learning.
95% indicated that they would use modelling materials.
94% said they were likely to use games.
92% said they would encourage self directed learning.

These are real options for the student-teachers after experience in applying the strategies in two different school contexts. They also indicate very different expectations of learning in mathematics from those they remembered from their own schooling.

89% indicated that they would develop programs of work in mathematics.
87% indicated that they would use direct instruction in some circumstances.
86% said they would use enquiry based learning techniques.

It was clear from the responses that the student-teachers had not merely adopted a new "method". All, with varying degrees of confidence, felt that they were able to use a range of teaching techniques as different situations and different learning needs required. One student commented:

> "I not only understand what to do in different situations, I also know how to do it and can explain why to anyone who asks".

Many had experimented with a wide range of activities in their efforts to facilitate an active role for all learners.

65% felt they would hold excursions in mathematics.
70% felt they would set writing tasks in mathematics.

73% felt it was likely that they would become involved in school policy making and curriculum development in Mathematics.

Despite their apparent confidence and demonstrated ability to facilitate active involvement in Mathematics classes, many were less positive about their own learning experience.

40% found working in a group difficult.
42% indicated that the tertiary support for their practicum assignment was unsatisfactory. Many more commented on the lack of support for investigative and enquiry based learning in schools (We now hold workshops for practicum supervisors and are taking steps to ensure their active involvement in supporting practicum in mathematics.

The discomfort and confusion that occurs during a major review of attitudes and beliefs is well recognized. (Mandler (1980), Gibbons & Phillips (1978)) These students were just beginning to gain confidence as teachers when their basic beliefs about what a teacher is and how learning occurs were called in question. They had wrestled with a very difficult educational task. It was not all enjoyable.

In contrast, 87% rated the practicum assignments positively.They recognized the value of the practical assignment which gave them a chance to explore the implications of what they had learned in a second school setting.

Interestingly, there was a significant ($p>.025$) positive correlation between formal achievement in mathematics and positive attitudes to group work in the classroom. Students found that they needed to be very clear about the mathematics involved in a theme to consult effectively with a number of small groups.

The experience of teaching this course suggests that providing opportunities for students to revise their beliefs about Mathematics Education is an effective way of enhancing the range and adaptability of their teaching practice. An opportunity to confront and review their strongly held beliefs about learning and teaching Mathematics is at least as important for student-teachers as a range of learning experiences in Mathematics. Perhaps the most important change for our students was the recognition of the practical implications of the fact that children come to the classroom with knowledge of their own and that they can use this as a basis for further learning. This understanding, seems a necessary prerequisite for a child-centred teaching style which responds to the needs, knowledge, purposes and priorities of the learners. In the process of schooling the teacher is a powerful element in the "activity system" of a classroom. Thus teacher perception of their role, their expectations of students and their needs and goals are major influences on student approaches to learning — influences on the quality of learning outcomes. In addition to knowledge of mathematics, it seems highly desirable that teachers leave pre-service education with a range of teaching strategies. Most importantly, as educators they need to have a strong professional rationale as a basis for deciding which teaching strategies are appropriate for different students and an understanding of the learning outcomes that are to be expected when particular "settings" are facilitated.

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# SCHOOL MATH TO INQUIRY MATH: MOVING FROM HERE TO THERE

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*The transition from a traditional classroom to an inquiry classroom is exceedingly problematic for most teachers. This study builds on an earlier study designed to examine the kinds of questions asked by three mathematics teachers attempting to adopt an inquiry approach to mathematics instruction. The focus of this paper is on qualitative changes in the questioning practices of these three teachers over that same year.*

Ushering in a new paradigm is never an easy task. Recent attempts to institute the instructional shift in mathematics education advocated by the NCTM Curriculum and Evaluation Standards for School Mathematics (1989) abound, but the transition from the traditional classroom which presumes a transmission view of knowledge to a classroom where students construct knowledge from genuine mathematical inquiry and discourse is exceedingly problematic.

It is our observation that inquiry-based curriculum and methods of instruction do not necessarily result in inquiry math discourse. In spite of the efforts to encourage teachers to foster such discourse, instruction may still bear many of the characteristics of school math. In an earlier paper (Davenport & Narode, 1991) we described the questioning practices of three teachers as they attempted to engage students in mathematical inquiry. We found that although the teachers in our study religiously eschewed the didactic approach to instruction in favor of inquiry, an analysis of the frequency and types of questions asked indicated that the ensuing discourse, at least during the first half of the year, was largely "school math". This paper attempts to look more carefully at patterns among the types of questions asked by these teachers over the entire course of the same year.

## Research Framework

The constructivist view of mathematics learning (von Glasersfeld, 1983; Steffe, Cobb, & von Glasersfeld, 1988; Richards, 1991) asserts that discourse is a universal and critical feature of concept development in mathematics. The construction of knowledge is idiosyncratic in that it is individual; it is consensual in that knowledge cannot stand alone, it is mitigated through social interaction and mediated through language. For discussion to occur, there must first develop a discourse community whereby the discussants implicitly acknowledge shared assumptions which gives the appearance that the discussants are acting in accord. Individuals exist in communities where, according to Richards (1991), membership is developed through a "gradual process of mutually orienting linguistic behavior". The shared community of the mathematics classroom presupposes that students and teachers accept implicit assumptions as to their roles and responsibilities. These form the basis of their linguistic behavior.

Richards (1991) distinguishes four mathematical communities where qualitatively different mathematical discourse occurs. The four different discourses are **research math**, or the spoken mathematics of professional mathematicians and scientists; **inquiry math**, or the mathematics of "mathematically literate adults"; **journal math**, or the language of mathematical publications which feature "reconstructed logic" which is very different from a logic of discovery; and **school math**, or discourse consisting mostly of "initiation-reply-evaluation" sequences (Mehan, 1979) and "number talk" which is useful for solving "habitual, unreflective, arithmetic problems." Bauersfeld (1988) also draws similar distinctions between what might be characterized as **school math** and **inquiry math** and identifies a **funneling pattern of interaction** that often comes into play when, in school math, teachers attempt to lead students to correct solutions. The distinction between inquiry math and school math is fundamental in appraising the success of present reforms in mathematics education.

## Research Methodology

The subjects in this study are three teachers who are part of an on-going project involving an effort to implement many of the recommendations contained in the NCTM Standards (1989) through the use of a curriculum developed with support from the National Science

Foundation. The teachers participated in 90 hours of staff development during the Spring and Summer in which they explored many of the activities included in the curriculum. Since this initial staff development, teachers have continued to meet with project staff on a regular and frequent basis to discuss issues relevant to implementation, with much of the focus on explorations of student thinking.

Two sources of data are examined in this study: (1) classroom transcripts and (2) teacher journals. Transcript analysis focused on the first two days of a three-day period of videotaping in the Fall and Spring. All 6th-grade teachers were teaching approximately the same lessons. Sequences of questions were examined for patterns which were suggestive of school math or inquiry math for all three teachers over time.

Journal analysis focused on passages pertaining to questioning and classroom discourse. The journals included reflections by teachers throughout the year as well as responses to more structured questions posed during staff development, including questions designed to be addressed as teachers reflected on the videotaped lessons of their classroom practice.

## Results and Discussion

In the Fall of 1991, sequences of questions asked by all teachers were highly reminiscent of "school math" as described by Richards (1991) and Bauersfeld (1988). Representative sequences include the following:

> T#1: (After placing the first three pattern block train on the overhead) *What will the fifth one look like, Marci?*
> SA: It will have 2 trapezoids and a hexagon then 2 trapezoids . . .
> T#1: *It will have how many pairs of trapezoids? . . . It will have 6 pairs of trapezoids and how many hexagons?*
> SB: (A student responds quietly.)
> T#1: *OK, so the 5th one will look like 3 sets of trapezoids and 2 hexagons. Can we say anything else about that? Maybe so we know how to arrange them?*
> SC: It would be a trapezoid then hexagon then . . .
> T#1: *What word did we use yesterday to describe things that go back and forth?*
> SD: Alternating.
> T#1: *Alternating. Can we use that word to describe this?* (Writing on the overhead as he speaks) *3 sets of trapezoids will alternate with two hexagons.*

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T#2: *Do you guys all agree that (the pattern block train) would start with a triangle? What would it end with?*
SA: mumble
T#2: *OK, here's what you guys can do . Here's the first train. The second train looks kind of like this. Can somebody describe it for me? Who wants to describe the second one? Trisha?*
SB: Another red one.
T#2: *How many sets of squares?*
SB: Three
T#2: *OK, the first train is 2 sets of squares and a trapezoid.*

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T#3: (Asking student to build the fifth pattern block train made up of trapezoids). *Yeah. What do you have there?*
SA: Two and a half.
T#3: *Do you see two and a half?*
SA: Um.
T#3: *It's five trapezoids, isn't it?*
SA: Uh-huh.
T#3: *How many trapezoids make a hexagon?*
SA: Two.
T#3: *OK, so for every two you're going to make a hexagon. Is that right?*
SA: I don't know.
T#3: *Look at it. For every two trapezoids, do I have a hexagon?*
SA: Yeah.
T#3: (walking up to the overhead projector) *We have how many?*
SA: Two.
T#3: *Two?*
SA: Two and a half.
T#3: *How many trapezoids?*
SA: Two?
T#3: *Trapezoids! Count.*
SA: Five.
T#3: *Count by two's. Pull them aside, go ahead.*
SA: Two, four, six.
T#3: *Count by two's, pull two aside. So how many two's did you take out of five?*
SA: Two.
T#3: *How many two's can you take out of 7? Go ahead, count.*
SA: 2, 4, 6.
T#3: *How many whole ones?*
SA: Three and a half.

This sequence is reminiscent of the traditional discourse in which teachers initiate, students respond, and then teachers evaluate for closure. Although there are some open-ended questions asked that might be suggestive of inquiry, on the whole, the discourse was tightly controlled and displayed, to some extent a funneling pattern of interaction. This is especially the case of in the dialogue with teacher #3 who worked very hard to lead a student to a correct solution.

This observed pattern of questioning is contrary to the ways teachers write about their practice in their journals. In the Fall, one teacher described the ideal classroom as follows:

> T#1 (Sept): Activities are structured so that students interact with the materials and have opportunities to explore and make connections. Process is emphasized over content and the teacher facilitates the learning with questioning and discussion rather than dispensing procedural knowledge. . . Lesson "plans" are by necessity more fluid and open-ended. . .

Journal entries for this time of year from other teachers contain similar remarks.

While teachers can describe their "ideal classroom" using language that is suggestive of mathematical inquiry, they acknowledge that such questioning practices are problematic for them. Teachers felt that students were not well-prepared for responding to more open-ended questions which probed their thinking:

> T#2: (Oct) These kids are not used to dealing with open-ended questions. It makes it tough for classroom management when you move to a setting that allows for a more open-ended approach. . . I think I am discouraged from asking these kinds of questions from the poor quality of response I often get on them.

> T#3: (Oct) I . . . saw that too many kids were not having success. . . I can't buy the idea that kids don't feel bad starting off with what they perceive to be failure. . . The kids need to succeed very badly. . . Once the kids have success, they will try harder and it won't need to be structured the same way.

Arguing the importance of providing students with success, the teachers justified a need to provide more structure for activities and explorations included in the curriculum. Structure was often interpreted by teachers to mean the use of questions which were leading and "set up" so that students were likely to respond correctly.

By Spring of 1991, the sequence of questioning had changed in some important ways for two of the three teachers, showing some movement from school math to inquiry math:

> T#1: (Two students at overhead sharing their solution to a problem they had all worked on in small groups.) *Can you explain a little bit about what you did there?*
> SA & SB: (Mumbling about 550's and 3)
> T#1: *What were you trying to do there? What was the purpose for doing that?*

SA: Add them all up.
T#1: *Why did you multiply 550 times 3?*
SA & SB: (Mumble)
T#1: *OK, and what if you had gotten a number like 2150. What would you have known?*
SA & SB: (Mumble)
T#1: *OK, any questions for what their method is here?* (Students ask questions and the teacher goes on to ask for other solutions which are then discussed.)

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T#2: *Would you please look at Casey's that she has in front of her? It's 4 cubes tall and 3 cubes long and 2 cubes wide. You should all have that in front of you . . . Once you've got that, count up the surface area.*
SA: I know what it is.
T#2: *OK, can you give me an explanation of how you got it? Also, find the volume. How many cubes are in that thing?*
SA: I got it. I know the volume.
T#2: *What is it?*
SA: 24.
T#2: *There are 24 cubes in it, aren't there? So its volume is 24. Now, I heard somebody say 46 on the surface area. That's close, but you'll want to check again.*

---

(Students have been working on problems in small groups. The teacher announces that she would like them to discuss the second problem. A student volunteers to come to the overhead and explain his solution.)
SA: (At the overhead) First I draw a circle and divide it into 5 parts. So there is 5 parts and I put 20% in each of them, OK . . . I put $1000 at the top and I put $200 under each of them (drawing as he talks) and that equals $1000.
T#3: *Robby, did you immediately think to draw a circle when you read the problem? What did you think?*
SA: A circle.
T#3: *Does anybody have any questions? Go ahead and call on people.*
SB: I don't understand (mumble).
SA: 'Cause there's 20%. He made 20% of each painting and you wanted to know how much the painting cost. Get it? . . . You get it now?
SB: No, I still don't understand why you put 20% in each space.
SA: You are making it very hard for me.
T#3: *Maybe he doesn't understand your question.*
SB: OK, you know how it says (mumble)? The part I don't understand is (mumble).
T#3: *OK Robby, how did you know to put 5 parts to your circle? Why didn't you put 3 parts and put 20% in each part? or 10 parts?*
SA: 'Cause, it's like, it equals a thousand.
(The teacher continues to ask for questions, then invites other students to share their solutions which are then also discussed.)

Here, in the discourse of Teachers #1 and 3, we begin to see features of inquiry mathematics.

·The discourse seems more genuine. Teachers are asking students to explain their thinking, to

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talk about their ideas, to explain their reasoning. Other students are encouraged to participate in the discourse and seem willing, even eager, to ask questions or share their observations. On the other hand, the discourse of Teacher #2 seems to have grown even more like school math over the year.

An examination of teacher journals over the year suggests a growing awareness of the limitations of their questioning:

> T#1: (Mar) I think I am trying to do more probing but that sometimes I miss opportunities to do so. I have tried to increase the questions phrased "What caused you to try that?", "what was in the problem that prompted you to do that?", etc. . . I need to be more alert to students who don't respond a lot in class and take advantage of their responses and questions to probe their thinking.
>
> T#2: (Feb) I was asking lots of questions. But as I wrote down the questions it seemed that almost none of them were probing student thinking. Rather, on many of them I had a specific answer in mind.
>
> T#3: (Nov) I think I probably become more directive if kids are off task or not responding the way I want them too. I am willing to risk empowering the kids with their own learning but I take back the power, not really consciously, whenever things don't go my way. That is not truly empowering the kids and believing that they can succeed, that they have ideas. . . If I really believe that the kids are capable, I will stop reverting to teacher-directed every time I feel insecure. . . I need to trust my kids more. They will learn. (Mar) This year I have really struggled with the questioning. I tell myself I am going to ask better questions, be less directive. At first I wasn't that aware of my questions but now I cringe sometimes at the questions I ask. I'm asking more genuine questions now, but not as many as I need to do.

Teachers #1 and 3 wrote extensively in their journals over the course of the year, looking critically at their own practice and talking with colleagues and project staff about the nature of their classroom discourse. Teacher #2, on the other hand, wrote little and seemed to be less engaged in looking critically at his own practice. In addition, Teachers #1 and 3 often spent class time helping students learn the behaviors that one might associate with inquiry math: sharing ideas, explaining one's thinking, asking questions, and looking for multiple solutions. Teacher #2 engaged students in these kind of discussions to a much lesser extent, and, one might note, did not actually support these behaviors in his classroom.

## Conclusion

The shift from "school math" to "inquiry math" is a challenge to instruction. Even in cases where teachers are working with curricular materials designed to reflect recommendations contained in such documents as the NCTM Standards (1989), genuine mathematical inquiry and discourse are problematic. Questions are not in themselves evidence of inquiry. One happy observation is that the teachers are becoming more aware of the problems surrounding their practice and some are making significant changes in their questioning. A better understanding of how that change occurs is central to the reform movement in mathematics education. Further discussions with teachers who are making those changes, as well as with those who are not, cannot help but inform our efforts to help teachers create an environment which supports genuine mathematical inquiry and discourse.

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# CUTTING THROUGH CHAOS: A CASE STUDY IN MATHEMATICAL PROBLEM SOLVING

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*I detail a case study of problem solving in an advanced mathematical setting. The study shows clearly the false starts and detours that occurred prior to a solution. It also shows the interactive catalytic effect of a group in problem solving. The study is presented in part as a counter-example to the notion that good problem solving abilities can be equated with the automation of domain specific rules. Such automation is important and, in most cases, necessary; it is however far from sufficient.*

INTRODUCTION In this article I detail the background to the solution of an elementary but important result in dynamical systems. The technical solution of this problem will appear in the American Mathematical Monthly (Banks *et al*, to appear). In what follows I have referred to the five authors of the Banks *et al* article as "A, B, C, D, and E" in a random order. I will consider the attributes of this group and its individual members that seemed to contribute to a successful solution to the problem. I have presented an account that attempts to reconstruct a successful group attempt at mathematical problem solving. The problem posed was novel for *any* person to whom it was posed. This is because it was at the time not only an unsolved problem but, as far as we are aware, one that had not even been previously posed because the connections it established were not suspected (a possible exception is evidenced in the article by Peters and Pennings, 1991, in which they speculate on the interdependence of the three conditions for chaos that we outline below).

The problem stems from a mathematical definition of chaos given by Devaney (1989). In order to discuss the problem, and the steps to its solution, a modicum of notation and basic concepts from dynamical systems is necessary. In Devaney's book a dynamical system is determined by a continuous function on a suitable topological space. In fact, in order to state the most important condition for chaos, Devaney assumes that he has a continuous function defined on a *metric space*. This is a topological space in which measurement is possible in very general terms, subject only to a few axioms, the most important of which is the triangle inequality. These axioms say that when we have a method for assigning to all pairs of points x, y from our space, a "distance" d(x,y), then the function d satisfies the following laws:

- $d(x,y) \geq 0$, with equality exactly when $x=y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

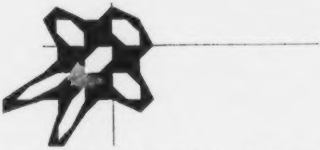
Probably the best known example is the metric d defined on the Euclidean plane by :

$$d(x,y) = \sqrt{(a-c)^2+(b-d)^2}, \text{ where } x = (a,b) \text{ and } y = (c,d)$$

A discrete dynamical system is then determined by a continuous function f on a metric space X. A good example to bear in mind for all that follows is the case when X is the Euclidean plane consisting of all pairs (a,b) where a, b are real numbers, d is the metric described above, and f is the function defined by $f((a,b)) = (|a|-b+1, a)$. This very simple function has quite complicated dynamics in the plane, Devaney (1983, 1988). These dynamics can be investigated empirically with a small computer and a simple programming language (as simple as Basic, for example).

CONDITIONS FOR CHAOS There are three conditions that Devaney (1989) requires for chaos. I will adumbrate these conditions in relation to the specific function f defined above.

The first condition is that f is *transitive*. In its simplest from this means that there is a point (a,b) for which the orbit of (a,b) - that is the set of points $(a,b)$, $f(a,b)$, $f(f(a,b))) = f^2(a,b)$, $f(f^2(a,b)) = f^3(a,b)$, ....., and so on - passes arbitrarily close to any prescribed point of the plane. The example f that I described above is *not* transitive in this sense. It is however transitive on the set X shown in black below.

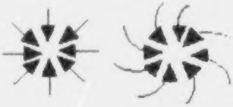


This set X, indicated in black, is *invariant* under f - that is, if (a,b) is a point in X then f(a,b) is also a point in the region X - and f is transitive on this invariant set. It is not immediately obvious that f is transitive on the set X shown, but in fact any point (a,b) in that set where a and b are irrational numbers will have a dense orbit - that is, an orbit that passes arbitrarily close to any point of the invariant set. Devaney (1984).

The second condition on the function f to define a chaotic dynamical system is that arbitrarily close to any point there is a *periodic* point. A periodic point is one that eventually returns to itself, under the action of the function, after a finite number of steps. For example the point (1,1) is a fixed point of the function f above (since f(1,1) = (1,1)), and the point (0,0) is periodic with period 6. Devaney (1989, p. 50) refers to the above condition -that is, density of periodic points - as "an element of regularity."

The final condition is widely thought of as the essential ingredient of chaos: the function f should have "sensitive dependence on initial conditions". This means, roughly, that there is some constant K so that if we take two distinct points x and y and iterate them under the function f sufficiently many times, we will get points at least a distance K apart. It is this condition that says that in a chaotic dynamical system small experimental errors are eventually magnified to large errors.Technically, the condition is as follows: there is a constant $K > 0$ so that if x is any point and n is a positive integer then there is a point y whose distance from x is $\frac{1}{n}$ or less, and an integer p so that the distance from $f^p(x)$ to $f^p(y)$ is K or more.

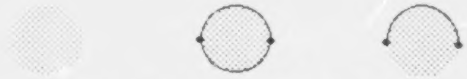
This third condition, of sensitivity to initial conditions, is different from the other two conditions in that it depends on the metric, and is not entirely a topological property. In the development of dynamical systems this creates a difficulty. The difficulty is that the best notion of equivalence of dynamical systems seems to be topological equivalence, and not the stronger notion of metric equivalence. The reason is simply that attracting behaviours such as those shown below are commonly thought to describe part of the same dynamical behaviour; however such dynamical systems can, in general, only be topologically equivalent, and not metrically equivalent.



SPECULATION ON A BASIC QUESTION  A question arose in the mind of **B** whether chaos, as the conjunction of Devaney's three conditions, is a topological property. If it were not this would be most unfortunate, because it would say that chaos was a metric but not a topological property, so that a dynamical system could be chaotic whilst another dynamical system essentially the same as it, from the topological point of view, might not be chaotic. This is a basic consideration in all structural mathematics: to determine what sort of mappings preserve a given property. **B** gave a nice simple argument to show that chaos *is* a topological property in the case that the underlying metric space X is *compact* (that is, when every sequence in X has a convergent sub-sequence). This includes such important metric spaces as closed intervals and closed discs; in general it includes all closed and bounded subsets of Euclidean space, of any finite dimension. However in many examples of chaotic dynamical systems the underlying metric space is not compact, so the more general question remained open.

**FIRST STEPS TO A THEOREM** As a result of **B**'s problem, **A** speculated that the metric property of sensitivity to initial conditions might be a logical consequence of the other two properties for chaos. This conjecture seemed surprising and somewhat naive to some other members of the chaos study group when it was presented. **A** proved, via a remarkably short and transparent argument, that his conjecture was correct in the special case when the space X is *unbounded* - that is when the set of distances d(x,y), with x and y points of X, is an unbounded collection of real numbers. This is a sort of opposite case to that when the space X is compact.

**CONVICTION** **A**'s argument, presented at a seminar, stimulated **D** to give a general proof by reducing the bounded case to the unbounded case. This is an *opposite* procedure to what is a common trick in analysis, so the idea came from a resonance with *bounded-unbounded* and seemed highly plausible, accompanied by a strong feeling of "I've seen this before." Consequently, **D** presented his proof to **B**, only to realise that the proof worked in detail only for those bounded spaces in which the diameter of the space is not achieved: spaces such as the open disc below, but not the closed disc, nor the half closed disc.



An open, closed, and half-open disc, respectively. The points shown in the closed and half-open discs are as far apart as the diameter of the disc. There are no such points in the open disc.

However it now seemed highly likely to **B** and **D** that **A**'s conjecture was indeed true, and that the condition of not achieving the diameter of the space was a technical hitch that could be patched up.

**A BREAKTHROUGH IDEA** A day or so later **C**, in contemplating **A**'s argument, presented the outline of an argument to show that the conjecture was true, at least in a fairly general and natural setting. I present below the first sentence of **C**'s statement because it shows the intuitive feel for being on the right track that characterizes creative problem solving in mathematics. It also shows too how one makes a leap of faith:

> "In the following $f: X \to X$ is continuous and X is some topological space with enough properties to make everything work (à suivre ... )"

What were the sufficient properties "to make everything work" and which were to (eventually) follow'? **C**'s idea, stemming form his work in differential geometry, was to show, by way of contradiction, that if the first two conditions for chaos held in conjunction with the negation of the third condition, then the period of any given periodic point would be forced to be arbitrarily long - a contradictory situation. His idea was to base an argument on volume estimates, assuming that volume could be measured in X in some way (for example, so that for each $\delta > 0$ the collection of balls $B_\delta(x) = \{y \mid d(x,y) < \delta\}$ had a measure that was bounded over x, the least upper bound for which tended to 0 as $\delta$ approached 0.)

**CRITICAL SIMPLIFICATION** Unfortunately it was not clear to which classes of metric spaces with well-defined notions of volume this argument would apply: in other words, it was not clear how general the argument would be. However it seemed then, and still does seem, a very potent idea that shed considerable light on the question. Then **E**, in trying to understand **C**'s written demonstration, concluded by a most pertinent but elementary argument that we could dispense with any idea of volume simply by interchanging the order of two operations, and we could simplify a technical part of the argument by an elementary but subtle use of the triangle inequality.

**THE FINAL ARGUMENT** The one catch was that Devaney (1989) actually had a somewhat more general notion of transitivity, of which the dense orbit notion is an important specialisation. We had therefore, in writing down a version of the proof of the conjecture for publication, to make an assumption that the metric space X had a special, but important, property (technically, it had to be a separable Baire space). After submission of the article for publication we worked individually, and as a group, for some time to rid ourselves of this restriction. A seminar visitor pointed out to us that over any metric space X there lived a separable Baire space to which a function on X could be lifted. This seemed to offer the hope we were after when the grim news arrived that the editor of the American Mathematical Monthly had rejected our article! The referee's remarks appear below. They are of interest here for two reasons. First because these remarks, in part, stimulated A to find an even better proof. Second, the referee's remarks indicate a genuine gap between a certain sort of applied mathematics, where one may use words in a somewhat loose way, and what is commonly thought of as "pure mathematics" where precise definitions are *de rigeur*.

> "The paper is a reasonable remark, which I believe is correct. I have read through the paper but not with a magnifying glass, and I can more-or-less imagine a direct proof. The writing is fluent (both in the sense of fluency of language - which shouldn't surprise us - and of fluency of exposition). I am not aware of any published proof of the theorem. I even suspect that it will be of interest to a number of Monthly readers.
>
> So why am I unenthusiastic? I think it's the first two pages, which seem to put the wrong stress on (sic). The popularity of Gleick's book (and I hope soon Stewart's book) and the wonderfully evocative buzzword "chaos" has inspired a lot of armchair scientists, and in in particular it seems to be the "in" thing to try to argue about the definition of the buzzword, chaos. (Imagine trying it with "art" or "democracy" or "truth" - you get chaos.) In order to make their comment weighty, the authors spend two pages discussing the "definition" of "chaos". I tend to yawn at such discussions, but I also wonder at their reliance on Devaney's text for the authoritative statement of such a definition. What do Collet Eckmann or Mane say? If you're going to discuss the "usual" definition, point to more than one source for it!
>
> I think, if it were pulled together a bit, the paper could be a perfectly reasonable note for the Monthly. The discussion motivating the theorem should be shortened considerably, and I suspect the proof can be done without invoking contradiction.
>
> I think you could reasonably either refuse the paper or ask for a rewrite. I don't favor publication as is."

Our paper had been deliberately written with the provocative title "What is Chaos?" We did this to highlight what seemed to us to be a fact: namely that no one yet had apparently come up with a satisfactory mathematical definition of chaos. This title, we concluded, had upset the referee, so we answered the remarks by changing the title and attending to a few other minor matters. Some of us were puzzled by what the referee referred to in the statement "the proof can be done without invoking contradiction", since logically, if not psychologically, a proof by contradiction is as direct as any other proof (simply change the statement of the result). The referee was also operating in a different theatre to us: he was apparently taking "chaos" as an intuitive undefined term in mathematics. This would be a revolutionary idea indeed, so we preferred to stick with the usual mathematical practice of making precise mathematical definitions. The definition of Devaney (1989, p. 50) was, as far as we know, the only *general* mathematical definition of chaos in print at that time, and we believed we had established an elementary but important result that showed an appropriate definition of chaos was still not yet clear. That is, our theorem was a *mathematical* criticism of Devaney's definition.

However A was moved, in part by the referee's remarks, to re-consider the entire proof and, using the same circle of ideas, came up with a shorter, more direct, and compelling argument in which we could use Devaney's more general condition of transitivity. The final argument had the compelling features of technical simplicity and complete generality. We sent the revised paper to the (new) editor, and were relieved to hear that it was accepted for publication.

## REFLECTIONS ON THE PROBLEM-SOLVING PROCESS

*Group work.* It would appear, even to a casual observer, that we understood the benefits of group work. We also seem to understand how to implement group work in practice. Indeed surges of excitement came in waves as we got deeper into the problem and the excitement of one member of the group spurred others on to better things. How did we cooperate - by writing, by talking, or both ? The answer is, of course, both. Our habitual way of working is to let one member of the group talk until we have a serious criticism, a misunderstanding that needs clarification, or until the speaker dries up. This speaking is not ordinary conversation: it is more like thinking aloud and is usually done at a blackboard This talking is almost always done in a room other than the speakers' room. The reason is that the speaker has an idea and an urge to talk about it. He goes looking for an audience, and so the talking begins. The listening is never passive, and sometimes it can be difficult to talk easily, especially if the ideas are only half-formed: at that stage the talker wants a critical but sympathetic audience.

When talking is temporarily done it is time for writing thoughts down carefully in a mathematical format, and time for reflection - on ideas just conveyed or on new ideas forming. In practice we seem to form most ideas alone, after much cogitation, or calculation, or both. This time is essentially time spent in finding quality data pertinent to the problem and the arguments we have used, or intend to present. But precious ideas need to be subjected to a searchlight of criticism, and that is where talking is essential, to us at least.

*False leads*  There were three obvious false leads that were important in the problem solving process. The first was the result that said the theorem is true for bounded metric spaces in which the diameter of the space is not achieved. Although this did not appear anywhere in the final theorem, nor did the ideas used there assume any importance later in the argument, this result on the way was a catalyst that stimulated us to look for a proof of the main result, *which a number of us now believed to be true*. In other words this subsidiary result, which we abandoned, gave us the feeling that we had to find a general argument for a palpably true result. This is a situation that mathematician's delight in, because feelings run strongly positively that success will soon follow

The second false lead was the excursion into volume arguments. This involved a beautiful circle of ideas that gave us great exultation at the time, but they rapidly became superseded by a very elementary argument, based simply on on the triangle inequality. As irrelevant as the volume idea was to the final proof it buoyed us up enormously, because we now felt that we had a deeper understanding of a reason why our hoped-for theorem was true, and we had a water-tight proof for some important special cases.

The third false lead was the simplification of the transitivity condition to that of the important subcase of a dense orbit, and the necessary assumption that we were working in a separable Baire space. For a long time we could not see how to weaken this condition, and it was, in part, the referee's comments which stimulated us to reflect sufficiently to give a proof in which these restrictive conditions were completely removed.

Whilst these three paths were eventually abandoned, they were each important in leading us to a completely general, simple, proof.

*Critical reflection*  Much of our time after the volume argument was spent critically examining our assumptions, and the restrictions we had imposed in order to get a moderately general result. In this period many original, and some fantastic, ideas were dreamed up to try to remove all restrictions in the statement of the theorem. All but two were abandoned as being without sufficient import. Only the suggestion of our seminar visitor, alluded to above, and the final argument of **A** resolved the matter, the latter most decisively.

*Some individual reflections on critical steps in the problem solving process*

I present below the re-collections of **A**, **C**, and **D** about the problem-solving process. The other members of the group either could not recall how they came to the arguments they did, or were not available for interview.

A:  "My argument came out of **B**'s question of whether topological equivalence was the appropriate equivalence for chaotic dynamical systems, or whether the conjunction of Devaney's three conditions was the appropriate notion. This was an obvious problem to answer - the whole notion of chaos in the sense of Devaney depended on the answer. After **B** gave his proof for the compact case I was looking at unbounded spaces as a sort of opposite to compact one. It was the result for the unbounded case that made me conjecture the result was true in general. I think I was just basically trying to produce some simple-minded proof that the three conditions taken together were preserved by conjugacy. It just popped out of that.

All the way through I'd been unhappy about the proof by contradiction. I didn't think it gave you much intuitive insight into what was going on. I tried to go through the proof by contradiction and convert it into a more direct proof. Arthur (*a colleague*) had said if you've got a fixed point then the map stays near there for a while (*thinking of flows*). I thought : yes, but that doesn't give you a number. On the other hand if you had *two* fixed points you'd be O.K! So I wrote the argument for the special case of two fixed points first.. I'd been investigating another matter related to periodic orbits. It occurred to me then that if you set things up the right way you could get the orbits separating properly."

C "I'm not sure what started me off on volume - I was trying to capture what was inherent in A's argument. I remember it was an interesting problem. The book had been around for a long time and it seemed that the other guys (**A**, **B** and **D**) might be right. My initial attitude was to find a counter-example. There was something similar in my past, but I wasn't conscious of it at the time. In my PhD I was looking at the problem of whether paracompact spaces are regular: the obvious argument didn't work, but a more delicate analysis - refining the ideas - did. I was just trying to make A's argument more subtle."

D "When A gave his argument in a seminar he made what seemed to me to be a very strange assumption: that the metric space was *unbounded*. This was the *opposite* sort of assumption to that normally made in analysis. As he talked I immediately had the realisation that a standard trick of passing from unbounded to bounded metrics could be used in reverse. All I had to do was to check that the three conditions for chaos passed from one case to the other. This I did very easily that evening. Unfortunately, **B** pointed out to me next morning, when I presented my argument to him, that I got an unbounded metric from a bounded one only when the diameter of the space was not achieved. Still, I had substantially broadened the spaces to which A's conjecture applied, and I now believed it to be completely true."

*Automation of domain-specific rules*

Sweller (Sweller, Mawer and Ward, 1983; Owen and Sweller, 1989; and Sweller, 1990; see also Lawson, 1990) has argued that, good mathematical problem solvers are good principally because they have access to relevant schemas and they have automated domain specific rules, so reducing cognitive load. My own view is that Sweller, like many psychologists who venture into a mathematical domain, may not be talking about problem solving in the way in which mathematicians and the mathematics education community in general understand problem solving. In one sense problem solving skills and strategies are what apply when automation of domain specific skills no longer helps. However let us look at what schema and domain specific skills may have helped in solving the problem reported here. *En route* certain specific techniques were important. First there was the idea that the bounded case could be related to the unbounded case via a specific trick in analysis. Then there was the idea of using volume estimates, with which one of us was quite familiar, to get a reasonably general argument. Then again there was a standard analytic technique of bounding a finite set of points away from another finite set. It is eminently reasonable therefore to argue that knowledge of specific analytic techniques proved very useful *en route* to a solution.

The results of this study are entirely in accord with Kilpatrick (1985), who said:

> "Studies of expert problem solvers and computer simulation models have shown that the solution of a complex problem requires (1) a rich store of organized knowledge about the content domain, (2) a set of procedures for representing and transforming the problem, and (3) a control system to guide the selection of knowledge and procedures. It is easy to underestimate the deep knowledge of mathematics and extensive experience in solving problems that underlie proficiency in mathematical problem solving. On the other hand it is easy to underestimate the control processes used by experts to monitor and direct their problem-solving activity." (pp. 7-8)

Our experience also supports the remarks of Thompson (1985) when he says:

- "Several studies in cognitive psychology and mathematics education have also shown the importance of structure in one's thinking in mathematical problem solving." (p. 195)

This is evidenced by the emphasis on such structural features as the distinction and connections between bounded and unbounded metric spaces, the rôle played by compact metric spaces, the rôle of volume in providing estimates on size, and the rôle of fixed points of continuous functions. The remark of C. quoted above, also emphasized heavily a structural approach to the problem.

So there is a sense in which Sweller's argument cannot be easily dismissed by this example of problem-solving at an advanced level. Indeed, in many respects it supports Sweller's thesis: knowledge of a subject and ready recall of pertinent skills *can* be of great assistance in solving mathematical problems. In practice however the converse is most often encountered: without the ready recall of pertinent skills the solution of genuine mathematical problems will usually be impossible. A terrible catch is: how do we know beforehand what is pertinent, or useful?

The problem considered in this study had a particularly simple conceptual scheme: A & B → C. However this logical formulation of the problem was of no assistance in telling us *why* we should expect the condition of sensitivity to initial conditions to be a logical consequence of the conditions of transitivity and density of periodic points. What we needed were useful ideas .

When we, as teachers, set difficult or challenging mathematical problems for our students we think we know what is pertinent. Davis' (1984) long term study suggest strongly that whilst for most students we are right, for highly capable students we are wrong. Pertinence is especially difficult to judge when the problem is unsolved; that is, when, as far as we are aware, no one knows a solution. Browder and MacLane (1978, p.) comment on pertinence, or usefulness:

> "The potential usefulness of a mathematical concept or technique in helping to advance scientific understanding has very little to do with what one can foresee before that concept or technique has appeared. ... Concepts or techniques are useful if they can be eventually put in a form which is simple and relatively easy to use in a variety of contexts. We don't know what will be useful (or even essential) until we have used it. *We can't rely upon the concepts and techniques which have been applied in the past, unless we want to rule out the possibility of significant innovation.*"(My italics)

However this question of pertinence, or usefulness, is of critical importance. It is a variable that needs to be considered deeply because it is at the heart of the process of creative problem-solving. Once the usual ideas and domain-specific rules seem to be exhausted, how is it that successful problem-solvers proceed? I believe they create. They create new ideas and concepts which they hope will be useful in solving the problem. The processes of concept creation, and its dual of concept annihilation due to the constraints of the problem and the critical comments of colleagues, is I believe, an example of evolution in microcosm. This it seems to me, is where mathematics is born, ever new, and this, I believe, is where we should concentrate our efforts on understanding the problem-solving process in mathematics.

Finally, the problem we worked on was a *universal* problem: it was a problem for *every* person to whom it was posed. The mathematics education literature has often had difficulty with the relative nature of "problems" - for whom is a problem a problem? - and many of the examples elucidated in Silver (1985), for example, are problems only for relative novices. I suggest that as a research community we will learn more about the important creative processes involved in problem solving when we concentrate on student/group interaction with universal problems: those that are known not to have been solved at a particular time. An example is the following:

> " A small boat has travelled 2 kilometres out to sea from a straight shore line. Fog descends, and visibility is almost nil. There is no wind and no current. The people on the boat do not know in which direction the shore lies. They decide to travel at constant speed to conserve fuel. What are the shortest path or paths they could take so as to be certain that they will reach the shore? "

This was an unsolved problem at the time of writing (Croft *et al* . 1991,pp. 40-41). Such problems, capable of being stated in elementary terms, are useful in that they largely dispense with the notion of utility or pertinence of an idea to a solution, *since no one knows what will be pertinent*.. The problem poser - usually a mathematics teacher - cannot then occupy a position of knower in respect of a solution to the problem. The advantage of such a situation is that it forces a teacher to *judge* proposed solutions for appropriateness to the problem at hand and for inventiveness, rather than scrutinise them as approximations to a "correct" solution. Since we don't know what will work we are obliged to take students *ideas* seriously and consider them carefully. I think by focusing on such problems we will learn much about mathematical concept creation in individual brains, and much about teachers critical faculties.

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# THE DEVELOPMENT OF PROBLEM-SOLVING ABILITIES: ITS INFLUENCE ON CLASSROOM TEACHING

by

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*Teachers' perceptions of the development of their own problem-solving abilities during a course on problem solving seem to be reflected in their perceptions of their students' development of problem-solving abilities. The data were collected during a 15-week course on the teaching of problem solving and consisted primarily of journal entries of reflections during the course. The subjects were all 6 students in the course (out of 18) who were also teaching full time. The results are presented in groups of 3 subjects each in which the subjects in each group were similar at the beginning of the course and during the course in their conceptions of problem solving and levels of confidence, and reported similar developments within their students in their own classrooms.*

Within the literature on mathematical problem-solving, few studies have studied the role of the classroom teacher in developing students' problem-solving abilities. Clark and Peterson (1986), after extensive review of studies in the broader educational literature on teacher thinking and decision making, concluded that teachers' theories and beliefs provide a frame of reference for planning and interactive decisions which affect their actions and effects in the classroom. Thompson (1985, 1988) reported a study in which teachers' beliefs about problem solving were changed and the changes in some subjects positively affected their abilities to teach problem solving. The purpose of the present paper is to explore the possibility that teachers' perceptions of the development of their own problem-solving abilities during a course on problem solving will be reflected in their perceptions of their students' development of problem-solving abilities. The data were drawn from a larger data set gathered to study the development of metacognition during mathematical problem solving (DeGuire, 1987, 1991a, 1991b).

## Method

### The Course and Data Sources

The data were gathered throughout a semester-long course (one 3-hour session per week for 15 weeks) on problem solving in mathematics. The course began with an introductory phase, that is, 3 sessions devoted to an introduction to several problem-solving strategies. The course then progressed from fairly easy problem-solving experiences to quite complex and rich problem-solving experiences, gradually introducing discussions of and experiences with the teaching of and through problem solving and the integration of problem solving into one's approach to teaching. Throughout the course, subjects discussed and engaged in reflection and metacognition.

A variety of data sources were used—journal entries, written problem solutions with explicit "metacognitive reveries," optional videotapes of talking aloud while solving problems, and general observation of the subjects. Subjects also wrote a journal entry each week. The topics of the journal entries were chosen to encourage reflection upon their own problem solving processes and their own development of confidence, strategies, and metacognition during problem solving. Each subject chose a code name to use for their journal entries. The code names of the 6 subjects for this paper were Apple, Euclid, Galileo, Hobie, Simplicius, and Thales. The data for the present paper were taken primarily from the journal entries, though their other sources of data were used secondarily.

## Subjects

The subjects in the entire data set were 18 students, all inservice and preservice teachers of mathematics, mostly on the middle-school level (grades 6-8, ages 11-14), but with some teachers on the intermediate level (grades 4-6) and some on the secondary level (grades 9-12). The subjects had chosen to take the course as part of degree programs in which they were involved.

The subjects for the present paper were all 6 students in the course who were also teaching full time. Of this subset, 4 (Apple, Hobie, Simplicius, and Thales) were teaching on the middle-school level and 2 (Euclid and Galileo) were teaching on the secondary level; all had substantial teaching experience, with 8 years being the minimum. Regarding their mathematics backgrounds, 2 (Galileo and Simplicius) had completed Masters in mathematics, 1 (Euclid) had completed an undergraduate major in mathematics, and 3 (Apple, Hobie, and Thales) had completed enough mathematics to be certified to teach mathematics in the middle grades (that is, about 7 courses on the college level, including at least 1 course in calculus and perhaps one course beyond calculus). All came to the course with some exposure to problem solving through inservice workshops varying in length from 2 to 10 contact hours, sessions at professional meetings, or professional reading; none had taken a problem solving course before. (Throughout this paper, direct quotes are from the subjects' journal entries.)

## Results

### Apple, Euclid, and Thales

The stories of Apple, Euclid, and Thales begin similarly in that each felt some apprehension about problem solving and a lack of confidence in their own problem-solving abilities. However, each also exhibited some growth in confidence quite early. Apple expressed her apprehensions and budding confidence as follows:

> When I first entered this class on problem solving, I was very apprehensive. . . . Now that class has been in session for two weeks, some of my fears have been alleviated. . . . I feel a certain excitement when I leave class, and my first inclination is to hide somewhere and work on the problems. . . . With every new technique and problem, my enthusiasm has increased.

Thales expressed feelings similar to Apple. "I consider my problem solving abilities to be minimal but increasing. In the past, when confronted with a problem solving task. . .[I would] panic. . . . My frustration levels are decreasing somewhat." Euclid summarized similar feelings in an interesting way. He said that "I felt somewhat that problem solving had to be caught rather than taught. . .and I seemed to not catch it frequently!"

All 3 subjects also began the course with very limited conceptions of problems and problem solving, conceptions that were rapidly expanded. In his very first journal entry (the second week of class), Euclid explained his expanded conceptions as follows:

> I'm not sure that I had the appropriate definition and understanding of the nature of problem-solving at the beginning of class. . . . My horizons have already been broadened and enriched from the distinction made between exercises and problem solving and the practice that we have had in problem solving. . . . My prior perceptions of problem solving centered on the word problem experience.

Thales expressed a similar conception of problems before beginning the course.

> My experiences with problem solving have been very similar to those discussed in this class as a misconception. As a student, I can remember many occasions when we were asked to solve a series of "problems" where the operations and procedures were evident. . . . I have had little experience solving actual "problems".

Thales soon realized in the course that, if she saw an immediate solution to the task, then it was not really a problem. Thus, her conception of problems had expanded. Apple initially expressed a similar misconception of "problem" by describing problems as "textbook word problems used to teach one basic particular skill; formulas with different arrangements of addition, subtraction, multiplication, and division; geometry and algebra word problems."

As the course progressed, each of the three grew in confidence and enthusiasm in their own problem-solving abilities. Apple chronicle her growth as follows:

> [ About a third of the way through the course:] I feel much more familiar with some of the problem solving techniques. Also I don't feat "not finding the answer" as much as I have in the past. I concentrate more on my attack to the problem. [About halfway through the course:] I am definitely more aware of the process going on in my head. [Towards the end of the course:] I understand a lot more than I did before. . . . I feel much more qualified to solve a problem now than I did. [After the final exam:] The answer that I'm about to write for that question [How confident are you now?] surprises me. Even one month earlier or possibly one week earlier, my answer would have been different. After having worked with the final exam, I feel a lot more confident. For some reason, ideas that I thought I had learned did not really become whole until that exam.

Apple's growth chronicled above was mirrored in the changes in her problem solutions; they became progressively richer in appropriate strategies and metacognitions and correct solutions, as well as in alternative and generalized solutions. Thales' development of confidence in her own problem-solving abilities is similar to Apple's but not as thoroughly chronicled in her journal entries. Her change in emotional response moved from "panic" at the beginning of the course to "enjoyment". Her confidence also grew.

> [About halfway through the course:] I think that during the last few weeks, my problem solving skills have improved. . . . This course is. . .making me a more confident problem solver. [Towards the end of the course:] I'm sure that my problem solving skills have increased over the last few months. However I'm still not an overly confident problem solver. There have been problems on each of the problem sets which I found to be particularly frustrating. However at least now I don't panic when I read the problem and a method for solving it isn't immediately obvious. [After the final exam:] My first reaction upon seeing the exam was panic. . . . However, as I began to more carefully study the problems, I became more confident. . . . I enjoyed working on the problems which I selected.

Euclid admits to some confusion on certain aspects of the course (especially metacognition), a confusion that was not completely cleared up even at the end of the course.

> [About halfway through the course:] I think I have become more aware of cognitive processes since the beginning of the course. However, I am not sure that I really understand yet. . .what I am trying to be really aware of. . . . Some of the early problems seem quite simple now. [After the final exam:] I feel as though I am definitely a better problem solver. . . . [Yet] I feel quite a bit of frustration. . . . I feel fairly comfortable with my success on the exam. . . . [Yet] I am still not as clear as I should be about the distinctions between teaching problem solving, teaching a problem, and teaching through problem solving.

His confusion was evident in his exam responses, both in his solution of problems (even though he had a mathematics major in college) and in his essay responses to items such as distinguishing teaching problem solving, teaching a problem, and teaching through problem solving.

By about a third of the way through the course, each of Apple, Euclid, and Thales began to report attempts to introduce problem solving into their own classrooms. The parallel of their perceptions of their students' success in and reactions to problem-solving experiences and their reports of their own development (as above) are striking and are even explicitly referred to by both Apple and Thales. Apple reported the following on her attempts to introduce problem solving into her classroom:

> [About a third of the way through the course:] One real effect that this course is having is that my reaction to my students has changed. I'm far more concerned with their attack on the problems than with their answers. . . . The students in my classes are experiencing a change in their success rate in solving problems. . . . [About two-thirds of the way through the course:] Lately, my students expect to be solving problems as a regular part of the routine. . . . Some very positive results seem to be happening. . . . I have to use different problems in different classes because the students get excited and tell each other all about the problems. [At the end of the course:] More than ever, I feel that I see a real difference in "understanding" as it is applied to me personally and in "understanding" as the work is taught. . . . I can see my strategies and my attitudes (enthusiasm!) picked up by the students. . . . My students have become very enthusiastic about problem solving. And successful too!

Thales admitted to mixed success in initial attempts to introduce her students to "real" problem solving. Several of her comments (underlined below) consciously mirror her own development:

> [About a third of the way through the course:] I have used some of the problems we have discussed in class with my seventh graders, with mixed results. . . . I'm sure that they have mixed emotions about their abilities now, just as I often have about mine. . . . It is not uncommon for many of my students to read the problem and immediately say, "I don't understand!" I know that what they are really saying is that they don't see an obvious solution and they're not sure where they should begin, a panic that I certainly understand well. [About halfway through the course:] At the beginning of this course, I viewed problem solving with some of the same wariness I now see in my students.

Thales did not make further comments in her journal about the success or enthusiasm of her students in problem-solving. Euclid did not comment in his journal on attempts to introduce problem solving into his classes until about a third of the way through the course.

> For years, the extent of my problem solving activities in the classroom involved word problems. I have taught a plan to try to solve these problems. Most students seem to have a great deal of trouble and difficulty with word problems. . . . I have consistently thrown in to the classes some problem solving activities. I have found that students enjoy them and gives [sic] a different pace to the classroom. However, success rates are mixed.

Euclid only very briefly included further journal references to his students' problem-solving experiences. They seemed to indicate mixed success and many reservations about the possibility of success for all students. Euclid's perception of his students' mixed success mirrored his own mixed success in the course.

## Galileo, Hobie, and Simplicius

Note: The stories of Galileo and Simplicius have been chronicled and their journal entries quoted extensively in at least two other places (DeGuire, 1991a, 1991b). Thus, due to space limitations here, conclusions about these 2 subjects will be cited here but supporting quotes will be limited. Hobie's story will be chronicled more thoroughly.

The stories of Galileo, Hobie, and Simplicius are also somewhat similar but quite different from those of Apple, Euclid, and Thales. Unlike the earlier 3 subjects, Galileo, Hobie, and Simplicius all began the course feeling quite confident about their own problem-solving abilities and very enthusiastic about problem solving. Galileo and Simplicius had extensive mathematics backgrounds and some previous problem-solving experience; they were both immediately very successful with the problems in the course. Their confidence, enthusiasm, and richness of solutions and metacognitions grew throughout the course. Hobie had the least mathematics background of all 6 subjects in this paper but had had some previous experiences in problem solving in a mathematics methods course the previous semester. She began the course feeling quite confident in problem solving but soon realized the limitations of her knowledge. She also consistently recorded enjoyment of the problem-solving experiences. She reported:

> [At the beginning of the course:] I felt okay about problem solving before this class or at least I thought I did. It is amazing what one can learn from just one class. I can already tell that my problem solving strategies were somewhat weak. . . . [About a third of the way through the course:] The more we get into the class, the more I realize how little I really did know. . . . Before the class, I had pretty much confidence in myself as a problem solver. After the first night, I had lost some of that. However, as each class ends and as I solve more problems and read more articles my confidence moves up a step again. . . . I am thoroughly enjoying these activities. [About halfway through the course:] I find that writing the metacognitive reveries in solving the problems has really helped me. . .to become a better problem solver. [Towards the end of the course:] I know I have become better at problem solving, mostly because I can take a problem apart, and concentrate on the process. [After the final exam:] I felt pretty good about the exam. It is amazing to me how sitting down and working on something can be so rewarding. . . . I feel like my problem solving skills have really improved.

Though Hobie's problem solutions were never as mathematically rich as Galileo's and Simplicius', they did exhibit mathematical richness in line with Hobie's more limited mathematics background. All three subjects also began the course with conceptions of "problem" and "problem solving" that were essentially congruent with the widely-accepted meanings of the words.

Just as all 3 of these subjects—Galileo, Hobie, and Simplicius—were consistently confident and successful in their own problem solving, so their perceptions of their students' confidence and success in problem-solving experiences was consistently positive. Hobie had already begun to implement some of problem-solving experiences into her classroom as a result of her experiences in the methods course the previous semester. Even at the beginning of the course, she reported her students' excitement about problem solving. Several of her journal entries make explicit statements (underlined) that consciously reflect her own development:

> [At the beginning of the course:] We have already gone over the problem solving strategies and have used several of them. So far, my students as well as me are very excited about it. [About a third of the way through the course:] I hope to become so confident when the class is over that some of it will spill over to my students. . . . So, far, I feel my students are really enjoying doing the problem solving, just as I am. [About halfway through the course:] Beginning this quarter, I am going to begin having my students write down their metacognitive reveries. . . . I think that this will really help them, just as it has me. [At the end of the course:] Not only has this course helped me, but it is doing wonders for my classroom.

From the perceptions that Hobie reports in her journal, it would appear that her students have become successful and enthusiastic problem solvers. Both Galileo and Simplicius make explicit references to the influence of the course on their teaching, with Galileo providing evidence of his students' development reflecting his own. In commenting on implementing a problem-solving approach to teaching, Galileo observes, "This almost becomes contagious to the student. I have noticed students beginning to imitate the very same processes which I utilize in confronting problems." As Simplicius expressed in her journal, "I feel that. . .my ability as a teacher has blossomed. I have definitely made more effort to incorporate problem solving into the curriculum. . . . I feel that this course has fundamentally changed my attitude toward teaching and what the focus of my teaching should be."

## Conclusion

The parallels between the subject's development of problem-solving abilities, confidence, and enthusiasm and their perceptions of their students' development of problem-solving abilities, confidence, and enthusiasm is quite interesting. The present conclusions have been based on self-report data. As with all self-report data, one must assume that, to a certain extent, the subjects reported what they feel the researcher wants to hear or read. Such data has many deficiencies and problems and is not here triangulated with other data sources. The problems and issues with self-report data have been discussed well in Brown (1987). It is unfortunate that it was not possible to follow these teachers into their classrooms to obtain independent, observation data on what problem solving they incorporated into their classrooms and how they did so. However, the data seem to present an interesting hypothesis for further exploration, that is, that students' development of problem-solving abilities, confidence, and enthusiasm will mirror their teachers' development of these qualities.

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# SELF-DIRECTED PROBLEM SOLVING: IDEA PRODUCTION IN MATHEMATICS

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*A study of productive thinking in elementary school and college students suggests (a) longitudinal effects of current traditional teaching methods on thinking habits and (b) the effects of self-directed strategies on thinking. A program was designed to diversify thinking by helping students produce ideas in a search for understanding, multiple formats, and connection-making. Research supporting the program includes studies on problem solving as a constructive enterprise, learning as a generative process, thinking perspectives, and metacognition.*

## AUTONOMY

For Students to develop insight and transfer knowledge to new contexts, they need to manage thinking consciously and stretch their own development. In this sense they share the direction of thinking with teachers.

> *For me to construct ideas, I must be in charge of my own thinking.*
>
> *For me to use* my *uniqueness to do* your *mathematics, I need to monitor the learning strategies I use.*

We know little about what students are able to do in this domain, especially through a deliberate cultivation of self-directed thinking. The work has begun, however. Notable examples of groundwork include the research of Feuerstein (1980) on student generation of new knowledge, Schoenfeld (1985) on student beliefs and Whimbey and Lochhead (1982) on collaborative problem solving that helps students use what they know. From a different perspective, Vosniadou and Ortony (1991) bring together diverse studies that reexamine the roles of analogical reasoning in learning for children and adults. This work includes self-direction insofar as it treats individual plans and goals, and thinking that students can initiate, e.g., the identification of surface features as cues to underlying structures.

Before children are introduced to academic learning, they use a global approach to learning language and certain quantitative relationships. With a sense of thinking autonomy during the early years (Kamii & DeClark, 1985), students engage in spontaneous qualitative activity to gain enough understanding to connect symbols to the world (Piaget, 1973). Although we do not expect young children to analyze thinking strategies, we do expect them to think in the ways that they are able. Children in the second grade, for example, can learn to strengthen their beliefs about the importance to conforming to the solution methods of others (Cobb et al, 1991). Teachers giving informal reports describe children who create ideas freely and decide consciously to use self-help aids instead of asking for help unnecessarily. These teachers say that they emphasize self-help because they cannot know the precise dimensions of thinking possible for every student at a given time.

Projecting self-direction to later stages of development, we might expect that children and adults would select different learning strategies for themselves. Children would choose to manipulate objects and interact with peers as aids to understanding mathematics and adults would use an even larger array of thinking strategies, including the manipulation of objects, drawing, and analysis represented by spatial patterns and symbolic equations. Yet we know that in school students rely on very few strategies, largely memory and speed that suppresses informal thinking (Resnick, 1989).

## A LEARNING BASE

This study emanates from three integrated research directions: (a) problem solving as a constructive enterprise (Steffe, 1990; Confrey, 1985), (b) thinking perspectives (Greeno, 1989) and (c) self-direction or metacognition (Schoenfeld, 1987; Lester, 1985).

As problem solvers, individuals interpret mathematical content, context, structure, and heuristics (Hatfield, 1984) and manage a repertoire of strategies to meet challenges. Performance weighs heavily on accessing learned content (Silver, 1982) and on searching and elaborating extensively (Mayer, 1985). This generative process is characteristic of learning (Wittrock, 1977; Dirkes, 1978) and of the thinking that students do to understand and solve problems. There is an expectation that students will construct possible models of reality without allowing perceptions of predetermined absolutes to restrict their thinking. (Glasersfeld, 1984).

According to Robert Davis (1984), the true nature of mathematics involves processes that demand thought and creativity. Doing mathematics means confronting vague situations and refining them to a sharper conceptualization; building complex knowledge representation-structures in your own mind; criticizing these structures, revising them and extending them; analyzing problem, employing heuristics, setting subgoals and conducting searches in unlikely corners of your memory. If this is so, students must assume an active role, one that they initiate and monitor.

Treating mathematics as an ill-structured discipline is a step toward both the dispositional and cognitive changes required for the construction of meaning (Resnick, 1984). For students working in familiar situations, algorithms and heuristics fit neatly into a structure. For unfamiliar and complex situations, however, students must not only create a plan to help them organize data and select mathematical strategies (Kulm, 1984), they also accept ambiguity, set aside time for problem solving, and find connections among possibilities that they produce.

Choosing to think and claiming the authority to produce ideas are commitments that rely on the development of metacognition, the awareness of mental functions and executive decisions about when to use them (Flavell, 1979; Sternberg, 1984). Metacognitive functions help students regulate (a) cognitive operations, e.g., recall, infer, and compare; (b) strategies, e.g., draw and list possibilities; and (c) metacognitive action to plan strategy, monitor it, and allocate time for thinking. These functions supplement what teachers do, beginning with the regulation of thinking unique to individuals.

## A STUDY AND A RESPONSE

SELF-DIRECTION IN MATHEMATICS (SDM) is a program designed to engage students productively in active problem solving from elementary school through college. Ten groups of college students enrolled in methods courses for teaching mathematics experienced the entire program for a semester and ten groups of students in grades two through seven participated in two or three sessions on idea listing. Numerous one-on-one interactions with students in public schools were also recorded for study.

Six components integrate the program.

1. MATHEMATICS Students construct meaning and problem solutions in response to a wide range of challenges, and teachers use oral techniques to prompt student connections. Current local and national recommendations direct the choice of mathematical topics and instructional strategies.
2. ROUTINE AND SELF-DIRECTED THINKING Students monitor and regulate thinking strategies and beliefs. They allocate time for thinking; produce alternate interpretations; and make connections among ideas, drawings and contexts.
3. IDEA LISTING Students produce ideas freely to tackle novelty and complexity, to clarify concepts, and create problem solutions. Resources include recall, observation, imagination and peer interaction. A checklist guides their thinking into mathematical concerns and informal prompts develop a climate for problem solving.
4. PROCEDURE For challenge problems students (a) list many ideas about given facts; (b) restate questions to insure meaning; (c) list many ideas in drawings, words and symbols that *might* lead to solutions; and (d) select their best ideas. They solve given word problem and those in which they add facts and a question. A modified version of this plan helps them

respond to social situations and, when needed, divert their efforts to skill development.

5. MATERIALS Diverse materials and technology stimulate thinking and multiple representations.
6. ASSESSMENT Productive thinking described in Components 3 and 4 reach beyond most paper-and-pencil instruments to support self-directed problem solving. Portfolios and two-stage tests show the development of thinking strategies, dispositions, and mathematical knowledge.

An examination of many idea lists shows that students in elementary and middle school can learn to access what they know and use ideas in new ways. With appropriate strategies, college students begin to use self-direction for thinking in mathematical situations and for managing learning. At first, their lists generally do not demonstrate more ideas or more quality ideas than younger students. Checklists that cue mathematical concerns and other strategies, however, improve the quality of their thinking and enlarge their perceptions of the nature of mathematics.

## ATTITUDES AND THINKING POWER

The SDM program uses *self-direction* because it is a term more familiar than *metacognition* and also suggests specifically that students be the ones to examine and regulate cognitive operations, strategies and metacognitive action. Monitoring their own thinking, students decide when to probe long-term memory and when to combine ideas into new inventions.

To optimize thinking that encompasses physical and social contexts as well as personal beliefs and understanding about cognition (Greeno, 1989), students construct concepts and solutions by connecting ideas within mathematics, other disciplines and life outside the classroom. An active assimilation of ideas prepares them to elaborate on what they know and develop representations that communicate their knowledge and problem solutions to others. Implications extend to what students believe about mathematics and themselves.

SDM activities center on an autonomous production of ideas in a search for new connections that develop concepts and solve word problems The word, *idea*, suggests that students expect to produce possibilities for answers or for the direction of complex problem solutions. Whereas *answers* are to be correct, the immersion of *ideas* into subject matter (Prawat, 1991) introduces problem solving that encourages growth and revision. *What is your idea, Susan? What else might be important? List many possible ideas. What do you want to revise?* The discourse created builds understanding (Lampert, 1989). Where expectations for a uniform development of meaning do not interfere, students take intellectual risks that reach beyond minimal prescriptions and perceive that an extensive generation of ideas is as much a part of school performance as the reproduction of definitions and algorithms.

Mathematical power comes with the direction of strategies. Students search for understanding by producing alternate interpretations stating questions and interpretations in their own words, and producing ideas as they reread to construct meaning. They demonstrate a willingness to think by speaking extensively, drawing and writing, and organizing ideas for future reference. Producing multiple ideas that *might* be connected is a process that complements long-term recall and making sense of mathematics. This is the kind of thinking students do outside of school.

When understanding is not forthcoming, a flexible production of ideas under deferred judgment, alone and in groups, breaks down barriers and suggests connections. Producing prior knowledge and new inventions, students unify their consciousness of facts, questions, and solution. Self-direction and understanding reinforce positive attitudes toward thinking so that they face novelty and complexity with a sense that *I can think* and *I always know how to begin.*

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# PROJECT DELTA: TEACHER CHANGE IN SECONDARY CLASSROOMS

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*This project investigates teacher change in intermediate and secondary classrooms. Using methodology consisting of interviews and observations, movement to a process teaching model is documented. Data have revealed that teachers can make behavioral changes but the richness of those changes is related to the match between teacher philosophical structures and the teaching approach. Additionally, materials supporting both the philosophy and specific pedagogical actions is an important contributing factor in the change process.*

Project DELTA (Determining the Evolution of the Learning and the Teaching of Algebra) is a research program investigating teacher change. The project focuses on teachers at the intermediate and high school level as they implement curricular materials and an associated teaching style. These materials were developed by the Hawaii Algebra Learning Project (HALP) (NSF grant MDR-8470273) and incorporate a process approach to teaching.

## Background and Premises

With recommendations that teaching move from a traditional or lecture approach to one with more student involvement, many descriptors such as process teaching, inquiry-based approach or problem-solving instruction have been tossed about, each with specific characteristics. Since mathematics educators, practitioners and researchers alike, do not agree on what secondary classrooms would specifically look like using nontraditional instructional methods, DELTA first sought to identify characteristics of process teaching to ease communications and to establish specific areas that are different from traditional instruction. These areas would then be associated with those most related to teacher change. This preliminary work was conducted in classrooms using the HALP curriculum (*Algebra I: A Process Approach*, Rachlin, Wada, and Matsumoto, 1992) as a means of assuring consonance between the process teaching method and materials supporting that method.

The HALP materials are intended to be a complete Algebra I curriculum for intermediate or high school grades. They were developed through a seven-year intensive classroom-based research program conducted with ninth-grade students. Curriculum developers served as classroom teachers and piloted draft materials in their classes. After each class, individual students were interviewed on a regular basis to "think-aloud" as they solved problems that would appear in the next lesson's problem set. These problem sets were constructed to model Krutetskiian problem-solving processes (Krutetskii, 1976). Combining these problem sets with a Vygotskian perspective on learning (Vygotsky, 1978), the HALP created an algebra curriculum that included materials and an intertwined teaching approach (process teaching).

In past curriculum projects, teachers have been afforded little or no implementation support. Because this curriculum presents algebra content in a different way and is based on a nontraditional teaching style, a 45-hour workshop was designed to help teachers use the curriculum. While it is required for those planning to use the curriculum, other teachers may also enroll.

During the workshop, participants read current articles about algebra, problem solving, and teaching. Homework assignments from the text are given to involve participants in thinking about algebra through a problem solving context rather than an algorithmic one. Participants experience the problem-solving processes of generalization, flexibility, and reversibility by solving problems that exemplify each process. Most importantly videos of individual high school students with varying abilities and of secondary classrooms are shown to stimulate participants to question their beliefs about algebra and its instruction. Even though instructors model process teaching through the workshop pedagogy, no explicit teaching methods are given to participants.

## Methodology and Results

The methodology was designed in three phases. Phase one focussed on ascertaining characteristics of process teaching and instrumentation. Phase two's purpose was pilot testing and phase three is currently concerned with study redesign.

### Phase One: Process Teaching Characterization and Instrumentation

**Methodology of phase one.** A member of the HALP team was chosen for pilot classroom observations. Her ninth grade, heterogeneous class was observed bi-weekly in consecutive three-day periods for two months. Scripted field notes and audiotapes were used to record class proceedings. The relatively set pattern of instruction and lesson format in traditional classes did not hold for process classes; they were much more complex. Even with audiotaping it was difficult to script everything that was occurring. An observation coding instrument (OCI) was constructed to ease data collection.

Its construction first required characterizing process teaching based on the pilot observations. The features of process teaching could be divided into quantitative and qualitative aspects. The quantitative features included time and frequency. The amount of time spent on the lesson segments of content development, seatwork, and management was particularly relevant. These three lesson segments appeared to be dramatically different from traditional lessons in that a much larger portion of the class period (42 minutes of 45 minute periods) is spent on content development and negligible time on seatwork.

Frequency collection documented the number of factual (ie., what is $3x + 2x$ ?), process (ie., is the answer unique and how do you know that?), and managerial (did everyone turn in a paper?) questions asked and answered by teacher and students. In traditional classes questions tend to be factual but in the process classes, process questions are more frequent. The person responding to questions was also different. In process classes students tend to respond more often than the teacher due to the active student participation.

It is not, however, just measurable aspects: the quality of responses and of content discourse is even more important. The OCI was constructed to allow for this documentation. Actual dialogue could be captured during observation periods or reconstructed with audiotapes of those sessions. Particular attention is given to the dialogues of each problem discussed during the lesson because the mathematical content that evolves in the dialogue comes from students and is, therefore, reliant upon the teaching method that allows for and encourages student input. This mathematical content could not be neglected since it affects, and is affected by, the teaching approach. The descriptions of developing algebraic ideas enhanced snapshots of the classrooms in the way in which students discussed particular ideas. For example, in one lesson, generalizations now carried the student's name that "discovered" it. These development ideas suggested richer views of the classroom culture and of the construction of mathematical knowledge.

While teacher and classroom behaviors are certainly one way to document changes, all teachers confronting change may not demonstrate it through their teaching behaviors. Based on previous work (Grouws, Good, & Dougherty, 1990), a semi-structured interview was considered to ascertain attitudes and beliefs about mathematics, algebra, their instruction, and individual and school demographics. The protocol questions clustered about four main research areas: (1) teacher views of algebra and mathematics, (2) teaching strategies and/or style, (3) student aspects including teacher expectations, and (4) enhancements to the change process.

**Results of phase one.** Preliminary data were analyzed with particular attention to characterizing process teaching and validating the appropriateness of the instruments. Important features of process teaching included: (1) class periods devoted to active discussion of mathematics, (2) "teacher talk" kept to a minimum, (3) questions from teacher and students were more related to "why" than to "how", (4) students assumed leadership roles in the learning process, and (5) mathematical content developed from a concept level to skill.

The instruments were modified slightly after initial data analysis. The methodology was reviewed by an external consultant and was determined to be appropriate for this stage of the study.

### Phase Two: Pilot Testing

**Methodology of phase two.** Phase two began with an HALP workshop offered in Honolulu, Hawaii. Teachers from across the United States were enrolled and those that indicated they were using the HALP materials in the next school year were asked to participate in the study. Six teachers volunteered and represented intermediate and high school grade levels, varying class types (ie., high ability, heterogeneous, and accelerated), and a range of teaching experience (3 to 18 years).

At the beginning of the school year (October) each teacher was interviewed with the protocol. Their responses were audiotaped and transcribed for later data analyses. At the same time their Algebra I classes were observed using the OCI. Classes were observed again in December and March. The last observation period also included administering the interview protocol again.

**Results of phase two.** Data analyses using Hyperqual showed interesting patterns within and across teachers. When both data sets were analyzed, teacher beliefs and teaching actions did not necessarily match. There were three cases: (1) beliefs were more traditional and teaching was process oriented, (2) beliefs were process oriented and teaching was more traditional, and (3) beliefs and teaching were process oriented. There were also noticeable differences in the ease in which teachers adapted to process teaching and the depth to which they were able to implement it in the classroom. These findings motivated a look beyond teacher beliefs and actions after the workshop.

### Phase Three: Study Redesign

**Methodology of phase three.** Data collection design was restructured to allow for capturing information about teachers prior to any workshop intervention. Two sites where workshops would be held in the summer were selected. Teachers pre-enrolled in the workshop in those locations were contacted. Thirteen teacher from a Midwest city and five from an Eastern site agreed to be involved in the study.

A graduate student from the University of Missouri worked with this phase in the Midwest as part of her dissertation. At the beginning this phase, she came to University of Hawaii where we viewed classrooms and videotapes of classrooms to establish observer reliability (.92). In May of the spring semester prior to the workshop, sample teachers were visited. Their Algebra classes (or another class if they were not teaching Algebra were observed over three consecutive days. These observations were anticipated to involve classes taught in a traditional manner and required the OCI be altered slightly to accommodate both traditional and process instruction

Also, an interview protocol that had been adapted from phase two was administered during these pre-workshop observations. The pre-workshop interview protocol was to be used prior to any workshop intervention so that questions related to how teachers were implementing workshop ideas were not asked.

Phase three also involved the workshop itself. Since explicit implementation strategies are not given in the workshop, we wanted to determine if workshop instructors were teaching in a way similar to what was expected o teachers after the workshop. The graduate student observed and audiotaped the Midwest workshop, including interviewing the instructors with the pre-workshop interview protocol.

The school year following the workshop, project teachers were observed in November and March for consecutive three-day periods in Algebra and other mathematics classes as available. The postworkshop interview protocol was also administered in March. This protocol was identical to the pre-workshop form with the inclusion of questions related to the workshop and to the implementation of the workshop ideas or materials.

**Results of phase three.** Analyses of data sets from both sites are currently being conducted but preliminary analysis on data from the Eastern site is available. Pre-workshop observations documented that all but one of the sample teachers taught in a traditional manner. At the Eastern site the amount of time spent on content development averaged 3.5 minutes in a 45-minute period and the average time spent on seatwork (starting the next day's homework) was 24 minutes. Student talk time to the whole class and student-student interaction was negligible. Questions involved students telling the teacher what "to do next." That is, students would give a step in the problem or answer a factual question such as "what is three times 21?" The mathematical content was introduced at the skill level and received developmental attention for one class period and again on the next day when homework answers were given.

Pre-workshop interview responses also indicated traditional views about mathematics, algebra, and their instruction. Teacher responses emphasized skill aspects with reference to rules, procedures, and application of those in appropriate ways. Expectations for the students except for one teacher focussed on the retention of skills. Problem solving was a separate topic included in the lesson as determined by the textbook's presentation of word problems.

One teacher, even though she discussed mathematics and algebra in a procedural manner, taught in a slightly different way than the others. She engaged the students in active discussion of topics and made an effort t connect the new material with something that students had previously done in class. Questions emphasized studen reasoning and alternative ways of approaching a problem. The classroom atmosphere was student-centered in that student ideas were regarded with respect by the teacher and other students.

Postworkshop observation data indicate varying levels of implementation of workshop aspects. Two of the project teachers are using the HALP materials. In one Algebra classroom, there was use of certain questions such as "did anyone do it a different way" or "is the answer unique." There was, however, a lack of depth in natural questions that arose from mathematical discourse. Students and teacher were discussing content, led predominantly by the teacher. While these instructional strategies do not precisely fit the process teaching model, there was a definite movement from the traditional style used prior to the workshop, albeit superficial. Her responses to interview questions related to belief structures were less rigid than comparable pre-workshop comments.

In the other teacher's classroom, her teaching style was more open than before the workshop. Previously she had encouraged student interaction under her direction but now students assumed more leadership in initiating questions and responses. She allowed the mathematical content to develop over longer time through the exploration of techniques and strategies in the developmental stage of new concept formation. Using prior knowledge to solve problems was encouraged. Her interview responses indicated movement to the creative aspects of mathematics and algebra.

Postworkshop observations of teachers in classes not using the HALP materials such as general mathematics or geometry are not consistent with the HALP classes. There is a tendency for all teachers to teach more traditionally, especially those that are not concurrently teaching any HALP classes. Teacher rationales vary; for some, they feel their students in those classes are different from those in the HALP classes. This is especially true for those teaching where students are tracked. Others commented that it is too time consuming and too difficult to change existing materials to fit a problem-solving approach to instruction. Restructuring homework assignments so that development over time can occur and students are exposed to new ideas through problem solving requires an expertise teachers felt they did not have.

## Discussion of Results

All data have not been analyzed, but the results from the second and third phases of the project suggest some interesting ideas about teacher change. The superficiality of changes in teaching strategies, while not an ideal application of the process teaching model, appears to be an important link in substantial changes. Using Hall, Loucks, Rutherford, and Newlove's levels of use (1975), one can find evidence of teachers at level III (uses new strategies while struggling with problem of classroom management related to implementation). However, there is more to change than the physical implementation of strategies. Teachers must also cope with the philosophy that underlies the specific teaching strategies. The crux of process teaching lies with students as they construct meaningful ideas about mathematics and teachers incorporating those ideas into the lesson. The use of student ideas that may be different from what has been accepted as conventional or traditional algebra content appears to be the most difficult aspect of implementing process teaching. The unpredictability for teachers not knowing what direction the lesson is heading challenges them to be flexible enough to recognize mathematical ideas used in creative ways. And, the worth of student ideas or their mathematical validity is almost inconceivable for those teachers who have constructed their own mathematical knowledge in a rigid fashion, especially as the amount of time increases since their workshop exposure.

Teachers are capable of using questions that are consistent from lesson to lesson such as "did anyone do the problem a different way?" but struggle with creating questions when mathematical opportunities present themselves in the lesson. HALP teacher materials have attempted to suggest appropriate questions but again, student experiences vary and often novel ideas appear in discussions. Some teachers have commented that their inability to perceive patterns quickly or to note subtle references to other mathematical ideas may account for difficulties in asking higher-level questions so they resort to factual ones.

The comfortableness teachers feel with the mathematics they are teaching is also another consideration. For example, in one classroom, students suggested that it may be possible to have three axes when graphing instead of two. The teacher ignored that suggestion because, as she indicated later, she was unfamiliar with three-dimensional graphing and could not think of how to pursue their ideas since she could not discuss it. Additionally if the teacher's mathematical knowledge is limited, it is difficult to assess the validity of student arguments. Rather than cope with that, some teachers opt to force the discussion in the way they feel most comfortable to handle. This may create an ambivalent classroom setting; at one time it is appropriate for students to guide the discussion and other times it is very directed by the teacher.

This ambivalence also occurs when teachers only implement the strategies and do not change other classroom aspects that support those strategies. The most common occurrence is to have student evaluation based wholly on tests and quizzes while ignoring other means. On the one hand, student discussion is encouraged, but i ignored in the evaluation process. This slows student adaptation to a different classroom environment and frustrates the teacher when students do not respond as they had expected.

Our study supports the obvious: if teacher beliefs are similar to the philosophy of a new curriculum, it is easier to implement change. But what about the teachers whose philosophy is diametrically opposed with that of a new curriculum? The day-to-day coping with the classroom forces a mechanistic application of strategies while teachers begin to bridge the chasm between the curriculum and their own beliefs. To a casual observer it would appear that implementation was well underway but closer inspection indicates a superficiality that may precede a return to previous teaching methods or movement toward a closer match with the process teaching model.

Three factors seem to influence the perseverance to move to a richer application of process teaching. Most important is the use of materials that support the instructional approach. More than ever, we are aware that pedagogy and content must be tightly intertwined. Secondly, a philosophical shift to match the teaching approach must occur. Finally, an integration of classroom practices into a global entity rather than isolated segments such as instruction and evaluation provides a cohesive environment that allows students to change and adapt to the classroom environment as teachers change.

Workshop data have not been fully analyzed at this point. However, aspects of the workshop will be tied to teaching actions and philosophical issues documented in postworkshop data collection. It is hoped that through this data analyses, workshop features can be modified to encourage greater success in the implementation stage.

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# REASONING AND REPRESENTATION IN FIRST YEAR HIGH SCHOOL STUDENTS

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*Ten first year high school students were asked to judge simple statements about combining odd and even numbers as true or false. They were also asked to give justifications or explanations for their decisions. All of the students initially reasoned purely inductively, appealing to specific cases and justifying their answers with additional examples when presses. However, three students went beyond this empirical reasoning and created idiosyncratic, personal arguments for their decisions. None of the students used algebraic notation in any of their reasoning. Two of the students used a visual representation of odds and evens in making their arguments.*

## Introduction

Generalization, and testing the limits of generalization through proof, may be said to be at the heart of mathematics. An acknowledgement of the importance of this kind of thinking in the mathematics curriculum can be found in the Curriculum and Evaluation Standards for School Mathematics, published recently in the United States by the National Council of Teachers of Mathematics (1989). In the Standards, mathematical reasoning is set forth as a goal for all students of mathematics, at all ages and levels. This term, "mathematical reasoning" is defined to include a range of capabilities. According to the Standards, students should be able to:

> " • recognize and apply deductive and inductive reasoning;
>
> • understand and apply reasoning processes,with special attention to spatial reasoning...
>
> • make and evaluate mathematical conjectures and arguments;
>
> • formulate counterexamples;
>
> • formulate logical arguments;
>
> • judge the validity of arguments ..."
>
> (NCTM, 1989, p. 81 and 143)

These skills have often been addressed only in geometry classes, in the context of carrying out formal, two-column proof on triangles, circles and other figures. Yet it has long been acknowledged that the teaching of proof in such classes is often unsuccessful, and may lead to shallow, syntactic knowledge rather than deep understandings of the mathematics involved (Schoenfeld, 1988; Hanna, 1983).

The study described here was concerned with mathematical reasoning and explanation outside of, and prior to, formal instruction in a geometry class. Instead, the focus was on the reasoning skills of ten first-year high school students, who were volunteers in a project on the use of computer-based microworlds for mathematics. In order to understand some of the difficulties involved in learning and teaching proof, it may be useful to look at the cognitive precursors to formal proof; that is, the kind of informal explanations that students offer when confronted with mathematical patterns or regularities. Such an approach, which takes a constructivist or genetic stance toward the development of students' reasoning abilities, may clarify difficulties and suggest instructional strategies for assisting students in learning this specialized kind of thinking.

## Objectives of the Research

The results reported here were gathered as part of a study of high school students' interactions with a computer microworld for transformation geometry (Edwards, 1990; 1991). The objectives of the research project as a whole were to investigate the kind of reasoning which high school students applied to situations involving composition of reflections, a task which had been previously investigated with middle-school students (Edwards, 1988). This task was determined to be useful in eliciting students' strategies for discovering and testing hypotheses, using the computer microworld, and for engaging in mathematical generalization (for a report of research addressing similar questions, using a different computer environment, the Geometric Supposer, see Chazan, 1990).

One of the research questions was whether the opportunity to use a computer microworld to generate and test hypotheses and conjectures would improve the students' abilities to reason mathematically. In order to test this, a simple task-based interview was carried out before and after the students' experience with the microworld. The objective of the interview was to discover the kind of reasoning the students already employed, in a domain unrelated to transformation geometry. If there was a change in this reasoning at the conclusion of the study, then it could be argued that the microworld was effective in helping the students to learn how to reason mathematically.

## Methodology

The students who participated in the study were 10 first year algebra students, ages 14-15, including four girls and six boys. The students worked in a small research lab at the university for a period of 5 weeks. During the first and last session, the students were interviewed individually using the task described below; for the remaining sessions, they worked in pairs with the microworld (written in Boxer).

The task used to assess the students' reasoning consisted of a set of statements printed on cards of the form:

"Odd plus odd makes even"

The students were asked to decide whether the statement was true or false, and then to tell the investigator why they made their decision. A final card was presented, showing the following pattern:

$1+3=4$

$1+3+5=9$

____________

____________

For this card, the students were asked to add two more lines which showed the same pattern, and to explain the pattern.

The sessions were video- and audio-taped, and transcribed. A full analysis of the protocols is still underway, but the initial analysis, which showed some surprising and intriguing results, will be presented here.

## Results

The pretest consisted of the following statements, two of which are true and one false:

"Even x odd makes even"

"Odd + odd makes odd"

"Even + even makes even"

The post-test consisted of the following statements:

"Odd + even makes odd"

"Even x odd makes odd"

"Odd x odd makes odd"

One unexpected outcome was that a few students (3 or 4) had some difficulty in establishing the truth or falsity of the first statement. This seemed to be attributable to two factors: first, many students answered very quickly, apparently without much thought. When they were asked, "Are you sure?" they quickly self-corrected. The other source of error on the first item, "even x odd makes even" was to interpret "even times odd makes odd" as a misapplication of the "rule" for positive and negative numbers: "positive times negative makes negative." For example, one student, when asked for a justification, stated:

> NR: A positive and a positive makes a positive and a negative and a negative makes a positive, uh, something like that, I don't know...

It turned out that the students had recently been studying positives and negatives in class, and that this "rule" was salient in their memories. This evidently interfered with their interpretation of the "odd and even" questions.

This result in itself was interesting, in that it indicated the syntactic nature of these students' learning in mathematics - while they might have remembered the form of a rule, they did not pay attention to its meaning. Nor did they attend to the meaning of the items presented in the pretest. Instead, they seemed to make a cognitive mapping, associating "even" with "positive" and "odd" with "negative," and then applying a rule they had recently memorized.

The experimenter modified the introduction to the pretest after this error appeared in the first two subjects. The interview was started with the statement, "These questions are about odd and even numbers. What are some odd numbers? What are some even numbers?" This prompt was effective in orienting the students to the question at hand, and the "positive/negative" error was thereafter not repeated.

A more significant pattern of responses was found in the students' explanations, provided after they had correctly decided whether a particular statement was true or was false. It was expected that at least some of the students, after a year of algebra, would use their algebraic knowledge in simple proofs for the statements which they stated were true. For example, when asked why "Odd plus odd makes even," it was anticipated that some students would present a proof such as the following:

"Odd numbers can be written as $2n+1$

$(2n+1) + (2n+1) = 4n+2 = 2(n+1)$

$2(n+1)$ is divisible by 2 and therefore even."

None of the 10 students offered an algebraic proof of this kind. In fact, all of the students initially offered a purely inductive or empirical rationale for their decisions. When asked why a statement was false, they would offer a counterexample. When asked why a statement was true, they would reply with statements to the effect of, "I tried it, and it works."

When pressed to justify their answers, most of the students simply tried more cases. For example, the following dialogue took place after the first item had been answered correctly:

JG: ...um, so even times odd makes even.
LE: Is there anything else you want to say or add about that,
or any way you could explain or prove to somebody that it was true?
JG: The only thing that I could do is just try a few...

In total, 7 out of the 10 students reasoned in a way which could be described as purely inductive or empirical.

## Beyond empiricism, before formal proof

The three students who offered explanations which went beyond simple induction did not use algebraic notation or appear to be using specific knowledge gained in their algebra class. Instead, each offered an idiosyncratic argument, which in two cases was based on a change in representation of the problem.

In one case, the student, CM, answered all of the questions quickly and accurately, working out examples mentally and only writing down the specific numbers he tried when asked to by the investigator. When pressed to give a reason or explanation for the fact that "Odd + odd makes odd" is false, CM offered an explanation based on sketches of tick marks corresponding to odd and even numbers, as indicated in Figure 1.

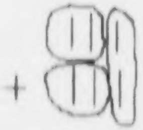

Figure 1: CM's sketch/explanation

CM explained that odd numbers always had one "left over", and showed with his sketch that when two odd numbers were combined, the "left overs" made up pairs, so that the sum would be even (a set of pairs).

This visual and verbal explanation indicated that CM was willing to go beyond empirical justification, and actually look at the structure of even and odd numbers in order to generate a valid argument for his decision. He used a similar argument for a number of the other items in the test.

In the second case, the student also used a visual representation to support her reasoning. She created a number line, and used a similar argument as that made by CM, involving "jumps" with gaps of two, or gaps of two and one more.

The final case involved a somewhat more complicated verbal argument, presented to justify the statement "Even + even makes even." In this argument, the student noted that all two-digit even numbers end in 0, 2, 4, 6, or 8, and since the sum of any pair of these single digit numbers is even, then the sum of any pair of even numbers must be even. This student did not present anything like this argument on any of the other items, instead appealing only to examples.

## Conclusions

Hanna has pointed out the importance of differentiating between "proofs that prove" and "proofs that explain" (Hanna, 1989). Before students are taught to prove, they can be provided with the opportunity to engage in less formal mathematical reasoning, by being asked to explain simple mathematical regularities. A well-noted difficulty encountered in this area is for students to see the need to go beyond empirical or inductive reasoning at all (Chazan, 1990).

The study described above suggests that some students at the beginning of high school, even without instruction in formal proof, will go beyond empirical reasoning and offer informal proofs or explanations of their findings. The results reported here are extremely limited in scope, and in fact, plans for the next phase of research are to extend the study both in duration (a school year) and population (two first year high school classes) in order to more fully investigate reasoning among students of this age. However, the results are consistent with previous findings for British students, working with a written test (Bell, 1976). It is interesting that for two of the students in this study, coming up with an explanation involved a change in representation of the problem. Each student "translated" the problem into a visual form in order to build his or her argument. This may have helped them to see better the structure of the mathematics underlying the simple regularities involved in combining odd and even numbers. In this sense, these were "proofs that explain," or at least, held explanatory power for the students concerned. In future research, the cognitive territory which comes before formal proof will continue to be explored, in order to provide a better understanding of how more sophisticated and powerful kinds of mathematical reasoning might be learned by students in secondary school.

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# PROBLEM-SOLVING IN GEOMETRICAL SETTING: INTERACTIONS BETWEEN FIGURE AND STRATEGY

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## Summary

*The analysis of the role of figure may explain some differences in problem-solving between the arithmetical and the geometrical setting. The aim of the study I am reporting is to begin an analysis of the interactions between figure and strategy in the resolution of problems in geometrical setting, with particular regard to problems related to the notions of area and perimeter of plane surfaces. The analysis of the protocols suggeststhat the perception of the figure as an object, autonomous from the graphic constructions performed, is achieved after a difficult and contradictory process. It suggests also that the ability at mentally trasforming figures may help pupils in planning and describing complex strategies in geometrical setting, since a figure may embody part of a complex procedure and thus contract its temporal dimension. It is also pointed out that a procedure may be grounded in a particular time without necessarily losing its generality.*

## 1.INTRODUCTION

### 1.1. Object of the research

In Bondesan and Ferrari (1991) some data are given that seem to stress the role of the figure in the resolution of problems in geometrical setting. In fact, it is reported that in geometrical problems children are more willing to search for alternative strategies and a larger amount of pupils who do not master verbal language in order to organize their reasoning can build effective strategies; moreover, the comparison of strategies, carried out in the classroom, gives rise to the diffusion of the ability at planning (or, at least, performing) complex strategies and the increase of the number of strategies produced for each problem. It is argued that the figure is crucial on account of its heuristic role in the search for a strategy, as pupils may 'manipulate' it (cutting, superposing, measuring, ...) by means of suitable representations. Moreover, it allows pupils to effectively represent the problem-situation (as far as it allows them to simultaneously perceive multiple relationships) as well as the resolution procedures (as far as it may embody the sketch or the record of a procedure). This seem to fit very well with learning processes based on verbal interactions among pupils.

The goal of the study I am reporting is to proceed deeper in the explanation of these phenomena, with particular regard to the interactions between figure and the construction of a strategy. In particular I was interested at testing the likelihood of my hypotheses, stating more precise ones and focusing some aspects of the subject. More systematic research is needed to validate the results presented here. The whole subject obviously concerns problem-solving in geometrical setting, but it may have implications for problem-solving in other settings, such as arithmetic, where representations seem to strongly affect the performances of pupils (in particular, low-level pupils).

### 1.2. Theoretical frame

In the last years the role of visualization in mathematics and mathematics learning has been widely analyzed (see for example Dreyfus (1991) for a review). The status of visual reasoning is not yet clearly explained, but a lot of studies has stressed the crucial role of figures in geometry. Figures are regarded as complex units, with both conceptual and spatial properties (such as Fischbein's 'figural concepts') and thus distinct from both pure concepts and drawings. Recently, research has pointed out the complexity of the interactions between different symbolic systems (such as verbal language and spatial representations) which have been regarded as a characteristic feature of learning processes in geometrical setting (see for example Arsac (1989), Caron-Pargue (1981), Laborde (1988), Parzysz (1988)). Computer models have been regarded as intermediate objects, different from both figures and drawings (see for example, Strässer and Capponi (1991)).

Related to the study I am reporting the results of Mesquita (1989, 1990, 1991) are quite interesting, in particular as far as they concern:

= the analysis of status of a figure (figures that are 'objects'-or models- in the sense that the geometrical properties used in their construction may be evinced, and figures that are only 'illustrations' if it is not the case)

= the analysis of the role of a figure (figures may only describe a problem-situation, as far as they supply a simultaneous insight of the properties involved, or may also promote the construction of a resolution procedure)

= the stress on pupils' representations of algorithms in geometry; three fundamental kinds of representation (figural, functional and structural) are recognized that do not depend upon age.

### 1.3. The role of figure: some hypotheses

Related to the issues mentioned in 1.1. I have stated the following hypotheses about the aspects of the status of figures that may affect performances in geometrical setting:

= a figure is an autonomous object on which pupils can operate and reflect; it can simultaneously represent complex systems of spatial relationships;

= a figure can represent complex resolution procedures; the temporal dimension of the procedures represented is contracted; this means that pupils who master mental manipulation of figures are expected to manage complex procedures better and more generally;

= pupils may perceive a figure and operate on it at different levels (material manipulation, measurement, symbolic manipulation, 'game of hypotheses',...; see also Mesquita (1991)); these levels may be simultaneously present (a pupil may use, at the same time, measurement arguments or more abstract relationships in order to discover or verify a property).

## 2. THE ORGANIZATION OF THE RESEARCH

The research that is reported is not a large-scale systematic one; I have gathered a large amount of protocols from 2 classes of grade 5 (about 40 pupils). These classes have experienced the Genova

Group's Project since first grade. The materials I have analyzed are normal working materials (pupils' copybooks, papers and so on) or assessment tests usually administered during the school year and concern the following tasks:

= find the area and the perimeter of a polygon (not necessarily regular nor convex) drawn on the paper;

= find the area of a region on a scale map (the pupils were given the map on a blank sheet);

= explain to some friend of yours how to fulfil the previous task.

For a general information on the Genova Group's Project see for example Boero (1989), Boero (1991) or Ferrari (1991). The concepts of area and perimeter have been introduced during grade 5 according to the following steps:

= discussion in the classroom of the meaning of words such as *area, surface, extension* in everyday-life;

= cutting (with scissors) or drawing on a squared sheet different shapes with the same extension, comparison of extensions by superposition and so on;

= doubling or halving the extensions of triangles and rectangles;

= measurement of the extension of rectangles by counting of the squares and using different units;

= construction of an area unit of one square meter;

= formula for the area of a rectangle;

= different ways to compare the extensions of plane surfaces: counting of squares, superposition, transformation, formulas;

= boundary of a plane surface; perimeter as the measure of the boundary of a plane surface;

= comparison of the boundaries of a plane surface;

= change of units of area and length;

= comparison of strategies in problems of area and perimeter;

= formula for the area of a triangle (by means of material and graphic transformations); heights of a triangle;

= measures with decimal fractions; change of decimal units;

= area of polygons (not necessarily regular nor convex) by (exact) covering with triangles;

= approximate area of geographic regions by approximate covering of a scale-map and balancing;

= formulas for the area of regular polygons (by means of graphic transformations) and of the circle.

The problem of the reliability of written reports related to the Genova Group's Project has been discussed in Ferrari (1991). For a general discussion of this issue see Ericsson and Simon (1980).

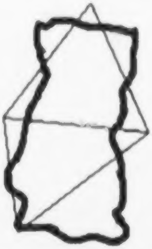
Throughout the paper by 'good problem-solvers' I mean pupils who are able to give acceptable solutions to most of the problems (either contextualized or not) they are administered during the year, not regarding too much the quality of the resolution processes or the reports. By 'poor problem solvers' I mean pupils

Nevertheless, these limitations do not seem to damage the skills at transformating the figure even mentally and planning complex strategies. There is also a number of pupils (about 20%) who manage to transform the figure (for example by decomposing it, or including it in other figures), but cannot use their constructions in order to solve a problem.

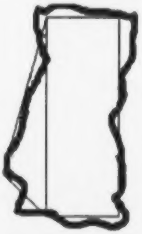
### 3.2. Figure and strategy

From a general analysis of the protocols concerning the approximate covering of a scale-map of a region with triangles or rectangles in order to estimate the area of the region, we have noticed three different kinds of constructions:

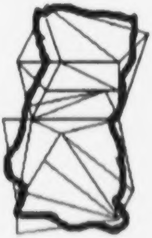
S1. the strategy is built according to some previous mental schema, without taking into account the specificity of the figure, in spite of contrary statements (for example, pupils who use only rectangles to cover a scale-map of Great Britain, or only triangles in order to cover a scale-map of Portugal or Sardegna);



S2. the strategy is built according to some previous mental schema which can be adapted to the specific needs (for example, pupils who change their strategy according to the map they want to cover, or who use both rectangles and triangles with the explicit purpose of reducing the calculations or the errors);



S3. the resolution is built by means of graphic operations without any strategy or schema previously thought (for example, pupils who cover the map with a large number of small triangles drawn at random, or who do not take into account the need for reducing the errors).



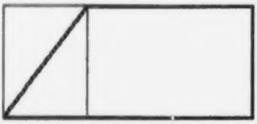
who almost never are able to design some strategy to solve complex problems and often meet with difficulties even when solving simple problems.

## 3. SOME FINDINGS

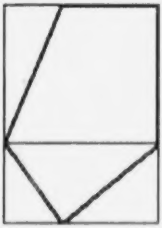
### 3.1 Figure as an object

Pupils succeed in perceiving a figure as an autonomous object only after a difficult process. At first they perceive the figure as the record of sequence of the graphic operations they have performed to build it. They write, for example: "*I change this triangle by putting another one at the side; so it is now a rectangle* ..." Only few pupils (less than 20%), in the first problems on triangles, seem to identify the figure as a product of their constructions, equipped with relations, which does not entirely depend upon the graphic operations performed. The elements of the drawing preserve the functions they have had in the graphic construction or in the manipulation, and are not included in a system of relationships. The height of a triangle is perceived (by about 90%) as "*the thing that allows me to divide the drawing*..." and the operation of drawing it is regarded as a transformation of the figure (as it is a transformation of the drawing). To the question "*why the area of a triangle is* $b \times h/2$ *and not* $b \times l$?" they (about 80%) give answers based on calculations or counting of the square ("*because* $b \times l$ *gives a wrong number*..."). In the successive problems, even if more complex, the figure as an object, with some relationships among it elements seems to appear.

When searching for the area of a trapezium most pupils work without any difficulty on the figure transformed by adding a small triangle on the left; this triangle (which allows pupils to regard the trapezium as a part of a rectangle) loses its procedural function of graphic construction and becomes a stable element of the new figure.



Some pupils (about 50%) begin to recognize some relationship among the components, such as the congruence of the small triangles, even if only few (less than 25%) explicitly recognize that the sides are pairwise equal.



Analogously, in the figure on the left, the operation of adding the triangle on the left-upper part is regarded by more than 60% of the sample as an operation on the trapezium, not on the whole figure.

Moreover, many pupils state (about 70%) that the quadrilateral they have built at the bottom is a rectangle, but very few use the fact that the short sides are equal when calculating the area of the external small triangles. Many of them (more than 50%) measure either side and someone even finds different values.

Pupils belonging to groups S1 or S2 seem to perform the graphic operations on the drawing according to some figural schema previously thought; in the whole sequence they have given significantly better results in tasks requiring mental transformations of figures.

Asked to deal with the error in their approximation, the following behaviours have been noticed in the first problems:

E1. pupils who do not realize the need for estimating the error;

E2. pupils who deal with the problem from a geometrical point of view, and try to improve their approximation by means of progressive refinements of the covering;

E3. pupils who deal with the problem from an arithmetical point of view and modify their covering according to calculations previously made (explicitly or not).

In the following problems of the sequence almost all pupils adopt this last procedure: they mainly use rectangles to approximate the area, and only few keep on using triangles; in the last problems no pupil use more than three polygons to cover the region.

The task "write down to a friend of yours how to find out the area of a geographic region by the approximate covering of a scale map, and to estimate the error" has provided some interesting data. All pupils have obviously given descriptions steadily grounded on time and organized as sequences of suggestions (or prescriptions) temporally structured by connectives such as 'before', 'afterwards' 'next' and so on; beyond this common feature, three different kinds of text may be recognized:

T1. pupils who reconstruct the procedure in a particular situation and time (about 20%); they use verbs and connectives that put the stress on the reconstruction of their own experience (*"suppose we must find the area of Argentina; now I draw two triangles here, I call this point A and this B; now I draw a line here..."*); all these pupils belong to the group S3 described above and their descriptions are always incomplete; no pupil in this group is a good problem -solver;

T2. pupils who reconstruct the procedure in a particular time (*"...now we draw,...and now we have almost get it...*) but do not refer to any particular situation (about 15%); they use verbs and connectives like T1 but the procedure is described in general; in this group have been found both good and poor problem-solvers;

T3. pupils who reconstruct the procedure as a process with the temporal dimension but place it in an abstract time (about 65%); they use connectives such as 'before', 'after' but never 'now' or 'at present'; in this group there are also the pupils who in the description of the procedure put some stress on aspects different from the temporal structure of the steps, taking into account the 'logical' organization or some constraints involved in the problem-situation; also in this group there are both good or poor problem-solvers.

## 4. FINAL REMARKS

= Pupils' representations of a figure are undoubtedly relevant related to the planning of a procedure; the one described in T1 (which seems roughly correspond to the attitude called 'figural' by Mesquita (1991)) seems correlated to a low ability at mentally transformating the figure. Nevertheless, the transition from a kind of representation to another is not clear-cut; it seems likely that a lot of pupils remain quite long in a level similar to the one called 'functional' by Mesquita and connected with the notion of 'schéma'; the emergence of the 'structural' (or 'algorithmic') perception of the figure is anyway difficult and contradictory;

= the ability at mentally transforming figures and regarding them as objects seem to affect the ability at representing and managing complex procedures, which become much simpler as far as figures may embody sequences of operations and substantially reduce their temporal complexity; good problem-solvers sometimes manage to go on even without it, but it seem crucial for weaker pupils, related to their problem-solving performances in geometrical as well as in arithmetic settings;

= pupils with a rigid ('geographic') perception of the figure meet with difficulties when asked to reconstruct some procedure in a general position; nevertheless the opposition between the figure (with its supposed specificity) and the algorithm (with its supposed generality, see Mesquita (1991)) is not entirely satisfactory; there are pupils who perceive algorithms in a close connection with time, and when describing them they seem to run over the steps again in a sort of identification; among these subjects there are also some very good problem-solvers; this way of perceiving algorithms does not seem an intermediate level between the understanding 'by examples' and 'structural' understanding, but a characteristic feature of a particular learning (and thinking) style, which does not seem to prevent the achievement of high level of abstraction.

= the trend of almost all pupils is to give simpler and simpler answers from the computational point of view (using mainly rectangles, and often only one) is most likely a consequence of the too rapid transition from area as a magnitude to area as a real number (see Douady and Perrin-Glorian (1989)); pupils' behaviour becomes more and more similar to their behaviour in arithmetical problems (very few alternative strategies, one strategy which spreads over the classroom, ...see Bondesan and Ferrari (1991)); even the lack of distinction between area and perimeter (which is more frequent among pupils who cannot transform figures mentally) may be explained by similar arguments.

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# A SEQUENCE OF PROPORTIONALITY PROBLEMS: AN EXPLORATORY STUDY

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*The report concerns an exploratory study performed about a sequence of 6 proportionality problems proposed in two classes by the same teacher over a period of about ten months. The problems concern different settings (geometrical setting and, after, arithmetical setting) and different contexts. The purpose of the study was to explore the transition to a multiplicative model, the conditions which may enhance it and the difficulties connected with the transfer of a model costructed in the geometrical setting to an arithmetical one.*

## 1.Introduction

The studies and surveys of the past decade concerning problem-solving have posed the question of the relationship between "laboratory" research on problem-solving and the study of the possible implications for teaching (in general, see Lester & Charles, 1991; as regards, in particular, proportionality problems, see Karplus & C.,1983; Tourniaire and Pulos, 1985; Grugnetti, 1991). We believe that this is a relevant question, as the research findings on proportionality problem solving do not seem in the least to have affected the most widespread teaching methods (consisting, in Italy and other countries, of training students to mechanically apply the A:B=C:X scheme).

This is an exploratory study of 7 teaching situations presented in two classes by the same teacher over a period of approximately 10 months. These teaching situations concern "paper and pencil explanation missing values proportion problems" (see Tourniaire and Pulos, 1985). The study involves the complete knowledge by the teacher-researcher-observer of the teaching activities carried out during the whole period considered. For this reason, we believe that it may provide reliable elements on which to base further studies concerning the "engineering of teaching" relevant to proportional reasoning and on the learning processes involved, even considering the limits ensuing from the small number of students and from the singularity of the experience.

This study is characterized by the following aspects:

- The first five situations concern geometrical proportionality problems referring to physical situations (sunshadows) evoked or directly experienced in real life (at first through problems without explicit numerical data). These problems,requiring a physical knowledge in addition to their proportional content, permit us to view separately, to a certain extent, students' difficulties and behaviours due to numerical values from their mastery of the relationships between the physical variables (see Harel & C, 1991). This choice appears to be significant in relation to the hypothesis that working with numerical values and the meanings of division may constitute in itself an element of difficulty. The problems posed permit, in particular, an exploration of the transition from the qualitative concept of dependence between proportional quantities ("if one grows then the other grows, too") to the quantitative concept ("if one goes into the other a certain number of times, then the other, too...")

The other two situations involve a change of context, the first one (body proportions; see Hoyles & C,1989,1991) still in the physical-geometrical setting (Douady,1985); the second in the arithmetical setting. These problems were proposed to explore the difficulties encountered by students in transferring, to more or less similar contexts, the models established in the first context. It should be noted that, in this sequence, the work in the geometrical setting precedes that in the arithmetical setting, and that numerical data never suggest easy, exact proportionality relationships (from this point of view, these problems may be classified as "difficult", according to Hart, 1981)

- In this study our focus ha been, above all, to the short- and long-term effect of particular teaching choices on the emergence and evolution of problem-solving strategies, and to the nature of such evolution. In particular, we have studied: 1) the effects of the presence of a real physical-geometrical situation, initially, and over a long period, experienced directly, referred to as an "experience field"( Boero, 1989) that provides meaning and consistency to the problem posed (determination of a height that cannot be directly measured); 2) effects of initially proposing problem situations without explicit numerical data; 3) the role of classroom discussion and of the active comparison of strategies (Ferrari, 1991) in overcoming the additive model and realising a conscious transition to the multiplicative model in the geometrical setting;4) the steps involved in this transition; and 5) the problems inherent in the subsequent transfer of the multiplicative model to other contexts and settings (especially to the arithmetical setting).

In our findings we have observed (see par.5) that real physical-geometrical situations directly experienced are not, in themselves, able to lead students (at the age of 11) to constructing proportionality relationships between the geometrical-physical variables involved, but that (if appropriately handled by the teacher) they may have an important role for many students in constructing such relationships(cfr. Karplus & C., 1983). However, a complete mastery of the multiplicative model - transferable to other geometrical contexts and well established over time - seems to require also the mastery of the link between geometrical proportionality relationships and arithmetical operations on the numerical values that represent the measurements.

An issue that we deem important and that remains an open question is the role of additive-type reasoning in the transition to multiplicative strategies. This problem appears to be more complex and, in part, different from what has been highlighted so far by the research on proportional reasoning. Another important question concerns the interpretation of the difficulties that students have in transferring strategies outside the geometrical setting.

## 2.Method

The study examined 37 students, of Grades 6 and 7, most of whom (30 out of 37) were between 11 and 12 years old at the beginning of the study. They were enrolled in two classes, of average level, of a school in Carpi (North Italy). The study was conducted from March, 1991 (Grade 6) to January, 1992 (Grade 7). It has also been possible to compare some of the data resulting from the observation of these two classes with data obtained from other classes of the same grade. All the classes we are considering are involved in the project of the Genoa Group for an integrated teaching of mathematics with the experimental sciences in the comprehensive school . The following characteristics of the project are relevant to this study :systematic work in "experience fields" (Boero, 1989) in the construction of mathematical concepts and skills as "knowledge tools"; systematic recourse to verbalization in problem-solving, and in comparing problem-solving strategies ; extended work on the (open) applied mathematical problems, including some problems in which numerical values are not made explicit;alternation between periods of individual work (e.g. during the resolution of mathematical problems) and of class discussions (e.g. during the comparison and evaluation of problem-solving strategies proposed in the classroom); systematic exclusion of the "automation" of the solution to proportionality problems through the adoption of such models as A:B=C:X.

The observations concern: individual solution of open questions, some asked as "story problems" (as in Situations 5 ,6,7); recorded discussions (in particular in Situations 1 and 2); reports by individual students (see Sit.2 and 4).

## 3. The sequence of teaching situations

These were the sole situations in which the two classes tackled the problem of the height of an object that cannot be measured directly, and of an additional situation of arithmetical type. During the period of the study (from March 1991 to January 1992) no other proportionality problems were posed.

Situation 1: the problem of height of the street-lamp ( during an outing; Grade VI, March)

The students go on a one-hour outing to observe sun shadows. During the outing the teacher poses the problem of determining the height of a street-lamp (almost 4 meters), whose shadow is seen on the ground. Near the street-lamp the students observe various shadows cast by objects of accessible height, in particular by fence-posts, just over one metre high; the teacher brings these shadows to the students' attention. This is a verbal arithmetical problem, without explicit numerical data and with the presence of a physical - geometrical reference that permits the students to tackle the problem without worrying about the actual calculation of the numerical result.

This problem situation falls within the teaching unit devoted to the phenomenon of sun shadows, which constitutes one of the most important parts, both in terms of content and of the time invested, of the Mathematics and Science activities of the project for Grade 6. In particular, the problem is posed after some observation and discussion of the "fan" of shadows during the day. During these activities, the students realize, among other things, that "when the sun is high in the sky, shadows are short; when it is low, shadows are long", and that "longer objects cast, at the same point in time, longer shadows".

In the process of "rationalization" of the shadows phenomenon, this problem situation represents the introduction to its quantitative analysis. If, with qualitative observation, a crisis was triggered with the model "strong sun - long shadow" that most of the students hold, with this situation we move to the quantitative aspect of the relationship "longer object - longer shadow".

Situation 2: the problem of the height of the street-lamp, in the classroom (the day after the outing):

*"On 4 March, between 11 AM and 1 PM we went out to determine the height of a street-lamp. Recount what happened and find a way to determine the height of the street-lamp."* (individual work).

Later, the teacher moves to the analysis and "active comparison" ( Ferrari, 1991) of the solutions produced: she selects two of the solutions produced by the students, one of multiplicative type (correct) and the other of additive type (incorrect), and asks the students first to determine which of these solutions their own strategy followed, then to follow the other strategy, and finally to evaluate both of them. Only after these activities are completed, are the measurements of the shadows and of the fence-post used to verify the different results produced by the two strategies and discuss in depth their correctness. Situation 2 required over three hours of work.

The work on shadows continues, with activities concerning parallelism and the movement of shadows on the ground (angles, and so on).

Situation 3: the problem of the two nails (as an evaluation test , a few days later)

*"The drawing represents, from above, the shadows cast at 11 AM and at 12 noon in Genoa by an 8-cm nail placed at position A. At position B there is another nail, 6 cm in length. Do you think you can draw precisely the shadows cast by the nail at position B, determining their lengths and positions? Explain your reasoning."*

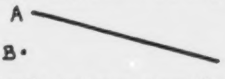


FIG. 1 (here reduced in scale)

The problem was posed to explore the difficulties the students experienced because of the presence of numerical data, and the text evoked the situation which they had previously experienced. The "a priori" analysis of the problem identified as additional difficulties those ensuing from the presence of a decimal ratio and, above all, from the fact that the unknown length was less than the known length.

The work on shadows continues, with activities concerning the (angular) height of the sun in the sky, the movement of the sun in the sky, and so on .

### Situation 4: individual report about the work performed during the year

During the summer holidays the students were asked to find and to reconstruct the main stages of the teaching unit devoted to the phenomenon of sun shadows (February - June), making explicit the knowledge gained and the difficulties encountered, so as to evaluate -in particular- whether the students are able to correctly "reconstruct" the experience of the street-lamp and the strategies that emerged from the discussion.

### Situation 5: the problem of the height of the clock tower of the Pio castle (October 1991, grade VII)

*"Yesterday I was in the square at Carpi and met one of the masons that are restoring Pio Castle. While we were talking he told me that the documents concerning the building of the castle tower (the clock tower) had been lost. Then, a little worried, he told me: "I have to call a crane, but it would be better to know to what height it must reach, to enable us to work on the clock tower." "If you want to know the approximate height of the tower, without measuring it directly, you can measure its shadow: that is much easier to measure! But you must also know the length of something else and of the shadow it casts, at the same moment" I told him. "Really?," he asked me, astonished, "let's try it, then!" We started to take measurements and chose, for comparison, my height. These are the measurements found:teacher's height: 1.60 m; length of the teacher's shadow: 2.08 m; length of the shadow cast by the tower: 32.5 m.*
*Can you determine the height of the clock tower? Explain and give the reasons for the method used."*

The problem was proposed to verify the medium-term persistence of the mastery of the multiplicative model in a problem situation given in the text, very similar to the "street-lamp problem", but with numerical data made explicit, this time.

### Situation 6: the problem of the height of the statue (December 1991, grade VII)

*"A recent archaeological excavation in Calabria found the remains of a Greek statue, probably of a warrior, that had stood in front of a temple. The only intact part of the statue is a foot, approximately 76 cm in length. We would like to know approximately how tall the statue was. We know the dimensions of Michelangelo's David, which are: foot length 54 cm,height of the statue 432 cm.*
*Try to find how tall the statue was. Explain your hypotheses and your method".*

The problem was proposed in order to verify the possibility of transferring strategies of the multiplicative type to geometrical situations that are partially different from previous ones, due to the different context. The elements of diversity essentially consist in the fact that they are proportional parts of the same "object", and not length relationships between different "objects" (object that casts a shadow, and its shadow), as in the previous cases. Moreover, the context of the problem may bring to mind a "natural" idea of proportionality to which the students may refer (see also Hoyles, Noss, Sutherland, 1989 and 1991).

### Situation 7: the problem of the jam (January 1992, during the four-monthly evaluation* - Grade VII)

"Last year Mrs. Pina made plum jam. She had 13 kg of plums, from which she obtained 5.5 kg of jam.
This year she wants to obtain 8 kg of jam. What quantity of plums does she need?
Explain and give the reasons for your procedure".

The problem was proposed to verify the transferability of multiplicative strategies to arithmetical problems without any immediate physical-geometric reference. Both the context of reference of the problem and the nature of the variables concerned are thus modified with respect to the previous problems.

Situations 3, 4, 5, 6 were proposed without any subsequent comment and explicit evaluation of the work done by the students. After Situation 7, approximately 6 hours of work were carried out (alternating between individual work situations and discussions), that lead the students to think (under their teacher's guidance) about the nature of the problems proposed, so as to recognize common aspects and possible common problem-solving strategies.

## 4. Analysis of the students' behaviour and evolution of their strategies

This table summarizes the results of the analysis performed on students' strategies:

| P\S | Sit. 1 | Sit. 2 | Sit. 3 | Sit. 4 | Sit. 5 | Sit. 6 | Sit. 7 |
|---|---|---|---|---|---|---|---|
| 1 | Ma | Ma D | Ma | R V | M | M | Mu |
| 2 | | A D | A | R | M | M | A |
| 3 | | A | A | | A | A | A |
| 4 | | A D | Ma | R V | M | M | A |
| 5 | | A | B | | M | M | * |
| 6 | | B | B | R | A | M | A |
| 7 | | A | A | | A | A | A |
| 8 | | A D | Mb | R V | M | M | Mu |
| 9 | | A D | Mb | R | B | A | Mb |
| 10 | | Ma D | Ma | R V | M | M | Mb |
| 11 | | A D | A | | M | M | A |
| 12 | | A D | A | R | M | M | Mu |
| 13 | | A | Mb | R V | M | M | A |
| 14 | | A | A | | B | A | A |
| 15 | | Ma D | Ma | R V | Ma | M | Mb |
| 16 | | A | A | R | A | M | * |
| 17 | | A | A | | A | A | * |
| 18 | | A D | Mb | R | M | A | A |
| 19 | | A | M-B | R | A | A | A |
| 20 | | A D | M-B | R | M | A | Mu |
| 21 | | A D | Mb | R | M | A | A |
| 22 | | B | B | | B | B | A |
| 23 | | A D | A | | M | M | A |
| 24 | | M D | M-B | R V | M | M | Mb |
| 25 | | B | Mb | R | A | B | Mu |
| 26 | | B | B | | A | B | Mb |
| 27 | | A | A | | A | M | A |
| 28 | Ma | Ma D | M | R V | M | M | B |
| 29 | | B | B | | A | M | A |
| 30 | | A | A | R | A | A | A |
| 31 | | A | M-B | R | A | A | A |
| 32 | | A | M | | B | A | A |
| 33 | | M D | Ma | R V | M | M | A |
| 34 | | A | B | R | A | A | A |
| 35 | | A | A | R | A | B | A |
| 36 | | A | M-B | R | M | A | A |
| 37 | | A D | Mb | | A | A | A |

A = additive ; M= multiplicative complete: Ma= multiplicative interwoven with additive considerations: Mb= building up (Hart, 1981); Mu = reduction to unity; B= blockage; M-B= begins with multiplicative considerations, then blockage; ; R V = remembers and verbalizes exaustively; R= only remembers; D= clearly distinguishes the two strategies ; * = absent

### 4a. Further information about students' strategies

(i) In Sit.2 , only 6 students proposed a correct strategy: purely multiplicative (M),or partially additive (Ma)- probably influenced by the strategies proposed verbally by their classmate in Sit.1.In Sit. 3,notwithstanding a more difficult problem than the previous one, 19 students seemed to have a clear idea of the proportionality relationship between the quantities.In other classes, in which no active comparison and evaluation of the strategies for* the "problem of the street-lamp" took place, less than 20% of students produced proportional reasoning in the "problem of the two nails".

(ii) In Sit.3, the analysis of the multiplicative strategies, complete or not, clarifies the nature of the difficulties foreseen in the "a priori" analysis:7 students, (M) or (Ma), solved the problem correctly and completely; 7 students (Mb) clashed with the decimal value of the shadow/nail ratio (19:8=2.3). They calculated it, made explicit that it was *"the times that the nail goes into its shadow at 11 AM"*, but did not identify the arithmetical procedure to be used. To solve the problem, they give *"a bit more than twice"* the length of nail B (for instance 6+6+1.5, or 6+6+2). This type of strategy is similar to that described by Hart (1981) and Lin (1989) called the "building up" method.Five students (M-B) followed again the strategy of the "street-lamp" problem and calculated the ratio between the two nails (8:6=1,3), made it explicit that this was *"how many times nail B goes into nail A"*, after which they did not manage to correctly use this ratio. They would need to use the inverse scalar operator (Vergnaud,1981),but the students did not succeed to give a meaning to "divide for a certain number of times".

(iii) In Situation 7 only 10 students solved the problem correctly:5 by using a strategy of "reduction to the unit" (Mu), calculating the weight of the plums needed to make 1 kg of jam (no strategy of this kind was performed before) ; and 5 by "building up" strategies (Mb)

(iv ) An analysis following the evolution in time of the students' strategies in the geometrical setting is particularly interesting:for six students the multiplicative model is present from the start (in Situation 2) and remains well established over time; for 8 students there is a progress, without lapses, from the additive model used at first to the multiplicative model. All these students (6+8) were able to recognize the model adopted for the solution of the first problem, and made explicit in this occasion, or later, the reasons why the other model was not valid.For ten students the progress from the additive model to the multiplicative model does not appear to be steady . Among these students, 5 had not been able to recognize with clarity, during the comparison of the strategies,the one they had used,neither had they been able to explain why the additive model"does not work". For seven students that had initially adopted an additive model,no progress is found;they were not able to recognize their strategy as analogous to the strategy selected by the teacher,and so much the less* to acknowledge that it was not correct.

(v)It may also be noticed that all the problems posed would permit to proceed both with strategies of the "Between or scalar ratio" type and with strategies of the "Within or function ratio" type (according to Vergnaud, 1981- see also Karplus & C.,1983). The choice between one and the other seems to depend on the context and on the relative size of the objects to be considered : in Situation 5 all the students except for 2 calculated the ratio between two shadows , while in Situation 6 the problem-solving strategies denote a different perception of the problem situation: 12 students out of 14 that had correctly solved also the previous problem pass from the evaluation of how many times a shadows goes into another to the evaluation of the ratio between the statue's foot and its height, while only one pupil operates the opposite change, and a second one applies the same type of strategy to both problems

(vi) An interesting fact emerged in relation to the evolution of strategies: several students combined (Ma) additive and multiplicative (of "scalar" and "function" type )considerations, in a step that may be considered as a transition to a coherent and completely multiplicative strategy. In the situation of the observation of shadows, the two students that identified a correct strategy, explained it to their classmates as follows:

*"There is a difference between the fence-post and its shadow as between the street lamp and its shadow, but in the street-lamp case the 'difference' must be longer, since the street lamp is longer. To make the street lamp equal to the shadow that it casts, I must take away a greater difference than in the post's case. I see approximately how many times the post 'goes' into the street lamp. I take away from the shadow of the street lamp the difference between the post and its shadow as many times as the post goes into the street lamp, so the shadow and the street lamp are equal and I can measure the shadow."*

This process is probably forced by the actual experience in which the students observe the "more" that makes the shadow different from the object that casts it. In Situation 2 this strategy is changed by two students, who replace it with a fully multiplicative one. In Situation 3, on the other hand, where the conflict with the arithmetical aspects of the problem is strong (the length to be determined is less than the given one), the correct strategies were of the type described above (except for two other students). In Situations 5 and 6 the correct strategies are all of the completely multiplicative type, save for one (in Sit. 5).

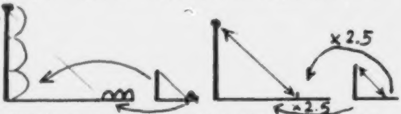
## 5. Discussion

The analysis that we have carried out shows how problem situations considered as such by the students, and relevant to a context in which the geometrical-physical aspect experienced directly is paramount, may be used by the teacher to motivate students' transition from an additive model to a multiplicative one. In this respect the "experience field" of the sun shadows, according to the analysis of the protocols produced, seems to have certain intrinsic characteristics appropriate to "force" the construction and the development of the students' strategies: the sun is the "cause" of the shadows; the relationship between the object that casts the shadow and the shadow itself cannot be modified by the observer. The system comprising the sun, the objects that casts the shadows and the shadows themselves is very "rigid": it evokes "contemporaneous" and "same-type" relationships between the heights of the objects and the lengths of the shadows cast by them, and therefore suggests the existence of the ratio as "invariant" (compare also Karplus&C.,1983). This aspect is particularly clear in several texts produced by the students in Situation 2:

*"The street lamp is much bigger than the fence-post, and its shadow must be much longer than the fence-post's. The difference between shadow and fence-post cannot be the same because the sun does not play favourites! Then I should know how many times the street lamp is higher than the fence-post: I have the shadows and I can know it dividing the shadow of the street lamp by the shadow of the fence-post".*

*"Everything has its own shadow and the difference between object and shadow changes from object to object. I cannot leave from the street-lamp's shadow the difference fence-post shadow because if I change the fence-post, for example, if I take a much smaller fence-post, this difference changes, and then also the height of the street-lamp changes, and this is impossible".*

The lack of explicit numerical data in the first two situations seems to have important effects, particularly as regards the necessity to graphically represent the problem-solving strategies, with positive consequences on the development of the reasoning based on "how many times... goes into ..."In this regard very effective representations were those given on the right, that were later taken up also by students other than those that had produced them.



The study reinforces the hypothesis (Kuchemann,1989) that the presence of numerical values superimposes specific difficulties inherent in the arithmetical processing of the data to the difficulties inherent in the conceptualization of proportionality as a ratio between quantities. In particular, a fact that stands out is that the "how many times a shadow goes" into another, or a fence-post into its shadow, does not correspond to the fact that the number of times may be determined calculating a division between the measurements. All this is evident when the problem is to determine the height of an object that is smaller than the object that casts the shadow(Sit.3)(see ii). The role of social activities in overcoming the additive model in the geometrical problems seems to be very important . In particular, the active comparison and the evaluation of strategies permit to overcome the limits inherent in the pure reference to "vision":in effect, even in the presence of other students that suggest recourse to multiplicative models and of a direct experience of "vision" of the fence-post and of the street lamp, only 6 students in Situation 2 produced a coherent and correct problem-solving strategy (while 26 propose a coherent additive model!). The situation improves considerably when the problem situation and the problem-solving strategies are represented and argued, and the students have to follow and evaluate the reasoning of their classmates. The reasoning and representations of the "best" students "mediate" the transition to more appropriate strategies.(see ii). The analysis also brings to light various problems:

- (see iii)For most students there is no transfer of the multiplicative strategy acquired in the geometrical setting to the arithmetical setting of Situation 7, and, when a transfer seems to take place, there is a change of strategy. This may be due, beside than to the lack of correspondence between arithmetical operations of multiplication and division and meanings of proportionality between quantities, also to precise characteristics of the "physical-geometrical" situation. In particular, in the "arithmetical" problem the reasons that induce some students to carry out the transition from the qualitative model "to grow with..." to the quantitative model of "equality of the number of times that..." fail (as may be noticed in the protocols mentioned before with reference to Situation 2, there appear extrinsic reasons of "balance" with respect to the mathematical structure of the problem and linked to the particular situation observed). Another failure that occurs is that of the direct perception of the simultaneity of the relationships between the elements compared. Another diversity is linked to the difficulty to produce appropriate external representations (like those produced for geometrical problems).
- Role of the additive strategies in overcoming the additive model and in approaching the multiplicative model . Although numerically limited and not long-lasting, some cases of interweaving (supported by external representations) between additive considerations and multiplicative considerations such as those exemplified at the end of paragraph 4 (see vi), suggest the idea (to be analysed on a wider scale and with a more appropriate methodology) that the multiplicative model in the geometrical setting may, at least in certain instances, result from an operation of "contraction" of additive reasoning, that would not thus constitute only an intermediate stage linked to the mental ripening of the subject , but a fundamental element of the transition to multiplicative strategies.

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# DISCREPANCIES BETWEEN CONCEPTIONS AND PRACTICE: A CASE STUDY

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*Recent work suggests that teachers' conceptions of the nature of mathematics, its teaching and learning, are not always consistent with their practice. This study is concerned with the reasons for these discrepancies. Inspired by Schön's Reflective Practitioner (1983), the author examined her own teaching practice while experimenting with a problem-solving teaching approach with college students. Tape-recordings of the teacher in class and a teacher-journal provide the basis for qualitative analysis which was conducted together with a second researcher. Partial results suggest that although class preparation follows teacher-conceptions, in class, spontaneous reactions differ. Subsequent analysis will look for explanations of these differences. On the basis of these results, the possibility of using reflection as a way of improving consistency between in-service teachers' conceptions and their practice will be examined.*

*(This paper will be presented in English at the conference).*

## Le problème

Les résultats de recherches en didactique n'atteindront la classe de mathématiques que si l'on passe également par l'un des principaux intervenants du système didactique, l'enseignant. C'est ce dernier qui en bout de ligne contrôle les choix didactiques et qui dans la mesure où il est maître de ses actes d'enseignement définit le cadre d'apprentissage de l'élève. Or ces choix, c'est l'avis de nombreux auteurs (Clarke, Peterson, 1985; Vergnaud, 1988; Ernest, 1989; Thompson, 1984), sont commandés par les conceptions de l'enseignant au sujet des mathématiques, de leur apprentissage et de leur enseignement.

Certaines études (Cooney, 1985, Thompson, 1984, Kaplan, 1991) laissent voir que les conceptions telles que déclarées par l'enseignant ne se transmettent pas toujours dans la pratique. La possibilité de prendre conscience de ses conceptions et de réfléchir sur sa pratique amènerait l'enseignant à améliorer la cohérence entre les conceptions et la pratique. Plusieurs interventions visant la formation des maîtres particulièrement, les maîtres en service ont expérimenté divers moyens pour susciter cette réflexion mais, la plupart des approches utilisées bien que fructueuses, demandent une organisation extérieure et ne se transportent pas nécessairement dans le quotidien des enseignants.

Nous avons donc voulu savoir

-Jusqu'à quel point la pratique reflète les conceptions exprimées?

-Comment expliquer les écarts, les divergences?

Et par la suite voir

-Quels moyens peut-on suggérer aux enseignants pour organiser leur réflexion en vue d'améliorer la cohérence entre leurs conceptions et leur pratique?

Notre objectif était d'ébaucher une méthode d'auto-analyse des actes d'enseignement qui permettrait à l'enseignant, de prendre conscience de ses conceptions et d'observer jusqu'à quel point sa pratique est cohérente avec ses conceptions.

### Les hypothèses

Au point de départ, nos lectures et notre expérience d'enseignante nous suggéraient certaines réponses que nous avons résumé en trois hypothèses que nous avons voulu vérifier.

H1) Certaines réalités comme les contraintes environnementales, les réactions des élèves ou encore, les modes de comportements habituels ou anciens sont plus fortes que les conceptions avouées et gênent la réalisation de l'enseignement tel que préconçu.

H2). Il est possible pour un enseignant d'analyser sa pratique.

H3) Le fait de réfléchir sur la pratique de façon quotidienne par la rédaction d'un journal amène des modifications à la pratique et aux conceptions de sorte à tendre vers un équilibre entre les deux.

### L'approche méthodologique choisie: sujet-chercheur

L'exploration de ce problème demandait une approche méthodologique originale. C'est l'interprétation des conceptions qui d'abord était questionnée. Cooney (1985) avait suggéré que des différences d'interprétations entre le chercheur et l'enseignant était peut-être ce qui expliquait les divergences entre les conceptions et la pratique. En choisissant d'être le sujet et le chercheur, nous pouvions interpréter sans biais nos conceptions. Nous les avons établies à partir d'écrits préalables à l'expérimentation.

De plus, le questionnement et l'observation par un tiers n'est pas sans influences. Nous avons voulu limiter autant que possible ces interférences dues au cadre de recherche pour mieux cerner les interférences entre les conceptions et la pratique. Nous avons remplacé la présence d'un observateur extérieur par l'utilisation d'enregistrements sonores et la rédaction d'un journal de bord. La lecture du journal de bord pourrait en plus, faire apparaître les modifications de nos conceptions en cours d'expérimentation.

Pour assurer la validité de la recherche, nous avons fait appel à la collaboration de d'autres personnes à différentes étapes du travail d'analyse. Une deuxième chercheure a travaillé en parallèle avec nous. Ayant pris connaissance de nos conceptions par la lecture de nos écrits préalables, ayant écouté et codé les enregistrements de l'expérimentation, elle pouvait corroborer notre analyse aux diverses étapes. En dernier lieu, deux autres personnes, l'une chercheure et l'autre enseignante ont vérifié si nos interprétations étaient soutenues par les données fournies. Nous étions de notre côté assurée d'une bonne connaissance du milieu et d'une présence suffisamment longue sur le terrain.

### L'expérimentation

L'expérimentation avait été précédée d'une exploration et d'une pré-expérimentation. L'exploration a permis de mieux cerner le contexte de notre expérimentation. Les élèves étaient des élèves ayant eu des difficultés avec les mathématiques auparavant, c'était en fait leur seul point commun. Les classes étaient non-homogènes quant l'âge, la provenance, la habileté d'apprentissage, les acquis et la motivation. Le cours devait combler les lacunes de ces élèves en passant à travers ce que l'on convient d'appeler les mathématiques de base: algèbre, fonctions, trigonométrie.

Forte des connaissances acquises lors de cette exploration, nous avons mis sur pied une approche pédagogique suivant nos conceptions de l'enseignement des mathématiques. Cette approche misant sur la participation active des élèves demandait la création de matériel didactique particulier. La période de pré-expérimentation nous donné l'occasion de construire ce matériel et de le tester auprès des élèves (Gattuso, Lacasse, 1989).

Nous avons pu par la suite passer à l'expérimentation elle-même. La clientèle étudiante était sensiblement la même et le matériel didactique utilisé à la pré-expérimentation a été repris après de légers réajustements. Pour les besoins de la recherche, nous avons ajouté l'enregistrements sonores des cours et la rédaction d'un journal de bord.

### L'analyse

L'analyse s'est déroulée en plusieurs étapes dont nous traçons ici les grandes lignes. Il a fallu d'abord établir nos conceptions afin de pouvoir construire une grille d'analyse. C'est ce que nous avons fait en utilisant comme source de données certaines de nos publications antérieures à l'expérimentation.

Nous sommes ensuite passée à l'analyse de la pratique en fonction des conceptions établies dans la grille. À cette étape, les données étaient tirées des enregistrements sonores et du journal de bord. Après le découpage et le codage des données, nous avons pu faire une compilation qui nous a menée aux résultats.

Nous avons par la suite examiné les réflexions notées en cours d'analyse afin de examiner si les conceptions se modifiaient en cours d'expérience.

En dernier lieu, nous avons regardé de façon critique le cheminement parcouru pour en tirer un cadre de travail que nous suggérons aux enseignants pour organiser leur réflexion sur leur pratique.

## Conclusions

Nous voulions en premier lieu mieux connaître les liens entre les conceptions et la pratique d'un enseignant de mathématiques et par la suite voir s'il était possible de mettre en place de façon profitable cette réflexion sur la pratique. Si nous regardons nos hypothèses de départ, nous avions prévu que certaines interférences dues à l'environnement, aux élèves et aux habitudes de l'enseignant interviendraient dans la réalisation de nos conceptions. Nous pensions également qu'il était possible pour une enseignant d'analyser sa pratique et que la réflexion quotidienne améliorerait la cohérence entre les conceptions et la pratique. Les conclusions auxquelles nous sommes arrivées nous amènent à préciser ces hypothèses.

### Des conceptions à la pratique

Ce travail nous a permis de faire un pas en avant dans la compréhension des éléments qui interviennent dans la mise en pratique des conceptions dans le cadre de l'enseignement des mathématiques.

Les résultats de l'analyse ont montré qu'il y avait une très bonne cohérence (82%) entre les conceptions telles qu'exprimées au départ et les actes d'enseignement observés. Ce résultat est sans signification si nous ne tenons pas compte des particularités de notre expérimentation. Nous avions voulu au point de départ alléger autant que possible les contraintes extérieures qui selon les auteurs consultés seraient des causes de discordances. La grande liberté dont nous jouissions au moment de l'expérimentation a surement joué. Nous avions pu définir l'approche pédagogique, choisir jusqu'à un certain point le contenu du cours et construire le matériel didactique en fonction de nos conceptions au sujet de l'enseignement des mathématiques. Notre expérience en enseignement et notre formation première en mathématiques nous garantissaient l'assurance nécessaire pour entreprendre une telle innovation. Mais il reste que certains obstacles demeurent. Certains sont exogènes et d'autres plus personnels à l'enseignant sont endogènes.

Nous avons pu voir que certains éléments dépendant de l'organisation scolaire gènent. Les plages horaires extrêmes, des locaux trop petits en sont des exemples. Le matériel didactique et le contenu mathématique amènent aussi quelques difficultés. Les protocoles d'activités comportaient encore certaines ambiguités et le contenu mathématique était parfois trop simple ou trop difficile pour les élèves. C'était alors difficile d'aller dans le sens prévu, c'est-à-dire, conduire les élèves à explorer les concepts et à déduire les connaissances à partir de leurs résultats.

D'autres entraves se trouvent chez les élèves eux-mêmes. C'est surtout leur manque de préparation au niveau des mathématiques et de la méthode de travail en général qui a contrarié la réalisation des conceptions qui visaient plus à soutenir la recherche de solutions qu'à expliquer comment faire le problème. Le temps pris par ces élèves moins préparés était trop important et gênait notre gestion du travail de l'ensemble du groupe.

Enfin, notre système de conceptions lui-même était en quelque sorte porteur de difficultés. Ce n'est pas que les conceptions se contredisaient mais elles pouvaient dans les cas limites être en conflit. Les conceptions concernant l'activité mathématique, ouverture, exploration, autonomie ont pris, sans que nous nous en rendions compte, le dessus sur les conceptions touchant à l'organisation du cours et à l'encadrement des élèves. Soulignons ici que la réflexion et le bilan qui s'en est suivi ont permis cette constatation dont nous avions jamais pris conscience auparavant. Signalons enfin que nous avons constaté que notre état d'esprit, fatigue, inquiétude, bonne humeur joue également sur nos actes d'enseignement.

En résumé, nous pouvons conclure que les liens entre les conceptions et la pratique sont forts et que s'il y a prise en charge consciente des conceptions et des moyens pour les mettre en oeuvre, le transfert des conceptions dans la pratique se produit.

### L'auto-analyse comme outil de réflexion

L'auto-analyse telle que nous l'avons pratiquée s'est avérée un outil de réflexion profitable et réalisable.

L'auto-analyse a donné lieu a un bilan professionnel qui a permis une prise de conscience intéressante et utile. Les résultats ont révélé certaines de nos faiblesses, ils ont exposé certains succès encourageants et indiqué des modifications dans nos positions. À la lumière de ces constatations, nous avons pu dans notre pratique déjà distinguer ce qui concerne l'encadrement des élèves et ce qui concerne la gestion de l'activité mathématique. Le fait de comprendre ce qui se passait a énormément facilité ces modifications. C'est un résultat important qui nous permet maintenant d'être plus précise dans nos demandes aux élèves, ce qui est profitable pour nous et pour les élèves.

Ayant réalisé le succès de nos efforts particulièrement ceux visant à amener l'élève à verbaliser ses démarches et à évaluer son travail, nous sommes encouragée à poursuivre dans ce sens et à rechercher de nouvelles solutions à d'autres points moins réussis comme le travail d'équipe par exemple.

Les hypothèses qui nous avaient conduites au départ à développer cette approche d'enseignement se rapportaient beaucoup à l'aspect affectif de l'apprentissage. Notre centre d'intérêt s'est déplacé, nous sommes maintenant beaucoup plus préoccupée par l'activité mathématique elle-même, les contenus, les activités de résolution de problème, le matériel didactique. En effet, l'activité mathématique se doit d'être intéressante et stimulante pour l'élève si l'on veut qu'il y prenne plaisir et qu'il l'attaque avec confiance.

Le fait d'avoir dégagé certains obstacles exogènes renforcera nos demandes auprès de l'administration scolaire au sujet d'horaires, de locaux et de regroupement des élèves par exemple, par ce que nous serons plus en mesure de les expliquer.

Soulignons finalement que cette prise de conscience n'aurait pas été complète sans l'étape de l'auto-analyse. La lecture du journal de bord a exposé certaines réactions comme la nécessité de plus encadrer les élèves. Mais la réflexion à travers l'action n'était pas suffisante et n'amenait pas une prise de conscience aussi complète. On peut lire dans le journal de bord des remarques par rapport aux problèmes vécus dans la classe et des idées pour tenter d'y rémédier mais, il n'y a pas d'analyse approfondie, faute de temps et de recul, ce qui fait qu'il n'y a pas de compréhension de la situation, on ne fait que la constater. L'analyse qui a suivi a eu tout autre résultat parce qu'elle a permis de voir ce qui se passait. Nous pensons particulièrement aux chevauchements parfois problématiques entre les conceptions qui ont été soulevés. C'est pourquoi ce qui est avancé dans notre troisième hypothèse est à compléter: la réflexion quotidienne est nécessaire mais il faut prendre un certain recul et faire un bilan pour arriver à une meilleure compréhension des phénomènes en jeu.

À la suite de cette expérience, nous pouvons dire qu'il est possible à un enseignant d'auto-analyser sa pratique. Le travail a parfois été difficile car il fallait constamment trouver des solutions aux problèmes méthodologiques qui se présentaient. Il fallait inventer et se réajuster. Nous avons pu à la suite de l'examen critique de notre démarche, suggérer des moyens que pourrait adopter un enseignant sans trop perturber sa pratique régulière et nous croyons que la démarche que nous proposons est considérablement simplifiée et tout aussi efficace. Nous avons conçu un inventaire de conceptions afin d'aider l'enseignant à établir une grille de conceptions qui lui permettra d'analyser ses actes d'enseignement à partir de l'enregistrements de ses cours. Nous proposons également une méthode simplifiée de compilation pour faciliter le travail d'analyse.

Il faut toutefois se garder d'attendre des résultats identiques chez toute personne qui s'engagerait dans une auto-analyse. Le terme l'indique, l'analyse est personnelle et les résultats seront surement en fonction du cheminement personnel de la personne qui l'entreprend. Toutefois le fait de s'engager dans une telle entreprise dénote une volonté de remise en question qui ne peut que se traduire par un avancement personnel.

### Implications

Bien que l'enseignant soit maître d'un grand nombre de choix didactiques, certains éléments sont hors de sa portée immédiate. L'administration et l'organisation scolaires devraient tenir compte des impacts de leurs décisions sur l'enseignant, l'élève, la classe et l'apprentissage du savoir. On devraient également apporter plus de soins aux questions qui touchent le regroupement des élèves. Sans viser nécessairement l'homogénéité des classes, il faut tenir compte de certains facteurs, notamment le nombre d'élèves dans la classe, le support didactique dont bénéficie l'enseignant.

Encore beaucoup de recherches doivent être menées en ce qui concerne les mathématiques de l'enseignement post-secondaire, particulièrement en ce qui est relatif au matériel didactique. Il y a peu de matériel disponible pour l'enseignant qui veut proposer à ses élèves des activités d'exploration ou des problèmes allant au delà de l'exercice de routine. Nous avons pu voir que le matériel didactique joue un rôle important dans les choix de l'enseignant, or, on ne peut exiger que chacun crée un matériel à sa mesure. Préalablement, l'étude d'un point de vue didactique des mathématiques enseignées après le secondaire est nécessaire et ensuite, il faudra faire faire appel aux enseignants pour expérimenter en classe des approches nouvelles et en examiner les résultats.

Cette recherche montre par ailleurs qu'il est possible d'innover en matière de recherche pour arriver à observer la classe de l'intérieur. Il faut de plus en plus s'assurer de la participation des enseignants à la recherche et profiter de ce point de vue différent. Les enseignants gagneront de leur côté une meilleure compréhension des phénomènes en jeu et seront plus disponibles pour expérimenter les modèles proposés par les didacticiens.

De plus, les résultats de l'auto-analyse portent à croire qu'il faut favoriser ce type de réflexion et la soutenir. Il faudrait poursuivre le présent travail et étudier les effets de l'auto-analyse chez d'autres enseignants. La nécessité de ce travail de réflexion pour amener une meilleur adéquation entre ses conceptions et sa pratique ne diminue en rien le besoin qu'ont les enseignants d'être plus informés sur les recherches spécifiquement en ce qui les touche de plus près, la didactique des mathématiques. Beaucoup de travaux sont menés sur les difficultés d'apprentissage des élèves, les causes d'échecs entre autres mais les enseignants ont peu de sources d'informations en ce qui concerne les approches, les présentations et les difficultés des contenus mathématiques qui sont enseignés au niveau post-secondaire.

En dernier lieu, on peut encore se demander où commence la boucle doit-on tenter de modifier les conceptions des enseignants pour finalement influencer leur pratique ou encore essayer de les inciter à modifier leur pratique pour susciter des évolutions dans leurs conceptions. Certains apports extérieurs peuvent influencer. Les informations sous forme de lecture, de présentations ou encore de formation peuvent agir sur les conceptions de l'enseignant et l'implantation de nouveaux outils, tel que l'ordinateur ou encore des manuels soutenant une approche innovatrice peuvent amener certaines modifications dans la pratique de l'enseignant. Cependant, le présent travail montre clairement les interactions entre les conceptions et la pratique. La clé qui selon nous peut intervenir dans cette interaction est la réflexion-sur-la-pratique qui suscite la confrontation entre les conceptions de l'enseignant et sa pratique. Nous nous devons de poursuivre les expérimentations en ce sens.

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# ANALYSIS OF STUDENTS' ERRORS AND DIFFICULTIES IN SOLVING COMBINATORIAL PROBLEMS

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## ABSTRACT

*The preliminary results of a systematic study of the difficulties and errors in solving a sample of combinatorial problems in two groups of pupils of secondary education are presented in this work. The analysis of the task variables of the problems constitutes a first approximation to the classification of the simple combinatorial problems and likewise enables the attribution of a content validity to the instrument developed, in order to assess the capacity to solve this kind of problems.*

## INTRODUCTION

In accordance with Piaget and Inhelder (1951), the development of the combinatorial capacity is one of the fundamental components of the formal thinking and can be related to the stages described in their theory: after the period of formal operations, the subject discovers systematic procedures of combinatorial construction, although in the case of the permutations, it is sometimes necessary to wait until they are 15 years of age.

However, more recent results, as Fischbein (1975) indicates, sustain that the combinatorial problem solving capacity is not reached in all cases, not even in the level of formal operations without specific instruction. Fischbein and Gazit (1988) study the relative difficulty of the combinatorial problems in terms of the type of combinatorial operation and nature and number of elements, in addition to the effect of the instruction on the combinatorial capacity. Other authors who in addition to those mentioned have investigated the difficulties of different types of combinatorial problems, are, Mendelson (1981), Green (1981), Lecoutre (1985) and Maury (1986).

In this work, we describe the results of a study of the effect on the relative difficulty of different combinatorial problems of several task variables of the same. Although the study carried out to date is limited, we consider it to be of interest to describe the classification carried out of the errors and the differences found in this distribution, between one group of pupils who have not had any previous instruction and another group that has. As an additional consequence we have a first version of a psychometric instrument available. This enables us to measure the "combinatorial reasoning capacity" of secondary students, and to diagnose the intuitions and types of error that should be taken into account in teaching. As Borassi (1987) affirms "errors can be used as a motivational device and as a starting point for the creative mathematical exploration, involving valuable problem solving and problem posing activities" (page 7.).

## DESCRIPTION OF THE PROBLEMS PROPOSED

The test consists of 9 problems some with several sections, in total 12 questions. As an example in Table 1, the statements of three of these problems are included that will serve to describe the different types of errors that the students have had during the solving process. The description of the characteristics of the problems, that are of two types: of enumeration and calculus, are presented schematically in Table 2.

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TABLE 1

1. Three boys are sent to the headmaster for stealing. They have to line up in a row outside the head's room and wait for their punishment. No one wants to be first of course!
(a) Suppose the boys are called Andres, Benito y Carlos (A, B, C for short). We want to write down all the possible orders in which they could line up.

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For example  A        B        C      we write ABC as shown below:
             |        |        |
              1st      2nd      3rd
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ANSWER: ABC / ____/____/____/____/____/____/____/____/____
Now write down all the other different orders.

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2. Calculate the number of different ways a class of 10 students can be divided up into two groups, one of them with 3 students and the other with 7.

ANSWER: There are      different ways.
Briefly explain the method you have used.

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3. An ice cream shop sells five different kinds of ice cream: chocolate, lemon, strawberry, apricot and vanilla. How many tubs of three different kinds of ice cream can be bought?

ANSWER: There are       different tubs.
Briefly explain the procedure that you have used.

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Inventory Problems:

We give the name problems of inventory to those problems like 1a) taken from Green's research (1981), in which the student is asked for an inventory of all the possible cases produced by a certain combinatorial operation, in this case, the permutations of three elements. These problems are ideal for our purpose of knowing the combinatorial capacity of the students before the instruction; on the other hand, in Navarro-Pelayo (1991) the little emphasis put on this type of exercise in the school books, has been pointed out.

Problems of calculation of the number of possibilities

In these statements, as in problems 2) and 3) the student is asked the number of possibilities without explicitly asking him for the inventory of the same, thus having to identify the combinatorial operation. This is one of the difficulties described by Hadar and Hadass (1981) to solve combinatorial problems.

Task variables considered

The task variables that have been taken into consideration for the choice of problems have been the following:

a) Type of combinatorial operation (permutations, combinations....). This variable has been one of the determining factors of the difficulty of the problems in Fischbein and Gazit's research (1988).

b) Context. Likewise, the previous authors distinguished the context of letters, numbers, people and objects; we have also included a problem in which undistinguishable objects are considered, since Lecoutre (1985) indicated the greater difficulty in employing these types of objects. Likewise, we have included a geometrical context, in item number 5.

c) Value given to the parameters m and n that have also been a factor of difficulty described by Fischbein and Gazit (1988).

d) Implicit mathematical model. According to what Dubois states (1984), the simple combinatorial configurations can be classified in three models: selections, that emphasize the model of sampling, distributions, related to the concept of mapping and partition or division of a set into subsets. We have considered these three models, in addition to that of simple ordering (arrangement) that can be considered as a particular case in any of them.

e) Help provided. To give an example or not in the statement, and in the case of giving it, whether a table or a tree diagram is used. This type of help was provided in items 1, 4, 5, 7 and 8.

The context, model, values of the parameters and combinatorial operations used in each one of the items appear in Table 1. The numeration of the problems does not correspond to the order of presentation in the questionnaire.

## RELATIVE DIFFICULTY OF THE PROBLEMS: EFFECT OF THE TASK VARIABLES

For this first pilot study we have preferred to choose an intentional sample that has been made up of a total of 106 pupils: 57 pupils from the 8th course of Primary Education "EGB" (14 years of age) who had not had any specific instruction in Combinatorics when the test was carried out and 49 pupils from the 1st course of Secondary education "BUP" (15 years of age) after the period of Combinatorics teaching. The percentages of the correct answers to each question of the two groups of pupils are presented in Table 2.

**TABLE 2**
**Percentage of correct answers in the different problems**

| *Problem* | *Context* | *Combinatorial Operation* | *Percentage correct answers* | |
|---|---|---|---|---|
| | | | *8th EGB* | *1st BUP* |
| 1a | Arrange people | $P_3$ | 89.5 | 95.9 |
| 1b | " | $P_4$ | 17.5 | 51.0 |
| 1c | " | $P_5$ | 7.0 | 46.9 |
| 2 | Partition (people) | $C_{10,3}$ | 0.0 | 6.1 |
| 3 | Select objects | $C_{5,3}$ | 8.8 | 20.4 |
| 4 | Throw coins | $VR_{2,2}$ | 56.1 | 67.3 |
| 5 | Select paths | Product Rule | 43.9 | 26.5 |
| 6 | Select people | $V_{4,3}$ | 1.8 | 49.0 |
| 7 | Distribute objects | $V_{4,2}$ | 38.6 | 67.3 |
| 8 | Select numbers | $V_{5,2}$ | 38.6 | 51.0 |
| 9a | Arrange letters | $P_5$ | 7.0 | 53.1 |
| 9b | Arrange letters | $PR_{5,1,1,1,2}$ | 0 | 12.2 |

We can observe that in practically all the items the percentages of correct answers are higher in pupils of "BUP". There is an exception in item 5, corresponding to the rule of the product, a type of problem that in our study (Navarro- Pelayo (1991)) we saw was little used in the text books. In this case one of the formulas corresponding to the combinatorial operations cannot be directly applied because it is dealing with the cartesian product of the two different subsets. The pupils of 14 (8th "EGB") who, in their majority have tried to solve the problem directly using an inventory, have obtained better results than the pupils of "BUP" who have tried to apply one of the formulas known for this case.

By considering the magnitude of the parameters we can clearly see the difference in difficulty when the value of the number of elements to be selected is 2 or 3. In all these cases there has been an important percentage of correct answers, even in pupils who have had no instruction, from which a good combinatorial capacity can be deduced when the number of objects is small. When this number grows the pupil of "EGB" has not been capable of satisfactorily completing a procedure and as he did not know the formula, has been unable to deduce in many cases. It is here where we can appreciate a greater effect of the instruction and the age.

Another item where there have been a significant number of correct answers, in spite of using value 4 for the parameter m, has been item 7 where a tree diagram was given as help. This agrees with the importance that Fischbein and Gazit (1988) give to the tree diagram as a model to solve the combinatorial problems. In general, providing the pupil with an example, has supposed a greater facility of the problem, especially in item 1a) where the percentage of correct answers has been surprising in the pupils of "EGB", taking into account, that Piaget's and Fischbein's theories point out the permutations as the most difficult combinatorial operation known before instruction. However, this percentage drops drastically when we pass to the permutations of 4 and 5 people, and it even drops (although not as drastically) in the pupils who have received instruction. The pupils of the first group lack the recursive capacity to form the permutations of 4 elements once those of 3 have been formed.

The difference of difficulty due to the type of combinatorial operation does not seem as big in our study as that due to the size of the parameters, since before instruction this has been the main determinant of success and after there is not a very clear difference.

By considering the mathematical model under which the combinatorial operation is presented, we do not observe important differences in the model of selection (items 3, 4, 5, 6 and 8), arrangement (items 1a, 1b, 1c, 9a and 9b) and the positioning or application (item 7), except in the case of the permutations with repetition that have turned out to be much more difficult. In this case, the main determinant of the difficulty is the fact that distinguishable objects appear mixed with indistinguishable ones. However, we have found quite an accused difference in item 2 referring to a context of partition of a set into two subsets in which only 14% of correct answers have been found after instruction, in spite of being a typical combinatorial statement.

## TYPES OF ERRORS IDENTIFIED

### Error of order

This mistake, described by Fischbein and Gazit (1988) consists of confusing the criteria of combinations and arrangements. For example, in item 3, when the pupil considers different tubs "chocolate with lemon and strawberry" and "chocolate with strawberry and lemon". This mistake has been found in 35 of the total of the problems solved by the children in "BUP", representing 16.5% of the total errors in this group and only in 8 of the pupils of "EGB" (2.6% of errors). From this result a greater relative incidence of this type of error can be induced in the pupils who try to apply one of the combinatorial formulas, which only occurs in the group who have received instruction.

### Error of repetition and exclusion

We have given this name to the case of the pupil who does not consider the possibility of repeating the elements or when there is no possibility to do so, the pupil uses it. For example, in item 1, when the pupil uses the formula of variations with repetition or repeats an element within the permutation.

In the case of item 9b), that deals with the permutations of 5 letters, two of them being the same: A, B, C, D, D, another option followed by some pupils is to exclude the repeated letter and form the permutations of the remaining ones, thus taking $PR_{5,1,1,1,2} = P_4$. We have called this mistake exclusion error. Letter D is considered to be fixed and its permutation with the remaining ones is not considered. This error has been committed mainly by the pupils of "EGB" (17 cases; 5.6% of their errors) and acquires special importance since it has only been given when associated to a particular problem, and thus seems typical of this type of problem.

In total there have been 53 errors of repetition in "BUP", which represents 25% of the total errors as opposed to 8 in EGB (2.6% errors), due to the fact that the first group prefer the use of formulas. We must also point out the greater importance of this error as opposed to the error of order, in the group of pupils with instruction.

### Non systematic enumeration

This type of error described by Fischbein and Gazit, consists of trying to solve the problem by an enumeration using trial and error without a recursive procedure that leads to the formations of all the possibilities. It has been one of the most frequent mistake in both groups, 24 cases in "BUP" (11.3% of errors), 96 (31.6%) in "EGB". This error has occurred specially before instruction, since the students have used the enumeration as the most frequent strategy in solving the problems.

We must point out that in our work (Navarro-Pelayo, 1991) we showed that the enumeration exercises are not usually proposed to the pupils since it is considered this is an ability that they have already acquired. However, we think that the results of this first sample confirm those of other authors like Mendelsohn (1984) that many pupils, although in the stage of formal operations, have difficulties with systematic enumeration. We have even seen that these difficulties continue in some of the students after their period of instruction in the first course of Secondary education "BUP".

## Error in the arithmetic operations used

The pupils of "BUP" have studied combinatorics and in some cases have identified the operation correctly, using the formula to solve the problem. On other occasions this operation has not been identified - or at least it has not been indicated explicitly - and they try to deduce the series of arithmetic operations necessary for the solution using a direct combinatorial reasoning. That is they try to find a formula, not valid for the general case, but that can be used in the given problem. This strategy is also quite frequent in the pupils of 8th of "EGB". In the case that a correct formula has not been found with this procedure we will say that there is an error in the arithmetical operations. This type of mistake appears in a total of 20 problems solved by the pupils of "BUP" (9.4% of the errors) and 56 for those of "EGB" (18.8% of the errors).

## Mistaken intuitive response

This error identified by Fischbein (1975), consists of not justifying the response, only giving a mistaken numerical solution. The frequency of this type of error has been 82 cases: 27% of the errors in "EGB" as opposed to 17.9% in "BUP"; this type of response is still important in "BUP".

## Other errors

- Badly applied formula, due to not remembering it, although the combinatorial operation has been correctly identified: 11 cases in secondary school pupils.

- Confusion of the parameters when applying the formula: 5 cases in secondary school pupils.

- Not remembering a property of the combinatorial numbers.

This error has appeared as associated to item 2, in which the pupil should realize that by considering 10 students, the same number of groups can be formed with 3 as with 7, so once one of these groups is formed the other one is determined. The pupils who do not identify this property add C10,3 + C10.7 to give the solution.

- Incorrect interpretation of the tree diagram (6 pupils of "BUP" and 9 of "EGB"). In spite of the importance given to this didactic resource by Fischbein as an aid in combinatorial problem solving, we have found ourselves with cases of bad interpretation of the diagram given in exercise 7, even in pupils who have been instructed in the use of this resource.

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# PROCESSES AND STRATEGIES OF THIRD AND FOURTH GRADERS TACKLING A REAL WORLD PROBLEM AMENABLE TO DIVISION

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*A random sample of 55 grade 3 and 4 children from six schools were observed while tackling five versions of a real world problem based on quotition division. The children were provided with simulated bottles of medicine (in tablet and liquid form), which showed the total contents and the amount to be taken each day, and were asked how many days the medicine would last. Calculators and concrete materials were provided, as well as pencil and paper. For all but the two most difficult questions, children overwhelmingly chose mental computation as their calculating device. Children predominantly used repeated addition (or subtraction) rather than division, which was almost always only used in conjunction with a calculator. Difficulties encountered by the children who used calculators confirm the mathematical sophistication required to interpret the answers thus obtained.*

## Introduction

There is now a substantial body of research into children's understanding of multiplication and division. Among major factors found to influence children's success in selecting the appropriate operations for word problems requiring division are the extent of familiarity of the context and the structural nature of the problem, with partition problems producing a higher rate of success than quotition or rate problems (Bell, Fischbein, & Greer, 1984; Fischbein, Deri, Nello & Marino, 1985). Prior to a study involving grade 5, 7 and 9 children, Fischbein *et al* hypothesised that children have intuitive models of division, based on both partition and quotition, which they can evoke as appropriate. Not only did partition problems prove easier than quotition, but grade 5 children performed considerably worse on quotition questions than older children. For example, for the question "The walls of a bathroom are 280 cm high. How many rows of tile are needed to cover the walls if each row is 20 cm wide?", although 44% of grade 5's correctly chose 280 ÷ 20 as the operation required, 41% chose 280 x 20. The success rate for grades 7 and 9 were 77% and 80%, respectively. This led the authors to conjecture that partition is the only intuitive primitive model, with children acquiring the quotition model with instruction.

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Kouba (1989) analysed the solution strategies of grade 1 to 3 children. She proposed three intuitive models for partition - sharing by dealing, repeated subtraction and repeated addition (using guesses for the addend). It is well known that children frequently resort to informal addition based strategies for a variety of problems (Hart, 1981, p.47; Bergeron & Herscovics, 1990, p. 32). Kouba found that children employed repeated subtraction and repeated addition for both partition and quotition problems, and hence questioned the separation of the intuitive models for these types of division.

Procedural knowledge without conceptual knowledge and the ability to use it in meaningful situations is of little use. This is particularly true in an age when reliable mental methods and an ability to use calculators (together with an understanding of the meaning of the operations and the real world problems which they model) are sufficient for all practical purposes (Hart, 1981, p.47; Bell, Fischbein, & Greer, 1984, p.130; Bergeron & Herscovics, 1990, p. 34). Yet many children are being taught to do calculations without being able to describe situations in which they are applicable and consequently do not find "real world" possibilities reflected in school mathematics (Greer and McCann, 1991, p. 85; Graeber and Tirosh, 1990, p.583).

Carpenter (1986) points out that "before receiving instruction, most young children invent informal modelling and counting strategies to solve basic addition and subtraction problems" (p. 114). Neuman (1991) reports on children's "original" informally developed conceptions of division, commenting that "young children who have not been formally taught division seem to believe that it is possible to solve all problems in some way" (p.76). Children were again found to use repeated addition and repeated subtraction for both partition and quotition problems. She questions the early introduction of the division algorithm as opposed to the elaboration of children's own informally developed thoughts.

Results obtained from a large sample of grade 5 and 6 children, using a pencil and paper test of problem solving (Stacey, Groves, Bourke & Doig, in press), indicate that most upper primary children do not use learned multiplication and division skills, with large numbers of these children still using repeated addition to solve a problem based on quotition (Stacey, 1987, p. 21).

While there is no centralised curriculum in Victoria (Australia), most primary schools base their mathematics policy on the state guidelines (Ministry of Education, Victoria, 1988). At grade 3, children are expected to learn number facts including division by 2, 3, 4, 5 and 10, as well as use calculators for computation, while at grade 4 they are usually introduced to division of 2 and 3 digit numbers by 1 digit numbers (p.97-8). Pencil and paper "long division" has not been included in guidelines for over 6 years.

This paper reports on the extent to which 55 grade 3 and 4 children, who were observed while tackling a "real world" problem based on quotition division, as part of a longer interview, were able to find correct or reasonable answers, the calculating devices they chose to use and the extent to which they made sensible and efficient use of calculators.

## *Method*

For the past three years, as part of the *Victoria College Calculator Project* and the University of Melbourne *Calculator-Aware Program for the Teaching of Number*, children entering six schools have been given "their own" calculator to use at all times. Teachers have been provided with a program of professional support to assist them in using calculators, not just as "number crunchers", but also as a means to create a rich mathematical environment for children to explore (see, for example, Groves, 1991; Groves, Cheeseman, Clarke, & Hawkes, 1991, Welsh, Rasmussen & Stacey, 1990).

In 1991, as part of an investigation of the long-term learning outcomes of the projects, over 430 grade 3 and 4 children at these six schools were given a written test and a test of calculator use. These children, who have not been involved in the calculator projects, form the control group for the study. In addition to the tests, a random sample of 55 of the grade 3 and 4 children were given a 25 minute interview, designed to test their understanding of the number system; their choice of calculating device, for a wide range of numerical questions; and their ability to solve "real world" problems amenable to multiplication and division, with or without calculators. Throughout the interviews, children were free to use whatever calculating devices they chose. Unifix cubes and multi-base arithmetic (MAB) blocks were provided as well as pencil and paper and calculators. Many of the questions were expected to be answered mentally. The tests and interviews will be carried out again at grade 3 and 4 level in 1992 and 1993. Among the hypotheses for the long-term study is an expectation that children involved in the calculator projects will perform better on the "real world" problems, selecting appropriate processes more frequently and making better use of calculators.

This paper focuses on interview results from the "real world" problem amenable to division - a simplified version of the question from the problem solving test referred to earlier (Stacey, Groves, Bourke & Doig, in press). The question consists of five parts, each with the same structure. In the first two parts, children were presented with clear bottles containing the appropriate number of white, medicine-like tablets (actually sweets). The bottles were attractively labelled with the contents and the amount to be taken each day - for example, in M1 the label clearly displayed "15 tablets take 3 each day", as well as the distractor "$7.43". For the remaining three parts, accurate volumes of coloured liquid were used with information such as "120 ml take 20 ml each day" and a price. For this example (the first using liquid "medicine"), 20 ml was poured from the bottle into a clear medicine measure. In each case, children were asked how many days the medicine would last. (For further details, see Welsh, 1991.)

As well as their answers, children's choice of calculating device were recorded. These were classified as calculator, written algorithm, Unifix or MAB, mental (which was further sub-classified to indicate automatic response and use of fingers) and other (such as drawing or the use of non-standard algorithms). Wherever possible, the mathematical processes used were also recorded. Original classifications of the processes included division, counting on (repeated addition), repeated subtraction, multiplication and other less frequently used processes.

### *Results*

Frequencies of correct and incorrect answers, use of calculating devices and solution processes for each of parts M1 to M5 of the medicine question are shown on the "double-sided" Table 1. The left side shows choice of calculating device against correctness of answer, while the right side shows solution processes. In those parts of the question where remainders occur (M2, M4 and M5), an extra category of answer is included to indicate answers which, while incorrect, give the correct number of whole days (or, in the case when the answer is 7.5 days, give 8 days). The categories for choice of calculating device and solution processes have been collapsed. Categories rarely used are included under "other" - e.g. standard written algorithms (which were never used successfully) and the (rare) attempts to use an incorrect algorithm (such as a single subtraction).

***Correctness of answers.*** Table 1 shows the dramatic decrease in correct answers when remainders are involved. Even for the relatively easy problem of "21 tablets, 4 per day", less than 45% of children give a correct answer such as "5 $^{1}/_{4}$ days" or "5 days with 1 tablet left over", although a

**Table 1: Frequencies of correct and incorrect answers, use of calculating uevices and solution processes**

| Question | Device[1] | M | C | O | NA | Total | AR | CO | DI | UM | O | NA | Process[2] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| M1 15 tablets, 3 per day How many days? | √[3] | 38 | 2 | 6 | 0 | 46 | 3 | 17 | 2 | 17 | 7 | 0 | √[3] |
| | X[3] | 7 | 0 | 1 | 0 | 8 | 0 | 1 | 0 | 6 | 1 | 0 | X[3] |
| | NA[3] | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | NA[3] |
| | Total | 45 | 2 | 7 | 1 | 55 | 3 | 18 | 2 | 23 | 8 | 1 | Total |
| M2 21 tablets, 4 per day How many days? | √ | 17 | 4 | 3 | 0 | 24 | 3 | 12 | 4 | 4 | 1 | 0 | √ |
| | •[4] | 5 | 3 | 3 | 0 | 11 | 0 | 3 | 2 | 2 | 4 | 0 | •[4] |
| | X | 10 | 0 | 3 | 0 | 13 | 0 | 4 | 0 | 6 | 3 | 0 | X |
| | NA | 0 | 0 | 0 | 7 | 7 | 0 | 0 | 0 | 0 | 0 | 7 | NA |
| | Total | 32 | 7 | 9 | 7 | 55 | 3 | 19 | 6 | 12 | 8 | 7 | Total |
| M3 120 ml, 20 ml /day How many days? | √ | 26 | 5 | 1 | 0 | 32 | 5 | 13 | 6 | 7 | 1 | 0 | √ |
| | X | 11 | 2 | 3 | 0 | 16 | 0 | 2 | 0 | 9 | 5 | 0 | X |
| | NA | 0 | 0 | 0 | 7 | 7 | 0 | 0 | 0 | 0 | 0 | 7 | NA |
| | Total | 37 | 7 | 4 | 7 | 55 | 5 | 15 | 6 | 16 | 6 | 7 | Total |
| M4 300 ml, 40 ml /day How many days? | √ | 6 | 8 | 1 | 0 | 15 | 0 | 6 | 7 | 2 | 0 | 0 | √ |
| | •[5] | 5 | 5 | 0 | 0 | 10 | 0 | 5 | 5 | 0 | 0 | 0 | •[5] |
| | X | 18 | 2 | 4 | 0 | 24 | 0 | 5 | 1 | 11 | 7 | 0 | X |
| | NA | 0 | 0 | 0 | 6 | 6 | 0 | 0 | 0 | 0 | 0 | 6 | NA |
| | Total | 29 | 15 | 5 | 6 | 55 | 0 | 16 | 13 | 13 | 7 | 6 | Total |
| M5 375 ml, 24 ml /day How many days? | √ | 0 | 13 | 0 | 0 | 13 | 0 | 0 | 13 | 0 | 0 | 0 | √ |
| | •[6] | 1 | 7 | 0 | 0 | 8 | 0 | 1 | 7 | 0 | 0 | 0 | •[6] |
| | X | 7 | 10 | 8 | 0 | 25 | 0 | 5 | 4 | 7 | 9 | 0 | X |
| | NA | 0 | 0 | 0 | 9 | 9 | 0 | 0 | 0 | 0 | 0 | 9 | NA |
| | Total | 8 | 30 | 8 | 9 | 55 | 0 | 6 | 24 | 7 | 9 | 9 | Total |

[1] M - mental; C - calculator; O - other (e.g. drawing, blocks); NA - no answer given

[2] AR - automatic response; CO - counting on/back (repeated addition/subtraction); DI - division; UM - unknown mental process; O - other (e.g. multiplication, single subtraction)

[3] √ - correct answer; X - incorrect answer; NA - no answer given

[4] • - incorrect answer with integer part correct (e.g. 5, 5+, 5 remainder 3, 5 remainder 25)

[5] • - incorrect answer with integer part correct (e.g. 7, 7+, 7 remainder 80, 7 remainder 2) or 8

[6] • - incorrect answer with integer part correct (e.g. 15, 15+, 15 remainder ?, 15 $^3/_4$)

further 20% give an answer involving 5 days. Incorrect answers for M2 range from 2 to 15 days, with 5 children giving answers of over 10 days. Given the provision of concrete models and the familiarity of the situation, these results confirm that many children find it difficult to relate school mathematics to real world problems. While the liquid medicine problem involving a whole number answer, M3, produced a high rate of success, it was anticipated that M4 and M5 would be much more difficult, as children would be unlikely to have the skills to find a solution except by using a calculator. (In fact, these parts were included specifically to determine the extent to which children can successfully use calculators to solve such problems.) The results from these parts confirm our expectations, with over half of the children who achieve a correct or reasonable answer for M4 using a calculator, and only one child succeeding without a calculator for the more difficult M5.

***Use of calculating devices.*** For all but M5, children showed an overwhelming preference for mental computation. Even for M4, where we had expected children to use calculators, 29 of the 49 children who attempted the problem chose to do it mentally. For all parts, approximately half of the children who used mental computation augmented it with the use of fingers. Only three children gave responses automatically to any questions - including one child who gave immediate correct answers to the first four parts and then used his calculator to incorrectly read the answer to M5 as "15 remainder point 625". Four children successfully used drawings or concrete materials for some or all of M1 to M3, but only one of these was successful in either of the other two parts - M4 using a calculator.

***Processes used.*** Only a small handful of children were observed giving "automatic responses" to the first three parts. For M1, a large number of children gave the correct answer after pausing to calculate mentally. It was often impossible to determine the mental processes used as time constraints did not allow for extended probing - hence the classification "unknown mental process". For the first three parts, counting on or counting back (repeated addition or subtraction) outnumbered all other known processes almost two to one. Children only began to use division when the numbers dictated the use of a calculator. In fact for M5 the only correct answers were obtained using division on a calculator.

***Effective use of calculators.*** While the first requirement for effective calculator use in problems such as these is to recognise the operation as division, it is also necessary to be able to make sense of the answer displayed. The difficulties were particularly apparent in M5, where 30 of the 46 children who attempted the problem used a calculator, but only 13 found the correct answer. Of the remaining 16 children, 9 were unable to read the number displayed correctly. Such difficulties are further highlighted by an earlier question on the interview. Children were shown 278 ÷ 39 and "the answer found by someone using a calculator" - i.e. 7·1282051. They were asked firstly to read the number and then to say "about how big" it is or give a "number close to it". Only 14 children were able to read the number correctly (i.e. say the words "seven point one two ..."), with 16 passing the question and the remaining 25 giving answers like "7 point 12 million ...". In response to the size of the number, 15 answered in the range 5 to 9, with 20 passing and 18 giving very large answers (e.g. 7 million). While this level of understanding is to be expected, it highlights the fact that calculator use will only increase children's facility with division if it is accompanied by considerable change in children's mathematical sophistication and overall number sense.

## *Conclusion*

These results confirm the fact that children are able to devise their own means of solving problems based on quotition - provided the numbers are not too difficult to handle. Their methods are predominantly based on repeated addition or subtraction. Wherever possible, children used mental computation in preference to calculators, with almost no attempt to use pencil and paper, except to draw diagrams. Nevertheless, the fact that several children consistently used completely inappropriate operations, such as a single subtraction, to arrive at blatantly incorrect answers such as 351, days, indicates the extent to which school mathematics is seen as completely divorced from the real world. Bell *et al* had predicted that calculators would allow a wider range of numbers to be considered earlier in primary school, but warned that "this still leaves the question of what meanings the pupils can attach to the operations" (Bell, Fischbein, & Greer, 1984, p. 130). The results obtained here confirm not only the importance of attaching meaning when using calculators, but also the necessity to develop skills such as estimation and approximation and a strong intuitive understanding of aspects of the number system such as decimals. Future results to be obtained from children with long-term experience of calculators will hopefully demonstrate the extent to which this can realistically be achieved.

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# PICTURES IN AN EXHIBITION: SNAPSHOTS OF A TEACHER IN THE PROCESS OF CHANGE

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*As teachers change their pedagogical practices to reflect current research on teaching and learning, the mathematics education research community has a unique opportunity to study this process of change. This paper will present results from one teacher, Margaret, in one project, the Atlanta Math Project (AMP), for one year, 1990-1991, as she attempts to modify her instructional environment to reflect current recommendations for reform. This pilot work lays the groundwork for future research on teacher change in AMP.*

Learning environments are emerging that are quite different from the lecture dominated mathematics classroom that many teachers and students have experienced. The Atlanta Math Project (AMP), a four-year National Science Foundation sponsored project in the second year of operation, is implementing a research-based teacher education model which assists teachers in constructing new knowledge about the teaching and learning of mathematics. AMP is studying how these teachers change their instructional practices over four years.

## Theoretical Orientation

The theoretical orientation of the Atlanta Math Project is grounded in the theories of constructivism and social constructivism (von Glasersfeld, in press; Wertsch & Toma, 1991) and of metacognition (Flavell, 1975). A more thorough discussion of the theoretical perspective of AMP and the framework for studying teacher change can be found elsewhere (Hart, 1991).

## Studying Teacher Change

This paper will explore aspects of change in the learning environment and in teacher knowledge for one teacher, Margaret, now in her second year with AMP. In particular we will discuss the following questions about change in classroom discourse and beliefs:

Whose ideas are being explored in Margaret's classroom?

What types of questions are being asked?

How is conflict resolved?

How is student thinking encouraged?

Is mathematical thinking modeled?

Who are students talking to?

How do Margaret's beliefs about learning mathematics and teaching mathematics change in year one?

How do Margaret's beliefs about mathematical tasks and content change?

## Methods

The data chosen for this report are two videotapes of Margaret teaching her grade 6 class in September and May of her first year with the project and responses to a project instrument completed by Margaret before and after year one. A research team composed of the two project directors, the assistant project director, a research associate and three graduate students have met regularly during year two to analyze and discuss the process for studying change. We have used Margaret as a first attempt to refine our methods. We will use this paper as an opportunity to share our struggles and achievements and to solicit feedback on our work.

## Margaret

Margaret is a sixth grade teacher in a rural middle school near Atlanta, Georgia. Identified by the mathematics coordinator for her school system as a strong teacher with leadership ability, she attended five days of staff development with AMP during the summer of 1990. She was introduced to the theories of constructivism and metacognition, the positions on reform set forth by the National Council of Teachers of Mathematics, and she experienced planning, teaching and reflecting from these perspectives.

Margaret is fairly new to teaching. Her first year with AMP (1990/91) was her third year of teaching. A mature woman with a family of her own, she turned to teaching later in her life. Margaret showed a great deal of maturity and intuitiveness during the summer workshops. She demonstrated caution in accepting without question the ideas we explored, but displayed a willingness to learn and try new approaches.

Snapshot one: July 31, 1990

At the beginning of the AMP summer workshop, Margaret was asked to respond to a set of statements designed to elicit her current beliefs. She was asked to describe a good mathematics teacher, a good mathematics supervisor, a good mathematics student, and a good math problem. Finally she was asked, how do middle school children learn? Her responses provided an opportunity to gain insight into some of her professed beliefs about teaching mathematics, about learning mathematics, and about worthwhile mathematical tasks.

Beliefs about teaching mathematics. Margaret stated that a teacher should be flexible in her/his thinking, creative, open, and display an enjoyment of mathematics. She felt having a background strong in content and knowing a variety of instructional strategies were important. She said teaching should be organized and relevant. Teachers should demonstrate respect for students and their ideas and should exhibit joy and interest for mathematics to students.

Beliefs about learning mathematics. Margaret gave a description of the learner which included inquiry, thoughtful, creativity, and enjoyment for mathematics. She said a "good" learner has a recognition for the relevance of mathematics. She said learners need to engage in both individual and group problem solving that relate to common, everyday experiences. Learning occurs through listening and discussing mathematics with others and when students reflect and organize their knowledge and use their knowledge about math in different ways.

Beliefs about worthwhile mathematical tasks. Margaret felt tasks should be relevant and require students to inquire. Tasks require discussion. Tasks should provoke thought, require reflection and synthesis of math knowledge, and tasks should cause students to think about and solve problems in more than one way.

Snapshot two: September 20, 1990

The first videotape of Margaret is of her teaching a lesson on estimation. The students are sitting in double-wide desks facing the front of the room in rows and columns. The class discusses the problem with Margaret at the overhead and students responding to questions by raising their hands and being called on one at a time.

The nature of the mathematical task. Margaret uses an experience of renting lockers the children had engaged in the day before. She poses the following questions, "Yesterday, about 700 students rented lockers. The lockers cost $2 each. How did that make you feel?" The students expressed strong feelings (e.g., it was too crowded, it was confusing, etc.) and this dialogue opened the floor for further discussion. Margaret continued, "How much money did the school take in?" Since lockers could be shared and exact numbers of students had not been determined, there were numerous opportunities for estimation.

The nature of classroom discourse. The direction of communication was always the same. Margaret would ask a question, a student would respond to the question, and Margaret would respond to the student. If a student disagreed with a statement made by another student the opposition was directed toward Margaret--not the other student.

Margaret's questions were a mixture of one-right-answer questions and more probing questions such as "Why?" or "Anything else?" Thought provoking questions were also raised by students. Very little time was given to think about questions. Margaret did not explore any of these questions in depth, but accepted partial thoughts and did not require justification, elaboration or explanation. The students did not talk to one another and were not asked to listen to the responses of others. Although presented, alternative ideas were not explored and as a result conflict was not explored or resolved. Some students were called upon more frequently than others. Not all members of the class were included in the discussion.

Snapshot three: May 3, 1991

Margaret's classroom had changed during the school year. The double-wide desks were now arranged with two desks facing each other forming small seating groups of four. The overhead remained at the front of the room.

The nature of the mathematical task. The tasks presented in this second lesson explored division of fractions. The children were first asked to contrast the division problem, 1/2 divided by 1/4, written with a division symbol and as a complex fraction. This was followed by four word problems that involved dividing with fractions. They were instructed to solve these by writing mathematical sentences. Finally the students were asked to count the number of "halves" and "fourths" in two inches, three inches, five and one-half inches, etc., to determine the pattern and "discover" the algorithm for division of fractions.

The nature of classroom discourse. Margaret used a cooperative learning technique of think-pair-share to begin each phase of the lesson. Students were asked to think about the problem(s), to share their thinking with a partner, and then to share their thinking with the class. Margaret's questioning was still a mixture of one-right-answer and more thought-provoking questions. Students still directed their responses and questions to her, but Margaret had begun to act more as a facilitator of the discussion. Consider the following dialogue about contrasting the two division problems.

| | |
|---|---|
| Brian | If you reverse the order of the numbers you get the same thing. |
| MARGARET | (repeats his comment) |
| Brian | yeah |
| Matt | No you won't |
| MARGARET | You don't agree Matt? |
| Matt | No you won't |
| MARGARET | Brian says if you reverse the order you get the same thing, Matt says no you won't. What do you think? |

Margaret did not, however, explore conflict in depth. As soon as a third student, Jennifer, suggested that you could not reverse whole number division, Margaret seemed satisfied that the argument was settled. Brian was not convinced and suggested fractions might be different. A comment was then made by John that you can reverse addition and multiplication, but not subtraction and division. Margaret disregarded these conflicting positions and simply turned to Brian and said "These properties [John just mentioned] knock this out." That was the end of the discussion.

### Snapshot four: June 15, 1991

At the beginning of the second summer staff development, Margaret was asked to respond to the instrument described in Snapshot One. Following is a summary.

Beliefs about teaching mathematics. Margaret stated that a good teacher is flexible in approach and content, open to new ideas, methods and challenges, creative, a good planner, and efficient user of time. They are well versed in current teaching strategies and materials and seek new avenues for exploring their own knowledge of teaching strategies and content. A good teacher is responsive to the needs of students by providing feedback. Good teaching requires being skilled at diagnosing student abilities and levels of mathematical knowledge.

*Beliefs about learning mathematics*. Margaret described the learner as being confident in ability, not afraid to fail, able to see relationships, prepared for class, open to new ideas from others, and motivated by questions or problems. She felt that students learn by doing mathematics, solving problems, listening to others, talking about mathematics. They need to find the mathematics relevant to their lives and investigate situations requiring mathematics.

*Beliefs about worthwhile mathematical tasks*. Good problems require creativity and may stimulate an extension in thinking. They should have a variety of strategies possible for finding the solution. They may have more than a simple solution and may prompt connections to other problems and/or life situations.

## Discussion

An analysis of the four snapshots of Margaret reveal many consistencies in her behaviors and her reflections. While the consistencies are of interest and necessary as we attempt to interpret the data, e.g., her consistent reference to mathematics needing to be relevant to real life, the brief space allowed here will only permit some discussion of change.

The beliefs Margaret expressed in Snapshot One, e.g, teachers need to respect student ideas, students need to work in groups, tasks need to provide opportunities for reflection and synthesis by the student, were not consistent with the classroom discourse observed in Snapshot Two. On the contrary, Margaret had the students working alone. She listened courteously, but did not explore student thinking. And, while the task certainly provided the opportunites described, they were not pursued. Margaret appeared to be focusing primarily on the lesson and on her teaching behaviors.

What is of interest, then, is the careful analysis of Snapshots Three and Four at the end of the year. It is here we notice more careful attention to student thinking and student organization. Margaret is displaying more of the characteristics she described at the beginning of the year. She is respecting students ideas and is open to their thinking. Students are more frequently working in groups. They are listening and discussing ideas, albeit they are still passing through the teacher.

This careful analysis of Margaret has confirmed informal observations the research team has made. Initially teachers acquire knowledge about alternative ways of teaching. As they put this knowledge into practice they focus on themselves and their behaviors. Overtime, they are more able to direct their attention to the student and student thinking. They begin to consider alternative solution paths. The quality of classroom discourse improves.

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# "CANCELLATION WITHIN-THE-EQUATION" AS A SOLUTION PROCEDURE

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Solving a first degree equation in which the unknown appears on both sides of the equal sign by formal methods involves two major cognitive obstacles. The first one is the students' inability to operate spontaneously on or with the unknown; the second one, perhaps even more complex, is the students' difficulty with operating on an equation as a mathematical object. The objective of the teaching experiment reported here was to overcome the second obstacle by a procedure based on decomposition of a term into a sum or a difference of terms (e.g. 5n + 41 = [illegible] + 5 ; 5n + 41 = 3n + 5n + 5) followed by cancellation of identical terms on both sides of the equal sign. While this procedure was adequate when a term was replaced by a sum, major obstacles were found in the case of decomposing a term into a difference.

In a previous paper (Herscovics & Linchevski, 1991(a)) we have tried to trace the upper limits of solution processes used by seventh graders prior to any formal instruction in algebra. Our investigation has shown that the students were able to solve successfully most of the first degree equations in one unknown. However the solution methods they used clearly showed that the notion of the didactic cut is valid (Filloy & Rojano, 1984).

The solution procedures which the students used were based exclusively on operating with the numerical terms , therefore when given equations in which the unknown appeared twice on one side of the equation or on both sides ($ax + bx = c$ ; $ax + b = cx$) the preferred mode of solution was that of systematic substitution. Therefore we proposed viewing the didactic cut in terms of cognitive obstacle (Herscovics, 1989) and defining it as the students' inability to operate spontaneously with or on the unknown.

## Preliminary considerations.

Viewing the didactic cut as a cognitive obstacle led us to consider various ways to overcome it. While students can develop meanings for an equation and for the unknown simply in terms of numerical relationship, this does not extend to operating with or on the unknown. Such operations have to be endowed with specific meanings of their own. This is what is achieved when the classical balance model is introduced to represent an equation. One can add or take away specific numerical quantities as well as quantities involving the unknown. Another model based on the equivalence of rectangular areas has been proposed by Filloy and Rojano (1989). The authors have pointed out that all physical models contain inevitable intrinsic restrictions regarding their applicability to various types of numerical operations on the unknown (Filloy & Rojano, 1989). One cannot represent $5n - 3 = 27$ on the balance because of the subtraction. The area model representation of $7n - 12 = 9n - 24$ becomes quite sophisticated.

While obviously lacking the relevance of physical models, numerical models do not have such restrictions (Herscovics & Kieran, 1980). In a teaching experiment based on the use of arithmetic identities, Kieran (1988) found that students who tended to focus on inverse operations had difficulties in accepting the notion of an equivalent equation obtained by operating on both sides of the initial equation. Perhaps these difficulties are even more

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complex than those identified with the didactic cut, as operating on an equation implies keeping track of the entire numerical relationship expressed by the equation while it is being subjected to a transformation. These considerations led us to design an individualized teaching experiment which would give us the opportunity to study the cognitive potential of an alternative approach (Herscovics & Linchevski, 1991 (b)). We prepared a sequence of three lessons, each lesson was semi-standardized. The lessons, as well as the pre-test and the post-test, were videotaped, and an observer with a detailed outline recorded all students responses. We chose six seventh graders, as described in the introduction, from three levels of mathematical ability: Andrew and Daniel were the top students, Andrea and Robyn average, and Joel and Audrey were weakest. The first lesson was aimed at overcoming the students' inability to spontaneously group terms involving the unknown on the same side of the equal sign. In the paper "Crossing the didactic cut in algebra : grouping like terms" (Herscovics & Linchevski, 1991 (b)) we gave a detailed description of this lesson. The teaching intervention was based on the students natural tendency, which had been found in our previous investigation, to group terms involving the unknown without any coefficient ($n + n = 76$, $n + 5 + n = 55$). We assumed that this tendency can be exploited by increasing the number of terms (e.g. $n + n + n + n = 68$) and relating this string of terms to the multiplicative term (e.g. $4n = 68$). The teaching experiment was successful. However, a problem of an arithmetic nature occurred. In jumping over terms in order to group , students were influenced by the operation following the term they started with.

In this paper we will describe and analyze lessons 2 and 3 which deal with equations in which the unknown appears on both sides of the equal sign.

## The Cancellation principle.

The notion of grouping like terms can be extended to decomposition of a term into a sum or a difference. Grouping and decomposition can then be used to introduce a relatively simple solution procedure based on **tranformation-within-the-equation.** For instance terms in $5n + 17 = 7n + 3$ can be decomposed into $5n + 14 + 3 = 5n + 2n + 3$. One can then appeal to a cancellation principle to simplify this to read $14 = 2n$, an equation that can easily be solved by all the students. Of course, the decomposition of terms can also deal with difference. For instance $13n - 22 = 6n + 41$ can be expressed as $7n + 6n -22 = 6n + 63 -22$ and cancellation reduces the equation to $7n = 63$.

From a cognitive perspective, the cancellation procedure which we refer to as **"Cancellation-Within-the-Equation"** might prove to be easier than the other procedures since the transformations are local, terms are grouped or decomposed into equivalent sums or differences without any operation on the equation as a whole.

## Preliminary assessment of comparison and cancellation procedures

Prior to introducing cancellation procedures in an explicit form, we wanted to verify the existence of a pre-requisite procedure, the comparison of corresponding terms. The first three questions given to the students were similar in form to those found in Filloy and Rojano (1984). The students were asked : "Just by looking at this equation can you tell me something about the solution?

1) $n + 25 = 17 + 25$

2a) $n + 19 = n + n$ 2b) Do you think that the other n must have the same value or can it be different?

3a) $n + 24 = n + 2n$ 3b) Do you think that the other n must have the same value or can it be different?

For equation (1) all the students except Audrey compared corresponding terms to conclude that $n = 17$. In (2a) 4 out of the 6 equated n to 19 and indicated that all the occurrences of n must be 19. This is in contrast to the results obtained by Filloy and Rojano. As for equation (3a) it was solved by comparison by 5 students.

In order to assess whether the students would use comparison to avoid unnecessary arithmetic operations we asked them: "What do you think would be a fast way of checking if the two sides of : $82 + 27 + 79 - 57 = 82 + 27 + 37 - 15$ are equal?" All six compared the two sides by simply performing the last indicated operation. This provides some evidence that the students can use comparison and develop procedural shortcuts.

The last two equations preceding the instruction were aimed at verifying if the presence of identical terms on each side of an equation might induce spontaneous cancellation.

The students were asked: "If you read the left side and then the right side of the equal sign what is the first thing you would do to solve the equation?"

1) $7n + 29 = 4n + 36 + 29$ 2) $3n + 4n + 21 = 3n + 57$

Andrew spontaneously cancelled the identical terms in both equations. Joel cancelled 29 in equation (1). The other four grouped the numerical terms in (1) and the terms in the unknown in (2). Hence we can conclude that apart from Andrew, the cancellation process had not yet been acquired.

## Lesson 2 - Cancellation of additive terms.

### Part 1: Introduction of the balance model.

We first presented the students with the equation $5n + 3n + 11 = 5n + 11 + 39$ and asked them if they could think of an equation as one side balancing the other. We then introduced little cutouts of each part of the equation which were put on the respective arms of a scale drawn on a worksheet. Students were then asked if removing the same weight on each side would leave it balanced, and if the same would be true with numbers. We used this model to introduce the principle that "Equals taken away from Equals leave Equals". We then suggested that they look at the scale and asked if they noticed any equal terms on both sides. They pointed at 11 and 5n. When asked if these could be taken away, 5 out of the 6 removed both 11 and 5n, while Joel removed only 11. The students were left with 3n and 39 and "solved" this "equation". Then the question of whether or not the solution they found ($n = 13$) would also be the solution of the initial equation was raised. All six were convinced that it was.

This introductory model had the distinct advantage of condensing the whole cancellation procedure and of offering the students a type of "inactive" mode of representation. However, as mentioned in the introduction, we did not want to build on this model because of its restriction (Filloy & Rojano 1989). Hence we proceeded to justify the whole process of cancellation on the basis of an "arithmetic" model (Herscovics & Kieran, 1980).

### Part 2 : Introduction of the arithmetic model.

We showed the students the equation $7 \times 9 + 11 = 74$ and constructed from this arithmetic equation an algebraic equation by hiding a number in turn by finger, place holder and finally by letter as in Herscovics and Kieran (1980). We repeated this transformation with the number 13 in $8 \times 13 + 11 = 5 \times 13 +$

50 in order to obtain an algebraic equation with the unknown on both sides: $8n + 11 = 5n + 50$. After pointing out that none of the procedures they knew could efficiently solve this type of equation, we told them that we would develop a new procedure and verified each step in this development by operating simultaneously on the algebraic equation and the arithmetic equation which we rewrote as : $8 \times 13 + 11 = 5 \times 13 + 50$ to remind ourselves that we have to imagine that the solution was hidden.

## Part 3 : The cancellation procedure.

In introducing the cancellation procedure, we had to choose between starting with the cancellation of the numerical terms or the terms in the unknown. The advantage of the latter is that the equation obtained can be solved by inverse operations (e.g. $3n + 11 = 50$). The disadvantage is that cancelling the terms in the unknown might seem arbitrary since it meant cancelling a generalized number before justifying the procedure with a specific number. Rather than creating the possibility of such a cognitive problem we decided on the longer process of starting with the first choice.

### Cancellation of identical numerical terms.

We started by asking the following questions: "When I look at the equation $8n + 11 = 5n + 50$ can I write it as $8n + 11 = 5n + 39 + 11$ ? Is this equation still balanced out ? Will the solution be the same ?"
We wish to point out that for all the transformations we introduced in the cancellation process, each one was accompanied by questions regarding the maintenance of numerical equilibrium and the invariance of the solution. These were always followed by a verification of the corresponding transformation on the arithmetic equation, whether the students agreed and responded affirmatively to each of the questions, or thought that the equality or the solution would be affected by the tranformations, or were not sure.
Then we asked the following question: "What if I take away 11 on both sides, do you think that both sides will still be equal ?... What is the new equation we get?... Do you think the solution is the same for both equations?" Five out of the 6 thought that removing 11 on both sides would maintain the equality. Regarding the invariance of the solution only two were sure. At this point with the help of Andrea we realized that they referred to another interpretation of the word "solution", the one usually used in arithmetic, the "answer" on the right side of an arithmetic equation.
We justified the transformation by showing the steps on the algebraic equation:

$$\begin{array}{rcl} 8n + 11 & = & 5n + 11 + 39 \\ \underline{\phantom{8n} - 11} & & \underline{\phantom{5n} - 11 \phantom{+ 39}} \\ 8n & = & 5n \phantom{+ 11} + 39 \end{array}$$

The students could check the validity of their operation by verifying it on the arithmetic equation.
After the justification and the verification we suggested a shortcut saying : "Let me show you a short way of doing what we just did. We start with the equation $8n + 11 = 5n + 50$ split 50 and replace it by $11 + 39$. We get $8n + 11 = 5n + 11 + 39$ and we simply cross out 11 on both sides : $8n + 11 = 5n + 11 + 39$. We called it "Cancelling 11 on both sides" or "Cancelling the addition of 11 on both sides".

### Cancelling the terms in the unknown.

After reducing the initial equation to $8n = 5n + 39$ we repeated the steps and the questions described above regarding the replacement of $8n$ by $5n + 3n$ and subtracting $5n$ from both sides. Two students felt that splitting $8n$ would

change the balance. Verifying their assumption on the corresponding arithmetic equation caused a change in their initial conception. We again suggested the shortcut $5n + 3n = 5n + 39$ calling it "cancellation of the same term on both sides". A summary of this lengthy introduction provided the opportunity to put together all the steps and to ask the students how would they choose the terms to be split up for eventual cancellation. They all used the criterion of "bigger" term to indicate their choice.
Following this reflection on the cancellation procedure we asked each student to solve $12n + 79 = 7n + 124$. All of our students except of Audrey solved it without any problem. Three started by replacing 124 by $79 + 45$ and cancelling 79. They then rewrote the equation and split $12n$ into $5n + 7n$, cancelled $7n$, rewrote $5n = 45$ and divided 45 by 5. The others started by replacing the unknown, cancelling and then splitting up the numeric term. Audrey, the weakest student, had to be shown the introductory example again, following which she rewrote the given example by decomposing $12n$ to $5n + 7n$, cancelled $7n$, rewrote $5n + 79 = 124$, and then used inverse operations. The next equation was $12n + 109 = 18n + 67$. All of the students used the same procedure they had used before. Audrey had to be guided in how to re-insert $12n + 6n$ into the eqution.

## Flexibility in the choice of sub procedures.

In order to verify if the student could solve the equation using other sub procedures, and in order to raise the question of the invariance of the solution, we asked the students to solve the same equation ($12n + 109 = 18n + 67$), but to start by decomposing a term other than the term they started with before. All of them were able to, and stated their conviction that the order of cancellation did not affect the solution. When asked to solve $109 = 6n + 67$ using another procedure, they used inverse operations.

More equations: The students were asked to solve :
(1) $19n = 13n + 72$ (2) $57 + 8n = 6n + 71$ (3) $12n + 30 = 13n + 19$
(4) $6n + 23 = n + 88$ (5) $71 + 12n + 38 = 13n + 67 + 5n$
Equation (1) was solved by all the students. In equation (2) Robyn spontaneously decomposed into sums both $8n$ and 71 and used double cancellation. Audrey split up 71 but did not know where to replace it, so we used an arrow to help her to remember the term she wanted to replace. In equation (3) to our surprise, all of the students split $13n$ into $12n + 1n$, and Andrea joined Robyn in double cancellation. Audrey, when ending up with $11 = 1n$ stated that it did not make sense. She had to be shown that $1n$ was the same as $n$ just as $1 \times 3$ is the same as 3. In equation (4) Andrea and Audrey got $6n = n + 45$, and were perplexed by the presence of a singleton. They overcame this obstacle when asked to write $6n$ as a string of additions (Herscovics & Linchevski 1991 (b)).
The last equation was intended to verify if students would first group and then decompose or would start by splitting up. Andrea and Audrey started by splitting up followed by immediate cancellation, while the others grouped first. Andrea, Robyn and Joel used a double cancellation in the solution process.

We ended lesson 2 with a short review. We presented the students with some equations, asking them to indicate which procedure should be used to solve. Lesson 3 started in the same way but this time we asked them also to solve the equations. Only Audrey had difficulties regarding rewriting the equation after transformations.

## Lesson 3. Preliminary considerations.

As in lesson 2 we chose to start by introducing the cancellation of the numerical term. In the pretest we had found that our students experienced some difficulty with the composition of consecutive subtractions. Some of the students did not perceive that $189 - 50 - 50$ was the same as $189 - 100$. Thus we decided to steer our students toward numerical situations which avoided this problem. We tried to achieve this by focusing their attention on the decomposition of a numerical term that was added. This also brought us to limit the scope of this teaching experiment to forms involving subtraction only on one side of the equation of either a numerical term or a term in the unknown as in $190 - 8n = 18n - 18$.

### Part 1 : Decomposition of a numerical term.

As in lesson 2 we built on an arithmetic model. The student constructed an algebraic equation from an arithmetic equation by hiding a number, but this time with subtraction on one side : $6n + 17 = 8n - 11$. We used decomposition of 17 into $28 - 11$. The stages of instruction were exactly as in lesson 2. This enabled us to highlight the basic principle "Equals added to Equals give Equals". We called this principle "Cancelling the subtraction of 11 on both sides".

During the summary review of this procedure we discussed with the students how to choose the term to be expressed as a difference. To assess how well our students had grasped our instructions we asked them to solve : $19n + 23 = 24n - 22$. The two top students figured out mentally the decomposition and immediatelly wrote : $19n + 45 - 22 = 24n - 22$ and solved the equation using two cancellations. However the other 4 students needed some guidance as 3 of them decomposed 23 into $22 + 1$. The next equation was solved by 4 students, and the two others needed some help in splitting a term into a difference. At this stage we presented our students with the equation $17n - 48 = 13n$. We wondered if after cancellation of $13n$ they would experience any problem. The results confirmed our conjecture, as four of the 6, after the cancellation of $13n$, did not know how to re-write the equation. Andrew, looking at $4n - 48 =$ stated : "All the weight is on one side and you don't have a solution". We reminded them that cancellation was justified by the subtraction of $13n$ from both sides.

### Part 2 : Restrictions on cancellation.

In order to verify if the students perceived the importance of not only "cancelling out" the same number but also the same operation, we presented the equation $15n + 18 = 17n - 18$ asking if we could cancel 18 on both sides. Five out of the 6 explained "If you want to cancel out, you must make sure it's the same operation". We recall that in the first equation when they were asked to solve $19n + 23 = 24n - 22$ three of them split 23 into $22 + 1$ and at that time we pointed out to them that one could justify cancellation only if the same operation on the term is involved.

### Part 3 : Decomposition of a term involving the unknouwn.

At this stage we began to observe some of the foundation problems we had observed at the beginning of the teaching experiment, which we previously called "a detachment" of an operation sign from the term (Herscovics & Linchevski 1991 (a) (b)). We gave them the equation $155 - 6n = 3n + 11$, which 4 of the 6 re-wrote as $155 - 3n + 3n = 3n + 11$ in order to cancel $3n$. Our teaching intervention was based on numerical examples, and on pointing explicitly at 3n to be split up. Only in the third equation of that type were all 6 students able to express a term in the unknown as a difference. The last equation to be solved was rather complex : $77 - 8n + 113 = 13n - 18 + 5n$. Andrew, Robyn, Joel and Audrey first grouped and then decomposed. Robyn and Joel reordered the equation before grouping. Robyn and Audrey used inverse operation when obtaining an equation with only a numerical term on one side.

## Post test.

The post test took place one month after the last meeting. The students had not done any algebra since the last lesson, and therefore we thought that some of the procedures would not come spontaneously to their mind. We thus had prepared two triggers, in order to jolt their memory and place them again in the framework needed for the solution, to be used only if necessary. The first trigger was a list with the procedures' names, and the second one was a ready-made right and wrong cancellation procedures.
We will discuss only the items of the post- test which are directly relevant to lessons 2 and 3.

### Comparison of algebraic equations :

We gave the same items as in the preliminary assessment of comparison. This time all of the students mentioned cancellation as the first procedure they would use except for Joel. In : $7n + 29 = 4n + 36 + 29$ he first grouped the numerical terms.

### Solving equations.

Due to space limitations we will not go into detailed description of sub procedures used by the students, but will comment that many interesting individual differences have been found. The equations were given one at a time.

### A) Single occurrence of the unknown.

(1) $13n + 196 = 391$ (2) $16n - 215 = 265$ (3) $12n - 156 = 0$
All the students solved by using inverse operations.

We must note how stable this procedure has remained over a period of 7 months (Herscovics and Linchevski, 1991(a)). Even after learning the decomposition of numbers into a sum or a difference, this new method did not interfere with the inversing procedure.

Results from parts (B) and (C) "grouping like terms" are given in Herscovics and Linchevski 1991(b).

### D) Unknown on both sides of the equal sign, involving only addition.

(1) $4n + 39 = 7n$ (2) $5n + 12 = 32n + 24$ (3) $12n + 79 = 7n + 124$
(4) $71 + 12n + 38 = 13n + 67 + 5n$
In equation (1) (2) and (3) Andrew, Daniel, Joel and Robyn immediately decomposed terms and solved successfully all of the equations while Andrea and Audrey had to be shown triggers (1) and (2). Evidenced by the comparison part of the post-test, both of them remembered cancellation, so probably they had forgotten the decomposition part.
The students had not lost their mastery of notation and could efficiently write down their steps. Also they were taught to cancel one term at a time. In the post-test Daniel, Andrea and Joel used double cancellation and Robyn and Audrey, after cancelling the term in the unknown, used inverse operations. Only Andrew stayed with the procedure we taught. As for equation (4) we saw grouping first and then double cancellation, as well as splitting up from the

very beginning. The students who started by splitting, after being prompted, willingly solved by first grouping and then splitting.

E) Unknown on both sides involving also subtraction.

(1) $19n + 23 = 24n - 22$ (2) $155 - 6n = 3n + 11$ (3) $17n - 48 = 13n$
(4) $89 - 5n = 7n + 5$ (5) $77 - 8n + 113 = 13n - 18 + 5n$

It is in this part that most difficulties emerged. Also some of the students avoided the need to decompose a numerical term into a difference by cancelling first the term in the unknown and then using inverse operations. All the basic problems mentioned previously, the detachment of the minus sign and jumping off with the posterior operation were observed.
The students tended to decompose "the bigger number" regardless of its sign (e.g. in equation (2) $155 - 3n + 3n = 3n + 11$) in order to obtain cancellation, or split a number into two numbers when the operation preceeding the new numerical term on the left was not the same as the operation preceeding the corresponding number on the right.

## Conclusion.

"The Cancellation Within the Equation" was accepted by the students as a smooth extension of their spontaneous ability to use comparison in the context of some specific mathematical equalities. This tendency was supported by both the balance model and the arithmetic model for justifying cancellation. The decomposition of a number was a natural complementary process to that of grouping like terms. Moreover, it was evident that the students were able to go beyond the instruction by themselves, inventing more efficient procedures. However, when a decomposition into a difference was involved, the cognitive obstacles we have mentioned in previous papers were found; the detachment of the minus sign and jumping off with the posterior operation. For some students expressing a number as a difference when the subtrahend is a given constraint was not a trivial problem. Although this procedure was addressed during the lesson, in the post-test they experienced the same difficulties. Some of them kept splitting the "bigger" number into two "smaller" ones. This seems to put in question the benefit of extending the cancellation procedure beyond replacing terms by equivalent sums.

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# EMERGING RELATIONSHIPS BETWEEN TEACHING AND LEARNING ARITHMETIC DURING THE PRIMARY GRADES

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*In this study, we examined relationships between instruction, students' understanding, and students' performance as they began to acquire computational strategies in multidigit addition and subtraction. We were interested in how conceptual understanding interacted with skillful performance as students received instruction on addition and subtraction with regrouping, and in how these interactions were influenced by different kinds of instructional activities. The results indicated that instructional activities which emphasized mathematical connections through the discussion of problems and alternative solution strategies were more closely related to the development of both understanding and skilled performance than were activities that emphasized procedural skills through paper-and-pencil practice. However, the relationships are not straightforward and several clusterings of individual cases are presented to reveal some of the complexities.*

## Background

The current reform movement in mathematics education in the United States is based, at least in part, on the belief that instruction should be redesigned to facilitate a higher level of conceptual understanding and to decrease the emphasis on drill-and-practice. Although such alternative approaches are widely presumed to promote a more flexible use of knowledge and better problem solving skills, we still have little evidence on the way in which understanding and performance interact and on the way in which alternative instructional approaches influence these interactions.

The notion of understanding has a rich history in mathematics education. Many of the psychological descriptions of understanding mathematics (e.g., Brownell, 1947) are based on the idea of establishing relationships between facts, procedures, representations, and so on. Our view of understanding is consistent with this perspective and with more recent discussions of building cognitive connections (e.g., Hiebert & Carpenter, in press). We believe that understanding develops as students establish connections of many kinds: between familiar ideas and new material, between different forms of representation (e.g., physical and written), between procedures and underlying principles.

In this study, we followed students during the first three years of school and examined the development of understanding and performance in multidigit arithmetic. We focused on the way in which different instructional approaches influenced this development. We were especially interested in the influence of approaches that emphasized the construction of connections.

## Method

Sample. Data were collected from an initial cohort of about 150 students during their first, second, and third years of school. Many new students entered the classrooms during the three years and some left, so the number of students and their instructional history depend on the time of assessment. The students attend suburban-rural public schools.

Instruction. Several different instructional approaches for place value and addition and subtraction were observed. During the first year, two of the six classrooms followed the textbook using relatively conventional instruction. The other four classrooms implemented an alternative approach during the five weeks of place value and addition/subtraction topics. The alternative approach was characterized by greater student use of physical representations, increased emphasis on translating between different kinds of representations (e.g., physical-verbal-written symbols), greater use of story problem situations, and fewer problems covered but increased time spent per problem during class discussions. Class discussions usually involved analyses of problems and sharing alternative solution strategies. (See Hiebert & Wearne, in press, for a more complete description of these classrooms.)

The following year the students were reassigned to six second-grade classrooms. Four of the six classrooms followed the textbook in a relatively conventional way and two classrooms used the alternative approach. The alternative instruction classrooms contained only students who had received alternative instruction in year 1. The alternative approach, an extension of that used during first grade, emphasized mathematical connections through class discussions of problems and solution strategies and through the use of different forms of representation. As in first grade, problems were situated in story contexts. Both approaches devoted about 12 weeks to place value and addition/subtraction instruction.

During the third year, the initial cohort plus about 75 new students were assigned to nine classrooms. Three classrooms used the alternative approach and six classrooms used varying textbook approaches. The majority of students who received alternative instruction during years 1 and 2 were in the alternative instruction classes in year 3. The classrooms devoted 10-14 weeks of instruction to place value and addition/subtraction.

Assessments. All students were given written tests three times each year--near the beginning, middle, and end of the school year. At the same time, about half the students were interviewed individually. The interviewees were randomly selected at the beginning of year 1--12 students from each of the six classrooms--and the same students were interviewed throughout years 2 and 3.

The tests and interviews were constructed to measure (1) students' understanding of grouping-by-ten ideas and of the positional nature of the written notation system; (2) students' skill in adding and subtracting with and without regrouping; and, (3) students' understanding of the computational procedures they used.

Classroom observations. During years 1 and 2, all of the classrooms were observed once or twice a week during instruction on place value and addition/subtraction. During year 3, all classrooms were observed for three consecutive days during relevant instruction. Field notes were taken on classroom activities and the sessions were audiotaped and transcribed.

## Results

We will focus on the data from years 1 and 2; at the time of writing, the data from year 3 had not been completely gathered nor analyzed. Year 3 data will be included in the conference presentation. Given space limitations, we will summarize the results; more detailed presentations are available from the authors.

Between-group performance differences. In general, students who engaged in the alternative instruction for two years performed better on all types of written test items. Specifically, they scored higher on items measuring (1) knowledge of place value and the tens-structure of written notation, (2) computation on instructed problems, (3) computation on noninstructed or novel problems, and (4) story problems. For most items, the differences in percentage correct between the two groups at the end of the second year ranged from 10% - 40%.

Profiles of emerging competence. Within-subject profiles helped to characterize the nature of the between-group differences in performance and probed further into the relationships between understanding and performance under different instructional conditions. For illustration purposes, we can consider two very different groups of students--those who entered the second year with a comparatively rich understanding of grouping-by-ten ideas and how these connect with the positional symbol system and those who still understood little about these ideas. Nine of the 65 students interviewed at the beginning of the second grade were relative experts, performing successfully and giving meaningful explanations. They all were highly successful on most addition, subtraction,and missing addend story problems during the second year, but their construction and choice of computation strategies showed several distinct patterns. Four students created decomposition strategies in which they dealt with the larger digits (e.g., hundreds) first, regardless of regrouping demands, and used these strategies even after they had been exposed (at home or school) to the algorithms. Three students developed the same decomposition strategies but switched to the standard algorithms once they had been exposed to them and used them successfully on all problems. Three students showed less evidence of using self-generated strategies consistently, switched to algorithms as soon as they saw them, and made some of the classic regrouping errors on the more difficult problems.

In contrast, 23 students began second grade with very little understanding of grouping-by-ten and place value ideas. Again, several different patterns of performance and understanding emerged. Some students were uniformly unsuccessful throughout the year, some students showed a sharp rise in computational performance after learning the algorithms (independent of understanding), and a third group showed more gradual increase in performance, based on invented strategies, that seemed to keep pace with their increasing understanding.

Interestingly, cases of these profile patterns occurred in both kinds of instruction. However, their frequency of occurrence differed. More students in the alternative instruction classes constructed and used their own computation strategies and depended less on the standard algorithms. For example, at the middle of the second year, before instruction on the standard algorithm for addition with regrouping, 81% of the correct responses of the alternative instruction interviewees were generated by self-constructed strategies compared to 39% for the textbook instruction interviewees. Standard algorithms (learned at home according to their users) accounted for most of these students' correct responses. Fewer students in the conventional classes used the understanding they possessed, even if it was substantial, to develop their own strategies or adjust taught procedures to solve new problems.

Links between instruction, understanding, and performance. In order to link learning with instruction, we were interested in the observed differences in instruction that might explain these different learning profiles. Both content and pedagogical differences were investigated. Content differences were not found in the scope of the curriculum but rather in the nature of the activities. More

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of the activities in the alternative instruction involved connecting procedures with conceptual underpinnings and connecting different ways of solving problems. For example, a great deal of time was spent in year 2 asking students to share invented strategies and then asking them to explain why the procedures worked and how they were the same as or different than other procedures.

Pedagogical differences are more difficult to summarize. In year 1, the alternative instruction (compared with the more conventional instruction) used fewer materials and used them more consistently as tools for solving problems rather than for demonstration; solved fewer problems but devoted more time to solving each problem; and delivered more coherent lessons. Details of these results are presented in Hiebert and Wearne (in press). In year 2, differences were found again in use of materials and the time spent per problem. The same material (base-10 blocks) was used consistently in the alternative instruction classrooms and was always available; a few different materials were used in the more conventional classrooms but only for one or two lessons each. During 40 minute lessons, the two alternative instruction classes averaged 12 problems per lesson in one class and 14 problems per lesson in the other class. The four more conventional classes averaged 24, 29, 36 and 38 problems per lesson. Finally, in the alternative instruction classrooms, students talked much more relative to the teacher and the teachers asked many more questions that requested analyses of problems, description of alternative solution strategies, and explanations of why procedures worked.

Conclusions

Relationships between teaching, understanding, and performance are extremely complex. Nevertheless, this brief summary of data hints at several links. First, the development of understanding seems to affect performance through the construction of robust strategies that are applied successfully across a range of problems. That is, understanding does not translate automatically into improved performance; the impact of understanding on performance is mediated by the kinds of strategies students use to perform tasks. If students are encouraged to invent and analyze strategies, it is likely their understanding and performance will be closely linked. This appears to be true for high and low achievers alike.

Second, instruction may be related most importantly to learning in terms of whether it affords opportunities for students to use their understandings to develop and modify procedures. It is clear that the relationship between understanding and performance can be fragile in the face of instructional demands. The data indicate that understanding does not necessarily translate into, or even inform, procedural skill. Further, taught procedures can take students well beyond their level of understanding. If students are to engage in productive interactions between understanding and procedural skill, instruction may need to focus on supporting students efforts to construct, analyze, and modify a variety of procedures.

A third conclusion, of a somewhat different kind, is based on the finding that routine procedural skills developed just as well or better in the alternative classes as in the more conventional drill and practice environments. Even though students in the alternative classes spent less time practicing routine skills on fewer problems, their performance did not suffer. This may be the most salient finding for immediate classroom application because it frees teachers to try their own alternative approaches, even if they are still accountable for high performance on routine tasks.

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# PRESCHOOLERS' SCHEMES FOR SOLVING PARTITIONING TASKS

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*We wanted to know what enabled young children to solve challenging dealing tasks in which perceptual cues were restricted. A sequence of partitioning tasks designed to progressively limit children's access to percetual cues was administered to 30 preschool children aged three to five years. An analysis of strategies used by both successful and unsuccessful children suggested that development of a stable pattern of operations having an iterative structure is critical. Further, reliance on sensory feedback as a means of monitoring commencement of internally regulated cycles seemed to constrain solution success.*

Young children have considerable informal and intuitive mathematical concepts before entering school (Gelman & Gallistel, 1978; Irwin, 1990; Miller, 1984; Resnick, 1989; Wright, 1991). One particular cognitive skill is an ability to equally divide a set of discrete items (Davis & Pitkethly, 1990; Hunting & Sharpley, 1988; Pepper, 1991). A common task used with preschoolers is a collection of 12 items -- sometimes food such as jelly beans -- which are to be shared equally between three dolls. Various names have been used to describe the process observed or infered from the behavior of the subjects studied: partitioning, sharing, dealing, or distributive counting. A feature of successful efforts to distribute items equally is a powerful algorithm leading to the creation of accurate equal fractional units. Three nested actions comprise the basic algorithm: (1) allocation of item to a recipient, (2) iteration of the allocation act for each recipient to complete a cycle, and (3) if items remain, repetition of the cycle (Hunting & Sharpley, 1988). The ability of young children to solve tasks of this kind is important for mathematics education because such actions can form a meaning base for the notation and symbolism of division, and for fractions and ratios. As Saenz-Ludlow (1990) says, "It seems that fraction schemes spring out of iterating schemes that lead to partitioning schemes" (p. 51).

Subsequent examination of children's partitioning behavior showed that some young children who used systematic methods varied the order in which items were allocated for each cycle of the procedure (Hunting, 1991). Also, some children were able to maintain the dealing procedure as they were carrying out a conversation with the interviewer, or re-establish the order of allocation after being distracted or interrupted. Pepper (1991) found that preschoolers' ability to succeed with dealing tasks was not related to their counting competence. In a follow up study, Pepper (1992) attempted to limit young children's use of pre-numerical skills such as subitizing (Kaufman, Lord, Reese, & Volkmann, 1949), visual height comparisons, and pattern matching, by including a task called Money Boxes, in which items to be allocated -- coins -- became hidden from view once placed in opaque containers. Of a sample of 25 four and five year old children studied, 16 succeeded with the Money Box task; and were evenly distributed across three categories of counting competence (rho=0.11, p=0.58). The most commonly observed strategy was a systematic dealing procedure where each cycle began with the same doll and money box. Two strategies were suggested by these results. First, a particular position of doll and/or money box served as a sign post to mark the commencement of a new cycle. Second, mental records of lots of three were used to regulate items as they were being distributed.

In summary, when given sharing tasks involving discrete items, in which the items are visible at all stages of the solution process, young children seem able to use different schemes as they work towards creating equal shares. These schemes include comparing heights of stacked piles, placing items in lines and comparing lengths or matching one-to-one across shares, successive comparison of items in each share using subitizing as items are allocated, counting, using one recipient as a marker, and mental records of lots corresponding to the number of recipients. Table 1 lists these schemes. Of interest was whether young children, if denied access to perceptual cues needed to use a particular scheme, could adapt by using a different scheme which relied on internal regulation of actions. We also wanted to explore more deeply by what means children succeeded with tasks such as Money Boxes where schemes seeming to depend on perceptual feedback, such as pattern matching, could not be used. We decided to examine in more detail the strategies used by successful and unsuccessful children on the Money Box task, and compare these with their methods for solving tasks that preceded and followed the Money Box task.

| Cognitive Scheme | Behavioral Indicator |
|---|---|
| Pattern matching/subitizing | organized display of items replicated across recipients |
| Measurement of height | stacks items, lowers heads to visually compare, moves stacks together |
| Measurement of length | sets out items in corresponding lines |
| Counting | counts in process, able to say how many in each pile at the end as verification |
| Mental grouping and monitoring of represented items | begins cycle at different points, organizes number of items for each cycle in advance, pauses in process/tolerance of distraction |
| Recipient as sign-post or marker | begins cycle at the same place |

Table 1: Possible schemes used to solve partitioning tasks

## Method

Thirty children attending the La Trobe University Child Care Centre were individually interviewed during November 1991. Children interviewed were from three age groups. Six children were from a three year old group, with mean age three years two months (3.2), median age 3.3, mode 3.3, and range 3.0-3.4 years. Fifteen children were from a group of four year olds, with mean age 3.11, median age 3.10, mode 3.10, and age range 3.6-4.4 years. Nine children were from a group of five year olds. Their mean age was 4.11, median age 4.11, mode 5.1, and their age range was 4.6-5.2 years. Children were selected on the basis of parents' consent to having their children participate. The children's parents were either students or academic staff of the University.

A set of partitioning tasks involving distribution of discrete items was administered. These differed in difficulty according to particular schemes it was thought children might employ in the course of their solutions. All interviewed children were given an initial task, called "Stickers", followed by the first Money Box task involving 15 coins and three dolls. The sequence of tasks which followed varied for each individual according to whether that child was successful or not. Figure 1 shows the flowchart governing task administration. The triplet of numerals in brackets represents the numbers of children from each age group who succeeded for each task -- the first numeral represents the number of youngest children.

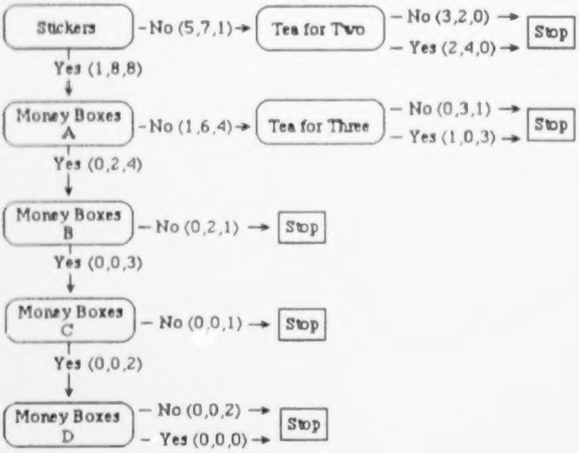
Stickers
- No (5,7,1)
Tea for Two
- No (3,2,0)
- Yes (2,4,0)
Stop
Yes (1,8,8)
Money Boxes A
- No (1,6,4)
Tea for Three
- No (0,3,1)
- Yes (1,0,3)
Stop
Yes (0,2,4)
Money Boxes B
- No (0,2,1)
Stop
Yes (0,0,3)
Money Boxes C
- No (0,0,1)
Stop
Yes (0,0,2)
Money Boxes D
- No (0,0,2)
- Yes (0,0,0)
Stop

Figure 1: Flowchart of interview tasks

The interviews were conducted in a small room adjacent to activity rooms that the four and five year old groups used. All interviews were video taped for later analysis.

A description of the tasks of interest in this report now follows.

Stickers. The child is invited to observe a sock puppet, operated by the interviewer, distribute 12 monochromatic stickers between two dolls. After preliminary discussion designed to put the child at ease, the child is asked to observe Socko give out the stickers to the dolls "so each doll gets the same." The interviewer says, "Socko isn't very clever at sharing out. I want you to watch what Socko does, and tell me if each doll gets a fair share." The puppet gives four stickers to one doll and eight stickers to the other in a non-systematic way. The child is asked if the dolls get the same each, and whether the dolls would be happy with their share. Regardless of the child's responses to these questions, she is then asked to teach Socko how to distribute all the stickers so each doll gets the same. After the distribution concludes the child is asked if the dolls got the same each, and for a justification.

Money Boxes (A). Three identical opaque money boxes are placed in a row on the table in front of each of three dolls. A stack of 15 twenty cent coins are positioned near the money boxes. The child is told: "Mum wants all the pocket money shared out evenly so each doll gets the same. Can you share the money into the money boxes so each doll has the same? How? Show me." The child is encouraged to distribute all the coins into the money boxes, and when the task is completed is asked: "Has each doll got an even share? How can you tell?"

Money Boxes (B). A similar task to Money Boxes (A) except for this task 17 coins are to be distributed.

Money Boxes (C). Five identical opaque money boxes are placed in a circle on the table. Nineteen coins are to be distributed.

Money Boxes (D). Five identical opaque money boxes are placed on a circular rotating platform known as a "lazy susan". The child is shown how the tray works, and it is explained that the tray will be rotated sometime during the allocation process. Again, 19 coins are placed in a stack for distribution.

Tea for Three. On a table are placed 18 items to be distributed between three dolls. On an adjacent table is a toy cook top with pot and spoon. In the pot are 12 white "crazy daisy" plastic items. The interviewer says the dolls are going to have their dinner, and indicates the items in the pot on the toy stove. The interviewer then says: "The meal is cooked and the dolls are very hungry. Can you serve out the food so all the food is given out and each doll gets the same amount?" If the child stops before all the items are distributed, the interviewer says: "Has all the food been given out? Remember the dolls are very hungry". The child is then asked to consider the allocation outcome with the question "Do you think each doll has the same amount? How do you know?" If the child disagrees she is asked: "Can you fix it up?"

Tea for Two. Children unsuccessful with the Stickers task are invited, in pairs, to set a table for two dolls, and distribute 12 items of "food". Results not reported in this paper.

## Results

Seventeen of the 30 children succeeded with the Stickers task. Of these, six succeeded with the first Money Box task. We first consider solution strategies observed for children who succeeded with the first Money Box task, and their behavior on subsequent variations. Then, we will examine solution strategies typified by children unsuccessful with the first Money Box task, and compare these with their solution strategies on the Tea for Three task, where items remained perceptually accessible.

**Solution strategies of successful children**

The first Money Box task required children to share 15 coins equally between three dolls. Two solution strategies were observed. The first is exemplified by Sharlene (3.7) and is represented by the following table, in which A, B, and C represent each of the money boxes, and the bullets, •, represent coins. The flow of action proceeds from left to right:

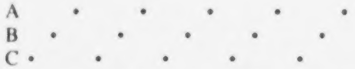

Figure 2: Sharlene's cyclic and regular solution strategy

Sharlene's strategy shows (1) cycles of three, in which each box is visited just once, and (2) the same money box used at the commencement of each cycle. The second solution strategy is exemplified by Jim (5.1). His strategy also involves cycles of three, but different money boxes mark the commencement of each cycle. Sharlene's strategy is cyclic and regular; Jim's strategy is cyclic and irregular.

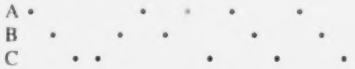

Figure 3: Jim's cyclic and irregular solution strategy

Of the six successful children, three showed cyclic and regular strategies (Sharlene (3.7), Aaron (4.9), Elise (4.7)); the other children cyclic but irregular strategies (Jim (5.1), Kalhara (5.1), Sophie (4.2)). Kalhara and Sophie showed at least three regular cycles of allocation.

For Money Boxes (B), in which 17 coins were to be distributed to three boxes, three children, after distribution, said the dolls did not receive a fair share (Jim (5.1), Elise (4.7), Kalhara (5.1)). Elise and Kalhara used the cyclic regular method; Jim used a cyclic irregular method as before. Sharlene (3.7) and Aaron (4.9) showed cyclic regular solutions but said the dolls got the same. Sophie (4.2) was unsuccessful. Her allocations followed a cycle of C, B, A, except for the second, which was C, C, B. She chattered to the interviewer during the allocations. Hesitation in fluency of her actions seemed to coincide with the onset of utterances.

Money Boxes (C) involved five identical boxes arranged in a circle. Nineteen coins were to be shared. Jim (5.1) and Elise (4.7) succeeded on this task; Elise on the second attempt. She was very uncertain on her first attempt, repeatedly asking the interviewer if she had placed a coin in a particular money box. She used a different approach the second time, taking piles of six, seven, three, and three coins from the stack in her left hand as she proceeded. The interviewer also advised her not to talk while she was working. Kalhara (5.1) said all the money boxes got the same. All children showed a one coin-one box, one coin-next box sequential strategy beginning at the box nearest them on the table, thus: A, B, C, D, E, A, B, C, D, E,...

Jim and Elise were given Money Boxes (D) where five money boxes were placed on a "lazy susan". The lazy susan was spun 2.4 times after the tenth coin had been placed. Jim commenced placing the 11th coin in the "right" box and continued sequentially; at the end saying "there's only four more left." Elise placed two coins in the third box even before the tray was rotated. After rotation she changed direction, using a one coin-one box sequential strategy. She was not successful.

A follow-up interview was given to Jim in which a task similar to Money Boxes (D) was given. The tray was rotated 1.8 times after the ninth coin was posted. There were 22 coins in all. Jim placed the tenth coin in the next box, despite the intervening rotation. He indicated the boxes did not receive the same number of coins, saying,"because this one didn't have any" -- as he touched the next box in the sequence after the last coin had been posted.

## Solution strategies of unsuccessful children

Three patterns of response were observed in the children who were unsuccessful with the Money Boxes (A) task. The first pattern was cyclic like that observed in the successful children. However, children who did this were not consistent in its use (Joshua (4.4), Vanessa (5.2)). The second pattern was to place a sequence of three or more coins in the same box. Six children did this (Leo (3.3), Anton (3.10), Blake (4.1), Julian (4.2), Tim (4.10), Carla (4.11)). Carla was the only child whose solution was exclusively of this sort (see Figure 4). A third pattern was to place a coin in the box adjacent to the box previously visited, like a zig-zag (see Figure 5). Three children's responses were predominantly of this sort (Justin (3.10), Tess (4.3), Brian (5.0)). Other responses were non-cyclic and irregular.

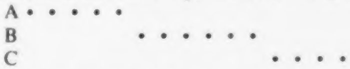

Figure 4: Carla's solution strategy

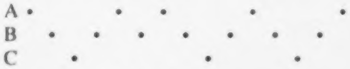

Figure 5: Tess's solution strategy

Tea for Three was given to children who were unsuccessful solving the first Money Box task. A significant degree of consistency of response was observed across these two tasks. Table 2 summarizes strategies used for each task. Tim and Brian were the only children who had success with Tea for Three. Tim counted the items onto the dishes. He knew each dish contained six items. Since he did not count all the items before, his estimate for the first dish was a good one. Brian was successful because his strategy was wholly systematic.

| Child | Money Boxes (A) | Tea for Three |
|---|---|---|
| Leo (3.3) | Non-cyclic, irregular | Non-cyclic, irregular: not successful |
| Anton (3.10) | Non-cyclic, irregular | Task not given |
| Justin (3.10) | Adjacent box strategy | Adjacent dish strategy predominantly: not successful |
| Blake (4.1) | Sequence of coins in each box, predominantly | Sequence of items for each dish: not successful |
| Julian (4.2) | Sequence of coins in each box | Task not given |
| Tess (4.3) | Adjacent box strategy | Task not given |
| Joshua (4.4) | First three cycles irregular | First four cycles irregular: not successful |
| Tim (4.10) | Sequence of coins in each box, predominantly | Sequence of items in each dish exclusively: successful |
| Carla (4.11) | Sequence of coins in each box, exclusively | Placed handfuls of items on each dish: not successful |
| Brian (5.0) | Cyclic with adjacent box strategy | Cyclic and regular: successful |
| Vanessa (5.2) | Cyclic predominantly | Cyclic mixed with adjacent dish strategy: not successful |

Table 2: Relationship between responses across Money Boxes (A) and Tea for Three tasks

## Discussion

A set of tasks involving Money Boxes was used to study partitioning schemes used by young children. These tasks restricted children's use of schemes dependent on perceptual cues such as comparison of heights of shared items, comparison of lengths, one to one matching across shares, and successive comparison using subitizing. What internal regulations of actions made it possible to succeed in these circumstances? The first requirement would seem to be a mechanism for monitoring "lots" or units of multiple allocations. Internally constructed units consisting of a temporal sequence of discrete counts or tallies replayed again and again could be needed. Alternatively, ability to visualize a spatial configuration corresponding to the number of Money Boxes, which can be "scanned" iteratively. Such internal constructions can be considered empirical abstractions (von Glasersfeld, 1982). Empirical abstractions occur "when the experiencing subject attends, not to the specific sensory content of experience, but to the operations that *combine* perceptual and proprioceptive elements into more or less stable patterns. These patterns are constituted by motion, either physical or attentional, forming "scan paths" that link particles of sensory experience. To be actualised in perception or representation, the patterns need sensory material of *some* kind, but it is the motion, not the specific sensory material used, that determines the pattern's character" (p. 196). The difference between Jim's and Sharlene's scheme for solving the first Money Box task was Sharlene's use of one box exclusively as a marker. This behavior indicates she relied more on sensory feedback located in the physical presence of the three boxes. In contrast, Jim's irregular starting points for his allocation cycles suggests greater confidence in a represented cycle and some cycle counter independent of the boxes. Jim's performance on more challenging tasks involving five boxes arranged in a circle showed he was capable of keeping track of the next box to receive a coin and monitor position reached in a cycle consisting of five elements.

The critical difference between successful and unsuccessful children on the first Money Box task was the development of a stable pattern of operations having an iterative structure. The role played by temporal or spatial representations of perceptual lots is unclear, as is the interaction between representational and direct sensory experience in the process of solving these kinds of tasks.

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# The Emancipatory Nature of Reflective Mathematics Teaching

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*Critical reflection on the act of teaching may be seen to be liberating for the teacher, who, as a result, has greater knowledge and control of the teaching act. This paper supports such contention where the teaching of mathematics is concerned by drawing on research with one teacher who might be seen to engage in critical reflective practice. It considers also how the researcher might influence the liberating process through which teacher-emancipation occurs.*

## Bondage

If the term *emancipation* - a state of being set free from bondage (Chambers' English Dictionary) - is applied to teachers, it might be inferred that the teacher who is not emancipated remains in some form of bondage - for example, the constraints of an imposed curriculum.

Anecdote abounds to support the frequency of statements from mathematics teachers in the vein of "I have taught them blank so many times and they still can't get it right", or "I should like to teach more imaginatively, but if I did I should never have time to complete the syllabus". Such statements typically come from teachers who are bound by tradition, convention or curriculum, and who fail to perceive their own power to tackle constraints. The result for pupils is likely to be a limited or impoverished mathematical experience.

## Reflective practice

Many educationalists have advocated reflective practice as a means of emerging from such shackles. I must make clear that the term *reflection* as I use it here has a critical dimension and is more than just 'contemplative thought'. Van Manen (1977) defines reflection at three different levels, the third of which, *critical reflection*, concerns the ethical and moral dimensions of educational practice. Boud, Keogh and Walker (1985) speak of "goal-directed critical reflection" which concerns reflection which is "pursued with intent". Smyth (1987) advocates "a critical pedagogy of schooling which goes considerably beyond a reflective approach to teaching", suggesting that the reflective approach is not itself critical. However, Kemmis (1985) brings these two elements very firmly together, as in

> We are inclined to think of reflection as something quiet and personal. My argument here is that reflection is action-oriented, social and political. Its product is praxis (informed, committed action) the most eloquent and socially significant form of human action (p 141)

It is reflection in Kemmis' sense which I address in this paper. I will make the case that reflective practice in mathematics teaching, which is critical and demands action, is a liberating force, and that teachers engaging in such reflection are emancipated practitioners.

## Teachers' *voice*

The emancipated teacher may be seen to be in theoretical control of the practice of teaching. This implies that the teacher explicates theories, or gives them 'voice'.

Cooney (1984) refers to teachers' "implicit theories of teaching and learning which influence classroom acts", saying further,

> I believe that teachers make decisions about students and the curriculum in a rational way *according to the conceptions they hold.* (My italics)

Although the classroom act itself may be seen as an explication of theory, teachers' thinking is often not explicitly articulated, and it is left to researchers outside the classroom to give voice to teachers' conceptions. Elbaz (1990) suggests that it has become important that researchers into teachers' thinking "redress an imbalance which had in the past given us knowledge of teaching from the outside only" by encouraging expression of teachers' own voice.

> Having 'voice' implies that one has a language in which to give expression to one's authentic concerns, that one is able to recognise those concerns, and further that there is an audience of significant others who will listen.

Smyth (1987) goes further in speaking of teacher emancipation, that only by exercising and 'intellectualising' their *voice*, will teachers be empowered in their own profession.

> To reconceptualise the nature of teachers' work as a form of intellectual labour amounts to permitting and encouraging teachers to question critically their understandings of society, schooling and pedagogy.

These notions pose a dilemma for theorists, researchers or teacher-educators proposing teacher emancipation, because to be truly emancipated teachers themselves must be their own liberators.

My experience as a teacher, and in working with teachers, suggests that *critical reflective practice* (which I discuss further below) can be a liberating process, but that it is actually very difficult to sustain if working alone. In my research with teachers I believe that I have, to some extent, facilitated their reflective practice by being there and by asking questions. I propose, therefore, that researchers working with teachers can be catalysts for liberation, through their encouraging of *questioning* of practice, and provision of opportunity for teachers to exercise their *voice*.

## The role of the researcher

Elbaz (1987), while acknowledging the "large gap between what researchers produce as reconstructions of teachers' knowledge ... and teachers' accounts of their own knowledge", nevertheless expresses the hope,

> I would like to assume that research on teachers' knowledge has some meaning for the teachers themselves, that it can offer ways of working with teachers on the elaboration of their own knowledge, and that it can contribute to the empowerment [of] teachers and the improvement of what is done in classrooms. (p 46)

The purpose of my own research with teachers was to attempt to elicit the deep beliefs and motivations which influenced their teaching acts. My methodology involved talking extensively with the teacher both before and after a lesson which I observed. Fundamental to any success I might have had in this was the development of a level of trust between teacher and researcher which would allow sensitive areas to be addressed. For the teacher cooperating in my research, and attempting seriously to tackle the questions I asked, a consequence was a making explicit of theories of teaching which could then be used to influence future practice. It was not part of my research aims to influence the practice of the teachers with whom I worked, it was an inevitable consequence that it did. However, change was effected by the teacher, and in this respect the researcher acted as a catalyst.

## An example of developing practice related to teacher-researcher discussion

The teacher was about to teach a lesson on vectors to follow up his introduction of vectors to his year-10 class (15 year olds) in a previous lesson. I had asked him to tell me what he would do in the coming lesson, and he replied that he wanted to "recap what a vector AB is". He referred to notes which he had prepared with plans for the lesson. The following piece of transcript records part of my conversation with the teacher before that lesson. (T - teacher, R - researcher, myself)

| | | |
|---|---|---|
| 1 | R | What do you mean by 'recap'? *You* recap? |
| 2 | T | Me recapping - well - me asking questions. "Now, what's meant by adding vectors? What's meant by taking away a vector, or a minus vector?" And then asking, "what's the difference between those three - the vector-AB, AB, and BA?" |
| 3 | R | Now is it your expectation that by asking appropriate questions you will get all that information from them? |
| 4 | T | Hopefully yes. And what I wanted to do today was not really concentrate too much on vectors, but say, "If we've got a vector (3,4) can we find the length of AB? *Hello* - we're back into Pythagoras - ha ha ha!" |
| 5 | R | Right |
| 6 | T | Okay? |

7 R Yes

8 T And then give them some questions, and then get them to check over their homework after they've sorted out - oh - one bit I've missed on here (referring to his notes) I want them to say what 2AB is - something else we talked about, and I want to talk about AB and BA as vectors, and AB and BA as lines.

9 R Right

10 T We're really talking about notation, aren't we, now?

11 R Right. - You said, give them ten questions. What sort of questions?

12 T They're going to be quite straightforward.

13 R To do what?

14 T Find the lengths of vectors.

15 R So, for example, "Find the length of a vector ...

16 T ... AB, if AB is 3,4."

... (slight digression here on what the 3,4 notation looks like)

17 R Or you could get them to invent some for themselves.

18 T Yes, that would be quite interesting, wouldn't it.

The conversation proceeded to more general aspects of teaching and learning, and at one point the teacher offered an anecdote from one of his lessons with lower-attaining pupils in which they had been invited to invent 'think of a number' games for each other.

19 T I started playing the game, like, "I'm thinking of a number. I double it and add three, and my answer is seven. What was the number I started with?" After a bit of practice, no problems. But the things they were asking each other were out of this world. If *I'd* asked them they'd have gone on strike!

20 R How do you mean?

21 T Well, they were saying "I'm thinking of a number, I've halved it, I've added three to it, I've multiplied by three, I take two away, I divide it by seven and my answer is twenty one. What number did I start with?" And they could actually solve them. Now if I went in and put that on the board for a bottom ability group they would go on strike.

22 R Yes, right.

23 T And when you actually got back to it, they had this inverse relationship all sorted out. They couldn't write it down, but they had it all sorted out. That's what, yeah, it's there isn't it, them setting their own levels. I don't do it often enough. I must do it more often.

You can have that for what it is!

24 R Thank you! How about doing it there (I pointed to his notes for the vectors lesson)?

25 T What, getting them to set their own?

26 R Get them to set their own.

27 T (Pause) I'll try.

28 R It'll be interesting to see if they only come up with questions of a particular type, because that will tell you something about the way they are thinking.

30 T Can I say, "Be inventive?"

31 R Sure!

32 T OK. We'll do that.

At statement 8, the teacher said he would give the class some questions. He then returned to talking again about his general lessons plans. I was interested in what the questions would look like and so I asked him (statement 11). His reply that they would be straightforward, was followed by a digression into forms of notation. I brought him back to the questions again with my statement (17) that he could get them to invent some for themselves. He acknowledged this, but little more at that point.

I was interested in what his questions would be, because I wondered what they would contribute to the pupils' perceptions of vectors. My remarks were a focusing device where our consideration of these questions were concerned. If I had not pursued them, the teacher may not have provided any more information. I had great power to focus in this way, although I did not at the time select explicitly this focus in preference to others. My suggestion was spontaneous. It was not my, or our, pre-planned intention to focus on pupils' inventing of their own questions. It arose in and from the context of the conversation, which was about the teacher's concerns.

As part of the continuing conversation, the teacher came up with the anecdote about pupils in another class setting their own challenges, and the value that he saw in this. It is my speculation that this was triggered by my suggestion, and that certain associations were set up in response to our talk. This analysis came some time after my work with the teacher, so I was not able to check its validity with him. However, his telling of the anecdote gave me opportunity to reiterate my suggestion (statement 24), and for the teacher to agree to try it out (statement 27). His questions at the end recognise that this is a suggestion from me, and seek in some way my clarification of the extent of invention I envisage.

Thus, I influenced the teacher's planning and execution of the vectors lesson more overtly than had been my intention. However, I feel that he was able to set pupils an open task of inventing their own questions because he could see this in the context of other open tasks which he had set, and which had been successful. Moreover, his style of working with the pupils was such that an activity of this kind was not unfamiliar territory to them. It is interesting to consider the extent to which my suggestion depended on my knowledge of his practice, and the extent to which his acceptance of it depended on his reciprocal knowledge. The developing trust between us made a significant contribution to our joint understanding of what was possible in his classroom.

In the lesson itself, he introduced that task with the words: "I would like you to make your own questions up and write your own answers out and then share your questions with a neighbour. Could you be inventive please. Don't put up a whole series of boring questions". I discussed aspects of this, and pupils' responses to it, in Jaworski (1991b) in another context, so I shall not repeat those details here. However, the outcome in terms of some pupils' questions and responses was very satisfactory. It opened up areas which the class had not yet addressed: for example, the special nature of parallel vectors, and the related notations for vectors of equal length albeit of different directions in different positions, both arose from pupils' own investigations. It provided the teacher with opportunity to address such questions in a way meaningful to pupils because they had arisen from the pupils' own thinking. Our retrospective reflection on this lesson, acknowledged the value and success of the activity.

## Critical reflection influencing the teaching act

I believe that this episode charts a stage in this teacher's own development as a teacher. For him, in this case, critical reflection[1] involved making explicit the value of occasions where he asked pupils to be inventive in setting their own challenges. It resulted in his becoming more aware of opportunities where he could encourage pupils in this. Three weeks later I saw a lesson in which he returned to pupils some tests which they had done and he had marked. Rather than present a set of correct solutions for them to compare, he offered a set of 'answers' of his own, all of which had errors in them. Their task was to spot the errors, and to explain, in discussion with neighbours, what would be correct. In this way he hoped to challenge them to work dynamically on their own solutions and errors, rather than passively to accept the teacher's 'correct' solutions.

I believe that enabling the pupils to take more responsibility for their work and thinking through setting their own challanges was an aspect of this teacher's philosophy and operation which developed during the time that I was working with him. I propose that this speaks to the emancipation of this teacher, in that he was actively seeking ways of enhancing pupils' learning, which brought him into a more acute knowledge and control of the teaching situation, and thus of his own direction and purpose. In this he was engaged in a process of self-liberation.

Our conversation often focused on the liberating process itself. On one occasion we discussed different sorts of decisions which the teacher had made in various lessons which I had seen, and the difference between responding to a pupil *instinctively*, and making a more informed response or judgment. The teacher commented, "I feel that responses are judgments that have proved right in the past and been taken on board." He went on,

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[1] I have explicated in some detail the stages of critical reflection which formed part of my analysis of conversations with teachers in my wider study. This is included in a paper "Reflective practice in mathematics teaching" which is currently submitted for publication.

> You've been through a lot of these situations before your responses. Don't they actually come from things which happen in the past and you're saying, I made a judgment then that was a good one, or saw someone do something that was good. And you actually take that on board. Isn't that what developing as a teacher is all about?"

Some manifestation of this general principle might arise after the vectors lesson and the asking of pupils to invent their own questions. Perhaps in some other lesson later, the teacher would recall aspects of this activity, and our subsequent analysis of it, and it would influence his teaching at that instant.

I have suggested (Jaworski, 1991a and b) that it is such in-the-moment recognition of choice of reponse, based on previous experience made explicit, that is the action outcome of critical reflection. I go further here in suggesting that this is the essence of the liberating process. The more critical such reflection is, in being disciplined about identifying the issues in a particular lesson, the choices taken, the decisions made, and their effect on learning and teaching, the more able the teacher is likely to be to act appropriately to what arises on a subsequent classroom occasion. Developing as a teacher is the result of such action. Such development is dynamic, and, if recognised and used deliberately, it can be liberating and empowering.

## Teacher emancipation

Teacher emancipation, according to sources quoted at the beginning of this paper, arises consciously from teachers becoming aware of their own knowledge and purpose through critical enquiry into their practice. Emancipation seems to be a state within the liberating process of action-oriented critical enquiry. In the case of mathematics teaching this involves questioning both pupils' perceptions of the mathematics on which a lesson is based, and the pedagogy to be employed in developing this mathematics. Teachers have to know what they hope to achieve in terms of the mathematical content of a lesson and their pupils' constructions of this mathematics, and also in terms of the teaching acts which will be employed. Although this content and these acts will be designed to fit some prescribed curriculum, they do not *need* to be conditioned or bound by it. The curriculum to which the above teacher worked required pupils' understanding of the elements of vectors which were being addressed in the lesson. It did not prescribe the means by which such content would be made available to the pupils, and it did not preclude the pupils coming to aspects of that content through their own investigations. The teacher's overt knowledge of mathematics and pedagogy, based on his own developing experience, as well as a confidence in his own ability to make appropriate choices and judgments, enabled him to construct suitable teaching acts. This meant that the teacher himself was in control of the learning environment of pupils in his classroom, and moreover that he could take responsibility for what occurred rather than blaming defects on pupils' inability to remember or retain, or on constraining effects of the curriculum. His awareness of this level of responsibility[2] and his overt exercise of control were indicators of his emancipated position. For the teacher I have described, encouraging *pupil-emancipation* might be seen as an element of his control.

I have indicated, where the above teacher was concerned, that my research presence had some effect on his developing practice. How does teacher development and subsequent emancipation depend on such presence, and how far is it possible for a teacher to achieve this alone?

I have no research evidence to present in order to address this question. The teacher group working together, perhaps in small-scale action research, to support and encourage such practice can be an effective sustaining medium (see for example, Kemmis 1985, Gates, 1989, Mathematical Association 1990). However, further research is needed into the development of the emancipated teacher through a liberating process of action-oriented critical enquiry, particularly where the teaching of mathematics and its effect on pupils' learning is concerned.

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[2] Evidence of this may be found in Jaworski (1991a, Chapter 7, and 1991b, page 219) A detailed account of the 'vectors' lesson is provided in Jaworski (1991a), and a curtailed account, more specifically related to *constructivist* aspects of the teacher's thinking, in Jaworski (1991b)

# REFERENCE, STRUCTURE AND ACTION: ELIMINATING PARADOXES IN LEARNING AND TEACHING MATHEMATICS

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*This paper starts by discussing a number of paradoxes to have recently emerged in theories of learning and teaching mathematics. These are found to make similar assumptions about the nature of mathematical knowledge and its epistemology. A detailed analysis of a transcript, recording the linguistic interaction between the researcher and a number of senior high school students, follows. This analysis traces the breakdown of a didactic contract (Brousseau) and its subsequent re-establishment; it also studies how the pedagogic sequencing facilitates learning the attainment of learning goals. The transcript is also used to exemplify the occurence of paradox in pedagogic situations. The paper concludes by adapting a model, drawn from the field of genre studies, in order to provide a theoretical account of linguistic utterances constitutive of pedagogic interactions and their epistemological implications.*

## §1 Introduction

In Plato's Meno (80e), Socrates presents a paradox which shows that a student cannot learn what he or she does not already know: For if the student had the knowledge there would be no need to seek it, and if the student lacked knowledge, then how would the student even know what to look for? The standard rebuttal of this paradox points to an apparent confusion about the meaning of words, for instance,"having knowledge". Nevertheless, paradoxes of this kind - the *learning paradox* (Bereiter, 1985) is almost exactly similar - bedevil modern theories of teaching and learning mathematics (Brousseau, 1986).

A motivating question for this paper therefore is: How may paradox free models of teaching and learning be constructed? The paper presents an interim report of a study into the meaning and use of words in pedagogical interactions in mathematics classrooms.

## §2 Paradoxes in mathematics education and some remarks on epistemology

The constructivist view that learning is a process in which the learner is actively engaged in a process of restructuring or organising knowledge schemata is widely held in information processing psychology (Resnick, 1983; Bereiter, 1985). This model for learning is , however, prone to paradox. Bereiter (1985), for instance, refers to the *learner's paradox* whereby

> ...if one tries to account for learning by means of mental actions carried out by the learner, then it is necessary to attribute to the learner a prior cognitive structure that is as advanced or complex as the one to be acquired.(p. 202)

More significantly for *educators*, the move to develop instructional procedures consistent with constructivist learning theory has also run into difficulties (Cobb, 1988, 1992: Kanes, 1991). Cobb argues, for instance, that Resnick's notion of "instructional representation", violates the autonomy of the individual constructor at a key stage in the pedagogic interaction. Resnick's procedure therefore erroneously reinstates as constructivist, a variant of the absorption model for learning.

Occurrence of paradox has also been noted in work proceeding on more general pedagogic grounds. Brousseau (1986) for instance argued that teacher and student enter a *didactic contract* in which the teacher must ensure that the student has an effective means of acquiring knowledge and in which the student must accept responsibility for learning even though not being able to see or judge, beforehand, the implication of the choices offered by the teacher. Brousseau argued that the contract is driven into crisis and ultimately fails, for

> all that he [the teacher] undertakes in order to get the pupil to produce the expected patterns of behaviour tends to deprive the pupil of the conditions necessary to comprehend and learn the target notion: if the teacher says what he wants he cannot obtain it. (p.120)

Similarly, Steinbring (1989) observes that the teaching process of making all meanings *explicit*

> leads to the effect that by the total reduction of the new knowledge which is to be learned to knowledge already known, nothing really new can be learned. (p. 25)

Obviously, in a few short sentences one is not able to treat the issues represented by these paradoxes exhaustively. However, it is interesting to note that the context within which each of these arise provides a similar epistemological stance. Each assumes that mathematical knowledge is essentially a matter of *content* and that, as such, is capable of being made, in principle at least, totally explicit. For instance, in the learner's paradox, knowledge is individually constructed as a representation of a knowledge target, and therefore, in a sense, is actually *derived* by the individual. It follows that this kind of knowledge can, and in pedagogic episodes should, be made explicit. To illustrate by a metaphor: Constructing a clock means being able, in principle at least, to make explicit each of the parts of the clock. When teaching clock-making the detailing of the clock's mechanism may, for the benefit of the apprentice, need to take place at a fine level. In that case, the clock maker is actually engaged in a process of re-presenting the clock to the apprentice as an articulation of its parts. In the same way, constructed mathematical knowledge is re-presentational and explicatable. This view, however, asserts an epistemology of *reference* over intention or transaction. For instance, the representational clockmaker is unable to convey the overall coherence of the form of the clock or the degree and manner in which its structure complements a certain aesthetic economy or style (expressive or intentional characteristics); likewise, she is unable to convey the actual experience of the actions of making a watch (transactional or pragmatic characteristics).

## §3 Analysis and discussion of a pedagogic episode

In order to provide an illustrative focus for the theoretical statements of the last section and those to come later in the paper, this section will present a linguistic analysis of a pedagogic episode. The sequence studied is drawn from a stimulated recall (Keith, 1988;Parsons *et al*, 1983). This method involved video recording a lesson in a naturalist context and, immediately after this, replaying the tape to a teacher-chosen subset of students. Students were asked to respond freely to the tape, the researchers reserved the right to ask probing questions. Those present include 6 students chosen from the class by the teacher, together with 2 researchers. The classroom teacher was not present. The discussion between students and the researcher is on the application of 'dummy variables' as indices in expressions involving complex algebraic manipulations.

In analysing this episode, it has been assumed that in order to recover the shifting epistemological positions of the Researcher and of the participating students, each utterance would need to be individually scrutinised for evidence of fine grain structure. The presumption has been, that only as the fruit of such an endeavour, would nuances indicative of the shifts sought, show themselves.

Note: In the following transcript 'R' represents the Researcher; Ms X is the regular classroom teacher; L8, L9 *etc* refer to lines 8 and 9 etc of the transcript s shown.

1 R: Now the very first step here, where you've got $\arg(z_1/z_2)$, Ms X wants you
2 to focus on $z_1/z_2$. Now the first thing that she did was to write that out in a
3 trigonometric form, or a polar form. And she wrote on the top line, what did
4 she write?
5 Sarah: $r_1$ ... (*inaudible*)
6 R: Outside of?
7 Alice: Inside the brackets, I think it's $\cos\theta_1 + i\sin\theta_1$

Assertions in L1-3 are followed by a single question in L4: in these, the researcher announces the theme of the inquiry. Primary focus is set on the structure of the mathematical steps Ms X performs, not their meaning or reference, nor any possible function they may perform. Further, in these opening utterances the Researcher is both signalling the attention the students give to Ms X as Teacher as well as displacing the Teacher in this triangular relationship of power. The utterance "Ms X wants you to focus ..." evidently means "I - the Researcher - want you to focus ...".

8 R: Why did she say $r_1$ and $\theta_1$?

In this utterance the Researcher inaugurates the main body of the episode. An inquiry concerning Ms X's *intentions* is opened. There are two parts related to this task: *content* (What is her meaning for $r_1$ and $\theta_1$?) and *function* (How do the nominated subscripts function in the mathematical procedures implied?).

9Students: (*Several students exclaim at once*) Because that's the modulus and argument for $z_1$!

Interestingly, a large proportion of the students answered immediately in this way. This response, however, only picks up the *content* aspects of the utterance (L8). That is, the students have only adopted the level at which the meaning, or reference of the symbols $r_1$ and $\theta_1$ is signalled. A study of the relationship between this semantic content, and transactional elements which could permit the capture Ms X's intentions is not considered, or if considered, not pursued. The emphatic tones and chorus like response of the students may also indicate growing resistance, even annoyance, on the part of the students. The Researcher has usurped the role of the teacher (we saw this in L1-7), but now seems unable or unwilling to take over the didactical contract (see above) originally forged in class between Ms X and her students. Once having gained admittance to the code, the Researcher seems to be consciously attempting to disturb it, threatening to bring about its collapse. Apparently, with Ms X, it is part of the didactical contract that teacher questions solicit information and that valid responses assume the referential mode. But the Researcher, by asking such an apparently straightforward question, now rejects in advance not only the answer presented by the students, but even the referential form the answer takes. Crisis in the contract is deepened further by the apparent lack of guidance to the students as to what alternative form a valid answer would take.

10 R: Sorry, just explain? Sorry who's talking?

The two questions here reveal a great deal. Both acknowledge the impasse in which the students have been placed by the Researcher, and hence each begins with "sorry". The repetition of "sorry", however, raises the ironic questions: Who is sorry? Who ought to be sorry? These signs also serve to reinforce a consensus view that the usurped contract has collapsed; and they herald a new phase, described by Brousseau (1986, p.113) as the interactive process of *searching for a contract*. In Brousseau's theory, knowledge arises precisely as the resolution of crises such as those described here. And indeed, in constructing a new contract the Researcher has already taken a lead: In each question the Researcher begins to suggest a *new* basis for interaction. In the first, "Sorry, just explain?", students are encouraged to treat the symbols as a prompt to perform an *action* of some kind rather than as a cue to passively provide information. The second suggested premise of a new contract relates to the form of admissible interactions within the social space of the episode: Researcher - Student interactions are to be one-to-one.

11 Alan: Because, well we've got subscript one, for $z_1$, we sort of use the same
12 subscript, probably.

Alan, identified by the teacher prior to this intervention as a quiet, co-operative student, is the first student to attempt to work within these shifting terms. However, as the colloquial expression ("sort of") and the terminating ("probably") would indicate, Alan is not certain of his ground, nor of the social topography defining the interaction. Alan has been very accurate, however, in picking up the clue provided in L10 as to what might constitute a successful response to the motivating question asked in L8. In his response, he switches away from the semantic or referential content of the symbols, and attempts to focus on the mathematical operations (actions) implied or controlled by the symbols. Nevertheless, Alan has only glimpsed the choice between reference and action which has just offered by the Researcher, and almost certainly has not yet grasped the consequences. The role of the Researcher in confirming or disconfirming the validity of Alan's gesture is now crucial in the process of establishing a new order in which this pedagogical crisis may be resolved.

13 R: Would it have mattered what subscript? If she'd written '2', would that have
14 been wrong? If she had written $r_2$ would that have been wrong?

Alan's contribution is implicitly accepted by the Researcher as valid. The first question plays a double role of reinforcing and extending the fledgling contract. Reinforcement is accomplished by verbal cues such as adopting Alan's reference to "subscript", the use of which had not hitherto become explicit. Non verbal cues such as tone of voice and the absence of a wait time (neither shown in the transcript) also implied Alan and the Researcher may be reaching common ground. Extension of the contract is also achieved by this question. This is done by switching attention from the action performed when using a given subscript, replacing this with a question directly relating the intention lying behind the *choice* of a subscript: "Would it have mattered what?" is made to read "Would it have been against her intention if?". This step in the pedagogic sequencing, if accepted by the students, represents a final transformation of focus which has travelled from *reference* to *action* and now settles on *intention*. Indeed, this is precisely where the Researcher wishes to end up, for this is the perspective sought from the students in L8 and either rejected or not observed by them in L9.

Note that the researcher is not modelling the 'correct' answer to the question posed in L8, instead however, the student's responses are being scaffolded with respect to their epistemological focus. Could the Researcher have been more direct here and merely asked "Would this have violated her intentions?"? By this time it should be clear why the answer is 'no'. Some students may have still thought that this question sought information about her *meaning*, whereas the question reaches much further than that, towards grasping the balance between the knowledge of what the relevant signs mean and the knowledge of what their functional significance in a mathematical procedure is *ie* a balance between epistemologies of reference and action. Such a misunderstanding would provide an example of Brousseau's so-called paradox of the 'devolution of situations' by virtue of which, the anxiety of the teacher to give the students what they appear to want - need - forecloses the possibility of them being able to directly obtain it. Instead, by shifting the pedagogy through an epistemology of *action* the teacher gains 'leverage' which may be employed to refocus student attention.

The second and third questions could be construed as attempts to lead the students through a thought experiment consisting of actions premised on a hypothetical condition. Note that question two is more general (suppose '2' is the nominated subscript) than question three (it would then follow that Ms X would have to write $r_2$). Each question requires the students to consider the consequences for the mathematical procedure if Ms X had nominated '2' as a subscript. By asking whether or not this hypothetical choice would be wrong the Researcher is asking whether or not the transactions implied by the symbols would disrupt the relationship between the form and content of the underlying mathematics, in other words: Would they be *syntactically* correct?

Alice: Only if she had have, it would have been confusing, because you've got $z_1$ and $z_2$, and then you've got, it would be easier to have $r_1$ and $\theta_1$ then they've got, it makes a link there so you have, you say that it's with the same, the same problem.

Alice, reported by the teacher to be a strong student, provides a relatively sophisticated response to L13-14. Her first utterance, "Only if she had have" clearly implies her answer to the second and third questions is in the negative. However, she does not leave the matter there. Her attention has been focussed on function, whether such an operation would be facilitated by a co-operation between the knowledge of form and content, as would be required by unproblematic mathematical transaction. At this point she identifies a disjunction which would arise between the form or structure of the mathematical statement and its content or reference: "it would have been confusing". An optimal match, facilitating action (eg mathematical manipulation) - "it makes a link" - is obtained by matching subscripts. Alice has arrived at a defence of Ms X's use of dummy variables which employs *both her knowledge of the form or overall structure of the mathematics together with her knowledge of it as an event.* Thus, she is able to express Ms X's intention to obtain or maximise clarity. Both the form and the intention of the utterance are co-incident. A response such as this was sought in L8.

## §4 Reference, Structure and Action in mathematics pedagogy

Working within theories of *genre* (Bakhtin, 1986: Holquist, 1990; Smales, 1990), Ongstad (1991) has provided a model which affords a starting point in understanding the theoretical relationships apparent in the transcript analysis conducted above. Seminal in this model is Ongstad's observation that in making an utterance

> "you are doing three things *all at once*, you refer, act and structure."
> (italics added,p.13)

In the accompanying table, terms relating to the analysis of utterances are arranged in a 3x4 grid. Read in columns, the terms, taken pairwise, are contrastive. Read in rows, the table sets out terms which correlate.

**Grid, setting out the key analytic categories in an adapted version of Ongstad's model**

| | | | |
|---|---|---|---|
| SYNTACTIC | STRUCTURE | INTENTIONAL (EXPRESSIVE) | FORM |
| SEMANTIC | REFERENCE | INFORMATIONAL (INDICATIVE) | CONTENT |
| PRAGMATIC | ACTION | --- | FUNCTION |

Since in Ongstad's model every utterance can be analysed in terms of *structure*, *reference* and *action* (2nd column), each of the 12 terms set out in this grid can be brought to bear on the analysis of any single utterance. The richness of this model allows us to trace the shift in emphasis of these terms amongst utterances which constitute any given linguistic interaction. Such an analysis was illustrated in the previous section.

Alternatively, the grid can be thought of as map on which may be traced 'pathways' for the development (both effective and ineffective) of mathematical knowledge. Each of the three sets of correlational terms might be said to support an epistemological viewpoint. Learner's normally need to have access to at least these three. For example, Cobb (1991) traces how the reflexivity between the syntax and semantics is obtained or mediated by the pragmatics inherent in consensual knowledge.

Ongstad emphasises (p.12) that contradictions and paradoxes arise when the multi-dimensional character of utterances is denied. Analyses offered in this paper amply substantiates this point. This does not mean, however, that utterances equally emphasise all the elements capable of influencing them. On the contrary, the selective emphasis a sender or receiver places on utterances lends a particular character to the interaction. Where, however, the task of the interlocutor is to alter or direct the interaction, as is the case for a teacher, the full range of perspectives is open in order to facilitate the development of a pedagogic strategy and student learning. Once again, the analysis of §3 provides a rich example of such a process.

## §5 Conclusion

In the last decade it has become more common for research to emphasise the consensual aspect of mathematical learning and teaching processes. Matching this has been a growing sensitivity towards epistemological questions. This paper firmly endorses both these developments. At the heart of the present work has been the suggestion that within the dynamics of microsocial interaction, epistemological shifts, as indicated by linguistic utterances, critically determine the character of pedagogic interactions. Arising from this, the presence of paradox in certain theories of learning and instruction may indicate an 'epistemological cramping' - or an over reliance on one view about what qualifies as mathematical knowledge. Paradox free theories of mathematical pedagogy depend, it would seem, on the disposition to retain, foster and protect a certain epistemological dynamism.

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INTERNATIONAL GROUP FOR
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MATHEMATICS EDUCATION

# PROCEEDINGS of the SIXTEENTH PME CONFERENCE

University of New Hampshire
Durham, NH (USA)
August 6 - 11, 1992

Volume II

330

INTERNATIONAL GROUP FOR
THE PSYCHOLOGY OF
MATHEMATICS EDUCATION

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Volume II

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# The Answer Determines the Question. Interventions and the Growth of Mathematical Understanding

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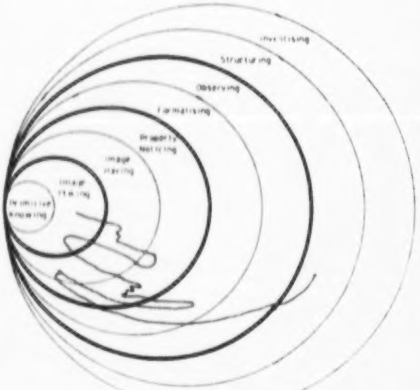
University of Alberta University of Oxford

**Abstract**: *Our work over the past four years has looked at the growth of mathematical understanding as a dynamic, levelled but not linear, process. An outline of our theory and its features is given in this paper before it goes on to address the question of how a teacher can influence an environment for such growth. We identify three kinds of intervention: provocative, invocative and validating and use these concepts in analysing interactions between a teacher and two students. Our contention is that for the promotion of growth the teacher needs to believe that it is the student response which determines the nature of the question.*

"The task of education becomes a task of first inferring models of the students' conceptual constructs and then generating hypotheses as to how the students could be given the opportunity to modify their structures so that they lead to mathematical actions compatible with the instructor's expectations and goals." (1)

"an organism has somehow to acquire the capacity to turn around on its own schemata and to construct them afresh. It is what gives consciousness its most prominent function. I wish I knew exactly how this is done" (Bartlett in (2))

Over the past four years we have been building and testing a theory of the growth of mathematical understanding which views mathematical understanding not as an acquisition (e.g. 3), nor as a developmental phase (e.g. 4), but as a dynamic process. Using this theory we have attempted to show that the growth of a person's understanding of any topic can be mapped on a model comprising eight embedded levels of understanding moving from initial primitive knowing through to inventising. ( fig. 1) We maintain that growth through such levels or modes of understanding is not in any way monotonic but involves multiple and varied actions of folding back to inner, less formal understanding in order to use that understanding as a springboard for the construction



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of more sophisticated outer level understanding. We are, of course, still in the ongoing process of elaborating the elements in such understanding. Our own understanding of the emerging theory is, itself, subject to constant acts of folding back with a view to gaining greater insight into the phenomenon of mathematical understanding.

The quotation from Bartlett, above, prompts us, too, to ask the questions, "how might such re-construction happen?" and "what roles might teachers play in bringing it about for their students?". In this paper we wish to consider the nature of some teacher interventions and their impacts on student understanding. More broadly, we also illustrate that such interventions do not have to originate with the teacher, although it seems likely that only the teacher is in a position to create such interventions deliberately. Theoretical descriptions of such interventions , which we call ***provocative, innovative*** and ***validating*** will be followed by analysis of an incident in terms of the effect of certain questions on the growth of understanding of a single student.

We do not claim to be alone in the field, attempting to answer the questions posed above, but to be taking a different stand point from which to analyse the phenomenon of growth of mathematical understanding. For example, Maher et al (5) consider the mathematical behaviour of one child sampled over four years and indicate in global or macroscopic ways the nature of change in sophistication of such behaviour.Our work differs from theirs in that it is driven by a particular, albeit developing, theory and tries to comprehend the dynamics of growth as they occur in local situations. It allows us to examine teaching strategies, interventions and effects in day to day classroom environments. Edwards and Mercer (2) do look in detail at interactions between teachers and students and their impact on understanding, but while our analysis shares with them the idea that contexts in which teachers and students exist are best understood in terms of their mental instead of physical features, our theory allows for a mathematical rather than general pedagogical analysis of classroom understanding.

## The Theory

Before defining our categories of provocative, invocative and validating questions, we will offer a very brief review of our general theory. Figure 1 offers a pictorial representation of the levels of mathematical understanding.The nature of the levels has been elaborated in (6), (7), (8), as has the notion of mapping the pathways of individual student growth, (9) but we would like to draw attention here to two of the key features in the model. The first is the notion of *folding back*, discussed in detail in a previous PME paper (10). This feature is crucial to the distinction between the types of intervention that we will be considering later in the paper. An illustration of folding back would be the student who, having spent some time working with physical shapes or accurate drawings on squared paper (imagemaking, image having and property noticing), derives the generalisation "length times breadth" for the area of a rectangle (formalising) but, on being asked what the area of a triangle might be, replies "I don't know - I'll have to go back to drawing some to see if there is a formal way of doing that too". The student is folding back to an inner understanding (image making) in order to extend her outer, formalised understanding. The second feature is the bold rings indicating 'don't need' boundaries. These are a vital element of the power of mathematics itself. They occur when one functions in a mathematical way that ignores the origins of current understandings. If the student above subsequently derives the generalisation "a half base times height" for the area of a triangle and then goes on to calculate areas using this formalisation with no further reference to counting squares or consideration of rectangles, then this student has crossed the don't need boundary between property noticing and formalising.

We return to the question of how growth in understanding might be promoted. Teachers have an important role to play in enabling students' personal construction of knowledge to occur. We see three kinds of intervention, provocative, invocative and validating, as crucial to the teacher's task of hypothesizing student constructs and proposing mathematical actions. A *provocative* intervention is one which points the student toward outer or more sophisticated understanding. An *invocative* intervention is one which makes students aware of the need to fold back to an inner level of mathematical understanding. A *validating* intervention is one which establishes that a student is working within some level of understanding with the effect of encouraging the expression, verbally, symbolically or figuratively, of current mathematical actions. The interesting and perhaps critical feature of these interventions is that *it is the student response and not the teacher question which determines the nature of the intervention*. The teacher may well offer advice intended to move the student on to outer understanding whereas in reality it can cause the student to fold back to earlier levels in order to make sense of the new situation. Consider the student above. The teacher may have posed the question about the areas of triangles in the context of the generalisation just formulated, with the intention of enabling the student to 'see' that the area will be half that of a rectangle (provocative). The outcome, however, was to cause the student to fold back and the intervention was therefore actually invocative.

## Analyzing the Dynamics of Intervention

We turn now to an example where we use this theory to analyse actual teacher-student and student-student interactions. The class was working on the task of finding relationships between the various different shapes in a set of "pattern blocks". The shapes are all based on a common length of side and comprise a green equilateral triangle, a red, 60 degree trapezium, an orange square, a blue, 60 degree, equilateral parallelogram, a yellow hexagon, and a plain wood, 30 degree, equilateral parallelogram. (Fig. 2)

The 14-year old students were given some time to play with the shapes, before being set the relationships task, as they had never previously seen or handled them. The task was introduced through the demonstration by the teacher that it was possible to build fig 3b on top of fig 3a. They were given no other guidance.

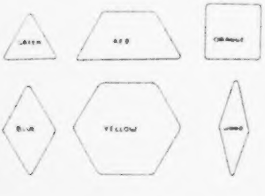


Figure 2 The Pattern Blocks

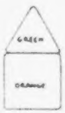


Figure 3a

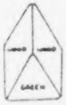


Figure 3b

Sara and Jen spent about ten minutes image making through building patterns and shapes beside and on top of each other. Their talk made it clear that they had the image of the shapes as "fitable" (their word) without gaps, although they did not explicitly comment on the sides being of equal length, and they noticed and recorded various properties such as "You can make a hexagon with a yellow or two reds or three blues". At this point the teacher came over to their table.

1 Teacher OK Sara where have you got to?
Sara So I've got: a red is 3 greens, a yellow is 6 greens a blue is 2 greens but the orange and wood aren't anything. (*said in a tone of finality*)
2 T Can you say anything about the greens themselves?
S Well they are triangles, aren't they..
3 T Can you say anything about the triangles?
S What .... you mean their angles and things? .... Well ... um ... If you put 3 together to make the trapezium they make a straight line like this. (*draws fig 4a*) and you get them together at the points so that is 180 degrees (*adds drawing to produce fig 4b*) so each angle will be 60 degrees.
.........

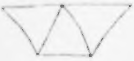


Figure 4a

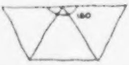


Figure 4b

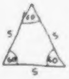


Figure4c

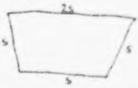


Figure 4d

(*the teacher moves away*)
........So the sides, the sides are s so the green goes ... (*draws fig 4c)* and the red is ..... (*draws fig 4d)* Triangles' areas ... um ... a half base times height (*draws fig 5a*) soooo ...... so ... (*writes area=1/2 s* ) ...

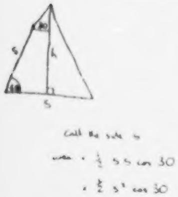
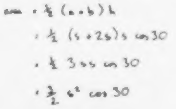


Figure 5a


area = ½ (a+b) h
= ½ (s+2s) s cos 30
= ½ 3ss cos 30
= 3/2 s² cos 30
Figure 5b

height , height what's the height?... s cos30 .. so (*writes scos30 = 1/2s²cos30 (fig 5a)*)
Jen Sara what did you get for the woods?
S Shush! I'm working.....(*writing fig 5b*)....Look.look! It works! The green area is a half $s^2$cos 30 and the red is three halves cos 30! So three greens make a red!
4 J Oh splot!(*a term of praise*) Well splotted! ..What do you get for the woods? How many greens? They're too big aren't they?

S You can't. You get orange plus green equals green plus two wood from her example. (*the teacher's: fig 3a and 3b*). So mathematically an orange is two woods, but you can't make it. You .. can .. also ... get an orange and a wood and a blue make a perspective cube (*builds fig 6 with blocks*) but it doesn't help. .... or you can ...(*building with blocks*) give me some more woods ..... make a star with just the woods (*finishes fig 7*)...and there's ..(*counting with her finger*). eleven, twelve, twelve of them, twelve woods, twelve pointy bits make the middle ..... wait a minute ..... 360, 360 is twelve so one's angle is ..360, twelves into ....30. Each one is 30, and you could cut the wood in half and get two triangles and they'd have area ... wait a minute ... (*she then proceeds to write fig 8*)

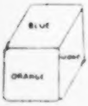


Figure 6

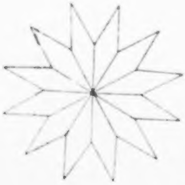


Figure 7

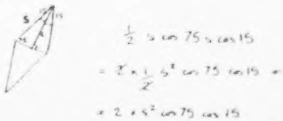

S
15
15
$\frac{1}{2}$ s cos 75 s cos 15
= 2 × $\frac{1}{2}$ s² cos 75 cos 15
= 2 × s² cos 75 cos 15

Figure 8

You've got a calculator, get a calculator, get a calculator...
J Hang on!... It's some where.....OK.(*Produces calculator*)
S Do cos 75...do 75, cos, times, 15, cos, equals...
J (*reading calculator display*) Point two five
S Times two.....
J Point five......
S It's a half, it's' a half, it's half the square!!
J Well it doesn't look like it!

In section 1, one sees a classic validating situation. The teacher has observed Jen and Sara working away with the blocks and suspects that they at least have an image of the idea of relationships between the different shapes. Since they have also been writing they have probably been recording some of the properties they have noticed. Sara's response confirms this and shows her ability to articulate her findings.

In section 2, therefore, the teacher aims to provoke Sara on to more formal comments on the relationships, BUT Sara treats the question as if it were a validating intervention and confirms that she knows the mathematical names for the shapes and is saying "greens" for convenience and not out of lack of mathematical understanding.

In section 3, the teacher rephrases her question and this time the effect *is* provocative. Sara moves from thinking of the shapes as visible concrete objects whose relationships to one another can be physically determined to a symbolic representation. She is not concerned with the actual sizes of the shapes, nor does she even need accurate diagrams; rough sketches suffice as her aide-memoire. She has crossed the 'don't need' boundary to formalised understanding. Indeed in some sense, she has 'proved' symbolically and formally the relationship she had earlier found by manipulation of the concrete materials.

In section 4, we see Jem, Sara's fellow pupil, in the role of instigator of an intervention. Jem at this juncture is talking in the language of the materials and referring to her physical inability to cover 'woods' with 'greens'. She has, in fact, covered an area of paper with tessellated 'woods' and built over this a tessellation of 'greens', many of the 'woods' sticking out from under the edges of the 'greens'. One could infer that Jem's intention was to provide a validating intervention. She probably assumed that Sara would have a legitimate solution, albeit the negative one given in her first response to the teacher. Sara, however, is no longer satisfied with her own null solution and recalls the original construction suggested by the teacher. She does not reconstruct it, but talks, one could say algebraically, producing as she says "mathematically" a solution that cannot be physically represented. She is not content to leave it there, however,and folds right back to image making again, building new figures, intent on finding a solution for the 'woods'. The intervention has been an invocative one for Sara. She creates a couple of new constructions before finding a way of calculating the area of a 'wood', returns to a formalised level of working, and again 'proves' the physical solution - to a sceptical Jem!

## Summary

What is the nature of the dynamics in the growth of mathematical understanding? What is the role of teaching interventions in this growth? Our theory has prompted us to posit three kinds of interventions - provocative, invocative and validating - which play a role in fostering the growth of mathematical understanding. Reflection on the incident given above suggests the power of our concepts of intervention in analyzing, elaborating and understanding the complex interplay between student and teacher in a mathematics classroom. That these are not isolated phenomena will be illustrated in our analysis of other incidents in our presentation. We hope that we have,indeed, shown, through the illustration of interventions and their intended and actual outcomes, that from the point of view of the student's understanding, "the answer determines the question".

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# ENCODING DIFFICULTY: A PSYCHOLOGICAL BASIS FOR 'MISPERCEPTIONS' OF RANDOMNESS[1]

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## Abstract

*Subjects' ratings of the apparent randomness of ten binary sequences were compared to the time required to memorize those same sequences. Memorization time proved a better predictor of the subjective randomness ratings than measures of the "objective" randomness of the sequences. This result is interpreted as supporting the hypothesis that randomness judgments are mediated by subjective assessments of encoding difficulty. Such assessments are seen as compatible with the information theorists' interpretation of randomness as complexity.*

Take a look at the two sequences below. Which sequence, [1] or [2], appears to be the most random?

OXOXOXOXOOOXXXXOXOXOO [1]

OXOXXXXXOXOOOOXOOOXXO [2]

Many will take objection to this question, and understandably so. A recent article by Ayton, Hunt, and Wright (1989) along with a set of published responses in the same journal (Vol. 4, 1991), include a range of arguments for those interested in exploring the debate about the meaning, theoretical status, and psychological investigations, of 'randomness'. We cannot address those issues here.

---

[1] To be presented at the Sixteenth International Conference for the Psychology of Mathematics Education. Durham, New Hampshire (August, 1992). This research was supported, in part, by grant MDR-8954626 from the National Science Foundation to Clifford Konold and by the Sturman Center for Human Development, The Hebrew University, Jerusalem. Opinions expressed are those of the authors and not necessarily those of the sponsoring agencies.

Mathematically, [1] and [2] have the same probability ($.5^{21}$) as any other ordered sequence of the same length of being randomly produced by, for example, flipping a fair coin. On this basis, they could be judged equally random. However, if we consider various attributes of sequences, more of the possible sequences are *like* [2] than they are like [1]. In this sense, [2] might be considered more characteristic of a random process than [1].

One such attribute is the probability of alternation between the two symbols. For every finite binary sequence, we can determine the relative frequencies of the two symbols and the conditional probability of change (or continuity) after a given character in the sequence. Given a sequence length of n, there are n-1 opportunities for a change in symbols. (All but the first character in a sequence can differ from a preceding character). The probability of alternation in a particular sequence, denoted P(A), is obtained by dividing the number of actual changes of symbol-type by n-1. The values of P(A) for [1] and [2] above are 0.7 and 0.5, respectively. When the probabilities of the two symbols are equal, the value of P(A) in large, random samples will tend toward 0.5. This result follows from the principle of independence — regardless of what has already occurred in the sequence, the probability that the next character differs from the previous one is 0.5. Sequences with values of P(A) other than 0.5 occur with less frequency. Additionally, deviations from that modal value are equally probable in the two directions. Thus, sequences with P(A) = 0.7, which contain more alternations than expected, have the same probability of occurring as sequences with P(A) = 0.3, in which there are fewer alternations (longer runs) than expected.

Sequence [2] is considered more random than [1] also from the perspective of information theory. Randomness, in this account, is defined as a measure of *complexity* (Chaitin, 1975; Fine, 1973, chap. 5). Despite the sophisticated computations used in information theory, the notion of randomness as complexity is straightforward: a random sequence is one that cannot be significantly shortened via some coding scheme. This notion can be illustrated with even a simplistic coding convention. For example, the

X O X O X O X O X O X O X O X O X O X O X [3]

perfectly alternating series above can be coded as 10XO 1X (10 repetitions of XO followed by 1 X). By forming the ratio of the number of characters in the *code* (where *10* is considered as one character) to the number in the *sequence*, we can express the complexity (or compressibility) of this sequence as 5/21 = 0.24. Using the same convention, [1] would be coded as 4OX 3O 4X 2OX 2O, for a complexity measure of

12/21 = 0.57. There are 18 characters in the code for [2], only slightly fewer than the 21 original characters; its complexity measure of 18/21 = 0.86 is much nearer the maximum value of 1. Using this coding scheme, [2] would be considered more random than [1], because compared with [1], it cannot be substantially compressed.

Despite mathematical reasons for considering [2] more random than [1], research has shown that most subjects hold to just the opposite. In selecting random sequences, people prefer sequences that include more alternations than typically occur (Falk, 1975, 1981; Wagenaar, 1972). The well-known gambler's fallacy, according to which tails is considered more probable than heads after a run of successive heads, may also be based on the belief that symbols in a random sequence should frequently alternate.

Kahneman and Tversky (1972) have explained these results by suggesting that people rely on error-prone "heuristics." In their account, the judgment that [1] is more random than [2] is based on an incorrect expectation that even small random samples will resemble their parent population (Tversky & Kahneman, 1971). [2] is judged less random because it contains longer runs (e.g., XXXXX) which do not capture or represent the equal distribution of symbols in the population. Random sequences, because they are random, must also avoid obvious patterns. The perfectly alternating [3] is accordingly judged to be less random than [1]. For a sequence to be considered maximally random, it must strike a balance between avoiding simple alternating patterns and maintaining a near equal number of symbol-types in any of its segments.

In the account summarized above, human judgments of randomness are based on the notion of *similarity*. Features of a sample are compared to the corresponding features of a population, and the more similar a sample is to a population, the more likely it is to have come from that population. Our research was designed to investigate an alternative hypothesis — that peoples' perceptions of randomness are based on assessments of *complexity*.

People might assess the complexity of a sequence by gauging how difficult that sequence would be to encode. We frequently are given information that must be copied or memorized. If that information can be reorganized into meaningful "chunks" (cf. Miller, 1965), it can be more efficiently memorized or copied. Chunking is obviously a way of compressing data. Therefore, assessments of "chunkability" are also judgments about difficulty of encoding. We are suggesting that people might make use of this type of assessment in judging the randomness of a sequence.

In the study reported here, we used the time required to memorize a sequence as a measure of encoding difficulty. We compared these times with results from prior research in which subjects rated the perceived randomness of the same sequences. If randomness judgments are rooted in assessments of complexity, we would expect that those sequences which were hardest to memorize would be perceived as most random. Such results would provide evidence of a psychological basis for people's "misperceptions" of randomness — that from the standpoint of human perception, sequences of P(A) = 0.6 *are* more complex, or difficult to encode, than sequences of P(A) = 0.5, and for this reason they are judged as more random. Furthermore, if people base their randomness judgments on the difficulty of encoding, the complexity definition of randomness might prove to be an intuitively compelling introduction to the concept.

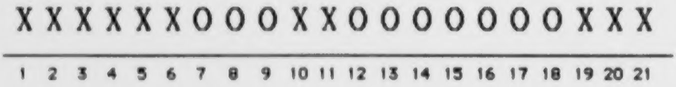
## Method

### Randomness Ratings

Data concerning apparent randomness were obtained in prior research by Falk (1975, 1981). Subjects were shown a set of 10 sequences, which included [1], [2] and [3]. These sequences were of length 21, and comprised two symbols whose frequencies differed by 1. The P(A)s of these sequences ranged from 0.1, 0.2, 0.3 . . . to 1.0. Subjects rated each sequence on a scale that ranged from 1 (not at all random) to 20 (perfectly random). Ratings were obtained from 219 subjects.

### Memorization Task

Ten different subjects were individually presented with the same sequences as were used in the rating task. The sequences were presented as shown in Figure 1 on a Macintosh computer.



| X | X | X | X | X | X | O | O | O | X | X | O | O | O | O | O | O | O | X | X | X |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |

*Figure 1.* Screen display of target sequence P(A) = 0.2

Subjects were instructed to study each sequence until they could reproduce it from memory. When a subject was ready to attempt recall, he or she hit the "return" key. This caused the target sequence to be masked. The subject could then enter a "response" sequence in a field provided on the screen, as shown in Figure 2.

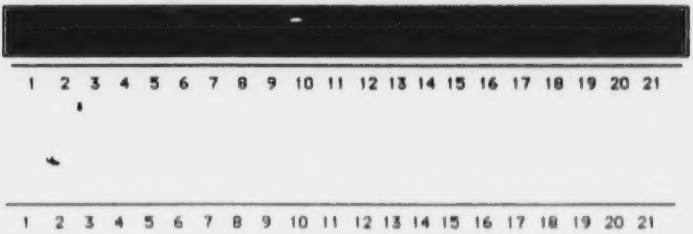


*Figure 2.* Screen display showing masked target sequence (above) and field for entering response sequence (below).

After entering a response sequence, the subject again hit the return key. If the response sequence was correct, both the target and response sequences were displayed together. The next target sequence could then be displayed by clicking on a "next" button. If the response sequence was incorrect, it disappeared, and the target sequence was again displayed. Subjects continued until they were able to enter the correct sequence.

Subjects were told the computer was recording the total time the target sequence was displayed. They were also told that time spent entering a response sequence was not being recorded and were shown how to use the delete key, which permitted editing a response sequence up to the time the enter key was depressed. They were instructed that the objective was to memorize the sequence as "efficiently" as possible, trying to minimize total viewing time.

The order of presentation of the ten sequences was randomly determined for each subject by the program. These ten experimental sequences were preceded by four practice sequences. The practice sequences had P(A)s of 0.2, 0.9, 0.5, and 0.3 and were always presented in that order. The subjects were not informed that these were practice sequences.

## Results

### Randomness Ratings

The randomness ratings (denoted AR, for "apparent randomness") for each sequence were averaged over the 219 subjects, and then linearly transformed to range from 0 to 1. Figure 3 shows these averages plotted as a function of P(A). This function peaks at P(A) = 0.6 and is negatively skewed.

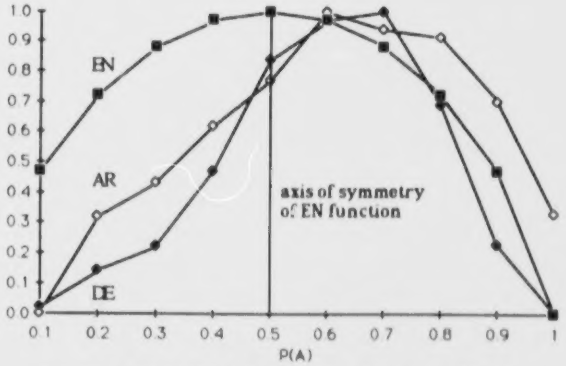


*Figure 3.* Plot of EN, AR, and DE as functions of P(A).

For comparison purposes, Figure 3 includes values obtained from an "objective" measure of randomness based on the "second-order entropy" (EN) of the sequences (see Attneave, 1959, pp. 19-21). This function peaks at 0.5, and is symmetric[1] around P(A) = 0.5. As reported in the introduction, these data indicate that subjects select as most random, sequences that include more than the expected number of alternations. Indeed, these subjects tended to rate sequences with P(A)s of 0.6, 0.7, and 0.8 as more random than the objectively most random sequence of 0.5.

**Memorization Task**

The times required to memorize each sequence were first standardized for each subject. For each P(A), we computed the mean of the standard scores over the ten subjects, and then linearly-transformed these to range from 0 to 1. This value, which is our measure of encoding difficulty (DE), is also plotted in Figure 3. The function of DE peaks at P(A) = 0.7.

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[1] Since in the family of sequences we used there is a sequence with p(A) = 1, but not one with P(A) = 0, the function in Figure 3 is not entirely symmetric.

If encoding difficulty mediates judgments of randomness, than we should expect measures of encoding difficulty to be better predictors of the subjective randomness ratings than are measures of objective randomness. Indeed, the correlation between DE and AR is .89, whereas the correlation between EN and AR is .54. In addition, difficulty of encoding, which was hypothesized to account for subjective randomness, is better correlated with AR (.89) than it is with objective randomness (.71) .

## Conclusion

The data presented here offer some support to the hypothesis that judgments of randomness are mediated by subjective assessments of complexity, an assessment that may be accomplished by judging how difficult the sequence would be to encode. The results of the memorization task are preliminary in that they involve only ten subjects, and these were not the same subjects who provided the randomness ratings. We are currently conducting a larger study in which subjects first rate the randomness of various sequences, and then either memorize or copy those same sequences. The copying task allows subjects to enter a sequence in "chunks," copying only what they can easily remember , thus reducing demands on short-term memory.

Though preliminary, our findings do suggest that human judgments of randomness are based in part on the formally sound criteria of complexity. Such a finding could have important implications for instruction. For example, introductions of randomness as a blind process of selection, or as statistical independence, may be difficult to comprehend because students lack prior intuitions into which these ideas can be integrated. Our results suggest that an interpretation of randomness as complexity may have more intuitive appeal to students, and therefore may provide the basis on which an initial understanding of randomness can be constructed.

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# EXPLORING BASIC COMPONENTS OF THE PROCESS MODEL OF UNDERSTANDING MATHEMATICS FOR BUILDING A TWO AXES PROCESS MODEL

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## ABSTRACT

*The purpose of this study is to make clear what kind of characteristics a model of understanding mathematics should have so as to be useful and effective in mathematics education. The models of understanding presented in preceding papers are classified into two large categories, i.e. "aspect model" and "process model". Focusing on the process of understanding mathematics, reflective thinking plays an important role to develop children's understanding, or to progress children's thinking from a level to a higher level of understanding. As a theoretical framework, a process model consisted of two axes is presented for further studies. The vertical axis in the model implies levels of understanding and the horizontal axis implies learning stages. At any level of understanding, there is three stages, i.e. intuitive, reflective and analitic stage.*

## I. INTRODUCTION

The word "understanding" is very frequently used in the descriptions of aims of teaching mathematics in the Course of Study (Ministry of Education, 1989) and in the teaching practices of mathematics in Japan. The putting emphaasis on children's understanding should be desirable in mathematics education, but what it means is not clear. Moreover, It is an essential and critical problem that what mathematics teachers should do to help children understand mathematics and develop their understanding have not been sufficiently made clear.

The key to the solution of these problems, in my opinion, is ultimately to capture what does it mean children understand mathematics and to make clear the mechanism which enables children's understanding of mathematics develop in the teaching and learning mathematics. In other words, it might be said to "understand" understanding. It is, however, not easy and we need our great effort to do it. In fact, as Hirabayashi (1987) describes, the American history of researches in mathematics education seems to be the struggling with interpretations of understanding. The problem of understanding is still a main issue buckled down by some researchers, especially from the cognitive psychological point of view in PME. As a result of their works, various models of understanding as the frameworks for describing aspects or processes of children's understanding of mathematics are presented (Skemp, 1976, 1979, 1982; Byers and Herscovics, 1977; Davis, 1978; Herscovics and Bergeron, 1983, 1984, 1985, 1988; Pirie and Kieren, 1989a, 1989b).

The purpose of this study is to make clear what kind of characteristics a model of understanding should have so as to be useful and effective in mathematics education.

In order to achieve this purpose, in this paper, the preceding researches related to models of understanding mathematics are summarized and the fundamental conception of understanding mathematics is described. Then, basic components substantially common to the process models of understanding mathematics are discussed. Finally, I present as a theoretical framework a process model consisted of two axes, what is called "a two axes process model" .

## II. FUNDAMENTAL CONCEPTION OF UNDERSTANDING MATHEMATICS

What do we mean by understanding? According to Skemp (1971), to understand something means to assimilate it into an appropriate schema (p.43). Haylock (1982) answers this question in the following: a simple but useful model for discussing understanding in mathematics is that to understand something means to make (cognitive) connections (p.54). These explanations of understanding are (cognitive) psychological and imply that to understand something is to cognitively connect it to a previous one which is called a schema or a cognitive structure. We could say that a schema or cognitive structure is a model of a nerve net in the brain of our human beings. In this sense, to understand something is substantially an individual internal (mental) activity.

Moreover, comparing the Piagetian cognitive structures with the Kantian schemata and categories, Dubinsky and Lewin (1986) describe that the Piagetian cognitive structures are constructed from the outset and undergo systematic changes of increasing differentiation and hierarchic integration (p.59). This suggests us that the understanding defined above is not such a static activity as all-or-nothing but a complex dynamic phenomenon which could change in accordance with the construction and reconstruction of cognitive structures.

Therefore, accepting such fundamental conception of understanding mathematics as an internal (mental) dynamic activity, we necessarily need some methods to externalize children's understanding of mathematics. A retrospective method, an observation method, an interview method, and a combination of these methods are promising and useful methods for externalizing it. It is, however, almost impossible for us to see directly understanding as the mental activity. Therefore, we need some theoretical framework. According to the definition of model by Gentner (1983), the theoretical framework for making clear aspects or processes of understanding mathematics could be called a model which has a mental activity of understanding as its prototype. In that sense, any model is indispensable for making clear understanding and the significance of building a model can be found in this point.

As already mentioned in the previous section, the various models of understanding mathematics are presented in the preceding papers. These models are, for example, including a discrimination of "relational and instrumental understanding" (Skemp, 1976), "a tetrahedral model" (Byers and Herscovics, 1977), "a 2•3 matrix model" (Skemp, 1979), "a 2•4 matrix model" (Skemp, 1982), "a constructivist model" (Herscovics and Bergeron, 1983), "a two-tiered model" (Herscovics and Bergeron, 1988) and "a transcendent recursive model" (Pirie and Kieren, 1989b). As Pirie and Kieren (1989b) point out, models can be classified into two large categories. The one is "aspect model" which focuses on the various kinds of understanding and the other is "process model" which focuses on the dynamic processes of understanding. The models presented in Skemp (1976, 1979, 1982) and Byers and Herscovics (1977) belong to the former and the models in Herscovics and Bergeron (1983, 1988) and Pirie and Kieren (1989) belong to the latter. We need both aspect model and process model to develop·children's understanding in mathematics education. They seem to be build mainly to describe the real aspects and processes of children's understanding and are very useful for us to grasp them.

It is, however, not sufficient to describe the real aspects or processes of children's understanding. Because mathematics education in its nature should be organized by both teaching activity and learning activity. Therefore, a model of understanding which is useful and effective in the teaching and learning mathematics should have prescriptive as well as descriptive characteristic. Namely, the model is expected to have the prescriptive characteristic also in the sense that it can suggest us didactical principles regarding to the following questions. What kind of didactical situations and how them should we set up to help children understand mathematics? Which direction should we guide children in developing their understanding of mathematics?

## Ⅲ. BASIC COMPONENTS OF PROCESS MODEL

In order to build such a model of understanding, we must elucidate the processes of children's understanding in mathematics. In this section, focusing on a process model, we explore basic components of it. For theoretically exploring them, we examine process models of understanding (Herscovics and Bergeron, 1983, 1988; Pirie and Kieren, 1989) and a model of learning mathematics (van Hiele and van Hiele-Geldof, 1958; van Hiele, 1986).

Herscovics and Bergeron have been buckling down to the difficult task of building and modifying a model of understanding in the processes of mathematical concept formation. They built "a constructivist model" of understanding mathematical concepts basing on the constructivist assumption that children will construct mathematical concepts. The constructivist model is consisted of four levels of understanding: the first one, that of intuition, a second one involving procedures, the third dealing with abstraction, and a last level, that of formalization (Herscovics and Bergeron, 1983, p.77). Then they modified this model and presented an extended model of understanding. This extended model is called "a two-tiered model", one tier identifing three different levels of understanding of the preliminary physical concepts, the other tier identifing three distinct constituent parts of the comprehension of mathematical concepts (Herscovics and Bergeron, 1988, p.15). Their fundamental conception underlying this model is that the understanding of a mathematical concept must rest on the understanding of the preliminary physical concepts (p.20).

Pirie and Kieren (1989) stress that what is needed is an incisive way of viewing the whole process of gaining understanding (p.7). And they present "a transcendent recursive model" of understanding which is consisted of eight levels: doing, image making, image having, property noticing, formalizing, observing, structuring, and inventing. Their fundamental conception of understanding underlying the model and the important characteristic of the model are succinctly and clearly represented in the following quoted passage.

> Mathematical understanding can be characterized as levelled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication. Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrainted by those without. (Pirie and Kieren, 1989, p.8)

We can see that these models of understanding are process models which have prescriptive as well as descriptive characteristic and involve some levels of understanding. There is, however, an objection to the levels of understanding. In fact, examining the Herscovics and Bergeron model for understanding mathematical concepts, Sierpinska (1990) describes that therefore what is classified here, in fact, are the levels of children's mathematical knowledge, not their acts of understanding (p.28). This criticism is based on the different notion of understanding that understanding is an act (of grasping the meaning) and not a process or way of knowing. It is worth notice but in my opinion there must be some levels, even if those are levels of children's mathematical knowledge, in the processes of children's understanding of mathematics. The process model of understanding mathematics should involve some hierarchical levels so as to be useful and effective in the teaching and learning mathematics.

The hierarchy of levels of understanding can be typically seen in a transcendent recursive model illustrated in Figure 1 (Pirie and Kieren, 1989, p.8). It remainds us of the van Hieles' theory of levels of thinking in learning geometry which was presented in their doctoral dissertation (cf. van Hiele and van Hiele-Geldof, 1958). In the theory five hierarchical levels of thinking are identified and five learning stages for progressing thinking from a level to a higher level are involved (van Hiele, 1986). We notice that these models are very similar to each other in two respects. The one is levels themselves set up and the other is the idea of progressing from a level to a outer (higher) level.

The first similarity can be recognized more clearly by illustrating the van Hiele model in Figure 2 (Koyama, 1988). In fact, ignoring somewhat difference in the scope and domain of learning mathematics, each two levels indicated by a thick circle in Figure 1 could be corresponded to each level in Figure 2 respectively;

(Doing, Image Making) ←——→ (Concrete Object•, Geometrical Figure)
(Image Having, Property Noticing) ←——→ (Geometrical Figure•, Property)
(Formalizing, Observing) ←——→ (Property•, Proposition)
(Structuring, Inventing) ←——→ (Proposition•, Logic)

[Note: The sign ←→ indicates the correspondence between levels and the sign • indicates a object of thinking in each level.]

The second similarity is more important in a process model than the first, because it is concerned with the crucial idea of developing children's understanding of mathematics. The idea of developing children's understanding in the Pirie and Kieren model is "recursion", whereas in the van Hiele model it is "objectification or explicitation". These ideas seem to be substantially same and might be said in other

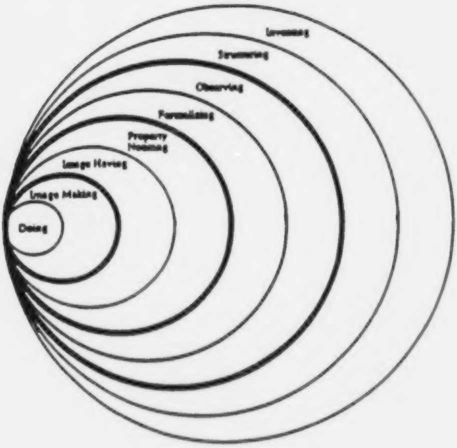
Inventing
Structuring
Observing
Formalising
Property Noticing
Image Having
Image Making
Doing

Fig. 1

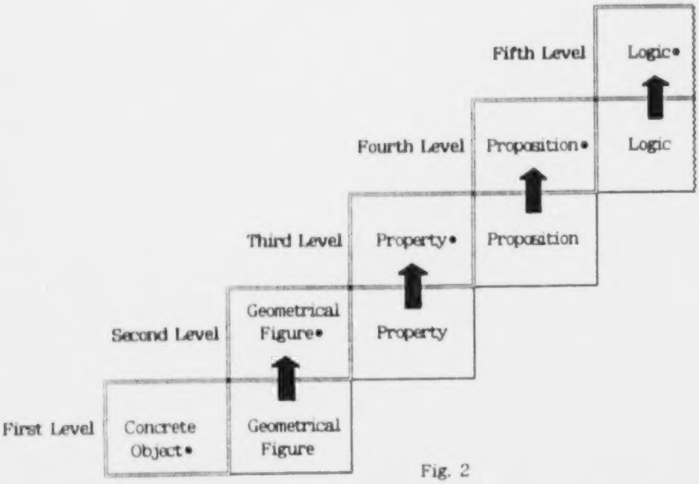
Fifth Level
Logic•
Logic
Fourth Level
Proposition•
Proposition
Third Level
Property•
Property
Second Level
Geometrical Figure•
Geometrical Figure
First Level
Concrete Object•

Fig. 2

[Note: The sign ⬆ indicates the objectification of a way of unconscious thinking.]

words reflective abstraction or reflective thinking. We can say that in the processes of understanding mathematics reflective thinking plays an important role to develop children's understanding, or to progress their thinking from a level to a higher level of understanding. Therefore these models suggest us that a process model should have learning stages involving reflective thinking.

After all, we identify such two basic components of a process model as hierarchical levels and learning stages. In the next section, a process model with these two basic components is presented as a theoretical framework for developing children's understanding in the teaching and learning mathematics.

## IV. A TWO AXES PROCESS MODEL

In order to build a process model which can prescribe as well as describe how the process of children's understanding of mathematics should progress, we must give serious consideration to the following questions. What levels should children's understanding progress through? How should children develop their thinking in any level of understanding? Relating to the first question, as already discussed, levels involved in the Pirie and Kieren model and the van Hiele model can be regarded as answers to it. Although we need to examine those levels and modify them in accordance with mathematical concepts intended in the teaching and learning mathematics, they form a vertical axis of the process model of understanding.

Relating to the second question, learning stages involved in the van Hiele theory (van Hiele, 1986) and in the Dienes theory (Dienes, 1960, 1963, 1970) are very suggestive. On the one hand, in the van Hiele theory five stages in the learning process leading to a higher level are discerned; information, guided orientation, explicitation, free orientation, and integration (van Hiele, 1986, pp.53-54). On the other hand, in the Dienes theory six stages in the mathematics learning are set up basing on four principles, the dynamic, constructivity, mathematical variability and perceptual variability principle (Dienes, 1960, p.44); free play, rule-bound play, exploration of isomorphic structure, representation, symbolization and formalization (Dienes, 1963, 1970). The stages in two models can be roughly corresponded like the followings; information to free play, guided orientation to rule-bound play, explicitation to exploration of isomorphic structure and representation, free orientation to symbolization, and integration to formalization respectively.

According to Wittmann's idea (1981), these corresponding stages are classified into three categories. He emphasizes that three types of activities are necessary in order to develop a balance of intuitive, reflective and formal thinking, basing on the assumption that mathematics teaching should be modelled according to the processes of doing mathematics (Wittmann, 1981, p.395). I modify his definitions of three activities a little in order to form a horizontal axis of the process model. At any level of understanding, there is three stages, intuitive, reflective, and analitic stage.

*Intuitive stage*: Children should be provided opportunities for manipulating concrete objects, or operating mathematical concepts or relations acquired in a previous level. At this stage they do intuitive thinking.

*Reflective stage*: Children should be stimulated and encouraged to pay attention to their own manipulating or operating activities, to be aware of them and their consequences, and to represent them in terms of diagrams, figures or languages. At this stage they do reflective thinking.

*Analitic stage*: Children must elaborate their representations to be mathematical ones using mathematical terms, verify the consequances by means of other examples or cases, or analize the relations among consequances in order to integrate them as a whole. At this stage they do analitical thinking and at the end they could progress their understanding to a next higher level.

Through these three stages, not necessarily linear, children's understanding can progress from a level to a higher level in the teaching and learning mathematics. As a result, a process model of understanding consisted of two axes, what is called "a two axes process model", can be led theoretically. In a two axes process model the vertical axis is formed by some hierarchical levels of understanding and the horizontal axis is formed by three stages in any level.

## V. BY WAY OF CONCLUSION

A model of understanding mathematics should have prescriptive as well as descriptive characteristic so as to be useful and effective in mathematics education as the integration of teaching and learning activities. We explored, basing on the assumption, basic components common to the process models presented in preceding researches and two basic components, i.e. hierarchical levels of understanding and learning stages for developing, could be identified. By using these components as its two axes, I built theoretically a two axes process model to elucidate the process of children's understanding in mathematics education.

The validity of this model can be assured indirectly to some extent by corroborative evidences in the preceding researches related to models of understanding. But the model is a theoretical one and a means to an end. Therefore, by using this model, to grasp the real processes of children's understanding in the teaching and learning certain mathematical concepts and to elabolate or mofify it is left as an important task.

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# POWERFUL TASKS: CONSTRUCTIVE HANDLING OF A DIDACTICAL DILEMMA

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*Mathematics instruction contains two conflicting demands: on the one hand the demand for economical efficiency and for well developed "motorways", and on the other hand the dema[illegible] that pupils should investigate and discover for themselves and should have the freedo[illegible] pave their own ways. It is argued that tasks with a certain richness and quality help to take some steps towards a constructive handling of this dilemma. Some examples of "powerful tasks" are considered and two pairs of properties are worked out. Much attention is paid to the social dimension.*

## Background

Discussing the importance of tasks in mathematics education has a long tradition. A detailed analysis of the so called *Task Didactics* (Aufgabendidaktik), which is one of the marked features of *Traditional Mathematics*, is given by Lenné (1969). *Traditional Mathematics* was the leading stream of mathematics education in Germany (and in a similar way in Austria) till the middle of this century, and then was progressively displaced by *New Mathematics* (which in turn is being pushed back more and more). *Task Didactics* is characterized by a partition of the mathematical subject-matter into specific areas (e.g. fractions, percentages, triangles, quadrilaterals). Each area is determined by a special type of task which was systematically treated progressing from simple to more complex tasks (combination of simple tasks). Cross connections (e.g. regarding fundamental ideas or structures) are not been worked out in detail. In general, the teacher taught theories and methods and the pupils had to apply them by solving tasks. How much has this situation changed?

Recent empirical research studies, like those of Bromme (1986) or Clark/Yinger (1987) show that even nowadays mathematics teachers plan and organize their instruction on a large scale with the help of tasks.

Research with regard to tasks takes different directions:

- There are many contributions to *general considerations* about tasks, for example: Wittmann (1984) views *teaching units* as the *integrating core of mathematics education*, incorporating mathematical, pedagogical, psychological and practical aspects in a natural way and therefore being a unique tool for integration. For Christiansen/Walter (1986) *task and activity* are basic didactical categories whereby tasks can yield an adequate reduction of the complexity of the interaction between teachers and learners. Bromme/Seeger/Steinbring (1990) stress the double-character of tasks as demands on teachers and learners by presenting different findings of empirical studies. Krainer (1991a) shows that tasks are *elementary building stones of didactical thinking and acting* with relevance not only in instruction but also in research and communication.
- A starting point of the *problem solving* movement (see ZDM, 1991) was Polya's book *How to solve it* (1945), distinguishing *routine-tasks* and *nonroutine-tasks* (later called *problems*). Questions concerning how heuristic strategies could be taught and how the teaching of them could help pupils become better problem solvers dominated the research.

especially in the 1960's and 1970's. Since the 1980's it became clear that problem solving research should include questions about the learner's prior mathematical experiences, his mathematical knowledge and beliefs, his needs and motivation (e.g. Schoenfeld, 1985; McLeod and Adams, 1989). In the last years more and more attention is being paid to studies and teaching experiments where problem solving is used as a teaching method. One example is the *open-approach method* by Nohda (1991) whose aim it is to foster simultaneously both the creative thinking of the students and mathematical activities. The fact that teaching by open-approach as a rule is done only once a month indicates difficulties of integrating *open problem solving* within the framework of normal teaching.

- Another direction of research deals with *systems of tasks*. Von Harten/Steinbring (1985) demand that tasks within such a "system" should refer to a common topic but should be related in the form of analogies. They discuss the features of systems of tasks in regard to probability instruction. Krainer (1990, see also 1991b) develops a system of tasks (with 69 tasks) for the concept of angle as a creative mode of organization for a *vivid geometry* (stressing a reflected relation between theory and practice).

This paper tries to develop some ideas towards a new culture of constructing and handling of tasks. First, a central dilemma in mathematics education is described. By means of some examples of *powerful tasks* it is argued that working with such tasks of a certain richness and quality is a flexible way of managing this dilemma. *Two pairs of properties* of powerful tasks are discussed and some links to other studies are made.

## 1. A central dilemma in mathematics instruction

How should mathematics instruction be organized? This is one of the most important questions to be dealt with in mathematics education.
There are two extreme answers to the above question:

- Mathematics is a highly complex and highly developed science which, however, in areas understandable for pupils offers polished and stable ideas and theories. Therefore, it is easy to build up well-established ("secured") courses for mathematics (*background theories* etc.).
- Pupils bring a variety of relevant and practical experiences, associations, intuitions etc. to mathematics instruction. If the spontaneity and creativity of the pupils is taken seriously it is - from a psychological point of view - necessary to have a certain *insecurity* of mathematics courses (formulated after Wagenschein; see Lenné, 1969, p. 65).

These responses constitute a dilemma in mathematics instruction:

| *security* | | *insecurity* |
|---|---|---|
| | versus | |
| of mathematics courses | | of mathematics courses |

This dilemma cannot be resolved by a didactical theory. The situation ever remains conflicting because both extremes embody meaningful demands: on the one hand the demand for economical efficiency and for well developed "motorways", and on the other hand the demand that pupils should investigate and discover for themselves and should have the freedom to pave their own ways.

For centuries it has been common to organize mathematics instruction in duplicating "ready" mathematical theories (as *Euclid's Elements* in geometry). Nowadays another extreme, namely to organize mathematics instruction with the help of open projects, is seen as an idea not to be excluded from the beginning by more and more people. After the wave of *New Math* - which can be seen as a step towards security - the pendulum seems to swing into the other direction. Notions like *problem solving* (Polya, Schoenfeld), *genetic method and operative method* (Wagenschein, Piaget, Aebli), *didactical phenomenology* (Freudenthal), *subjective fields of experiences* (Bauersfeld), *intuitions* (Fischbein), *discovery learning* (Winter) etc. indicate this pendulum swing, which in part is accompanied by constructivist views. All of these didactical programmes take into account the complexity of mathematics instruction and stress the active confrontation of pupils with mathematics.

In the following we try to take some steps towards a constructive handling of this dilemma. We start with an intuitive approach: We regard the way of one task to a *powerful task* and the interconnections of this task with two other tasks.

## 2. Three powerful tasks and their genesis

Let us try to put ourselves in the place of a teacher who wants to begin with the topic "functions" in the near future. He looks for ideas for an introduction which on the one hand leads directly to the mathematical subject-matter but on the other hand leaves scope for pupils' creative productions. Concerning this dilemma he has an interesting discussion with a good colleague. She gives him the advice to glance through the script "Sketching and Interpreting Graphs" (SHELL CENTRE). At home he reads this script and immediately the idea "Hoisting the Flag" (p. 6) attracts his attention because he remembers that it was his class that constructed the crank-handle for the school's flagpole. So his pupils would already have a lot of pre-experiences and (positive) associations.

Now to the text of "Hoisting the Flag" (incl. fig. 1):

*"Every morning, on the summer camp, the youngest boy scout has to hoist a flag to the top of the flagpole.*

*(i) Explain in words what each of the graphs below would mean.*

*(ii) Which graph shows this situation most realistically?*

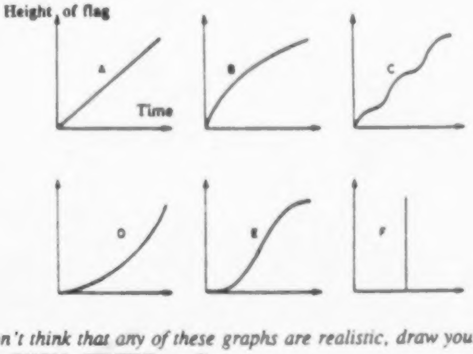

fig. 1

*If you don't think that any of these graphs are realistic, draw your own version and explain it fully." (SHELL CENTRE, p. 6).*

The teacher likes this task but he wants to modify it, especially question (ii): it depends on the specific situation which graph is more or less "realistic". He exchanges graph F for another one (fig. 2).

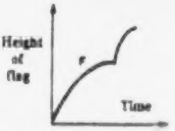

fig. 2

The new graph F refers to a situation in which the boy apparently made a mistake while hoisting the flag, resulting in a short-timed sinking of the flag. (Foremost in the teacher's mind is a similar mistake which happened to him in the schoolyard in front of the whole class. What a yelling will arise when they recognize the "disgrace-graph" of their math teacher! And his hidden triumph in turning now this case with gentlemanly ease into a mathematical problem!)

The teacher's modified task contains the following text (with the same graphs as before, only graph F replaced by the "disgrace-graph"):

*The 5A class has assembled in the schoolyard testing their self-constructed crank-handle for the school's flagpole. As usual, the math-teacher tries to take a mathematical view on this situation: he draws a system of coordinates and describes the attempts of six persons graphically.*
*Describe in a few lines, how (with what method) the six persons have hoisted the flag! Are there other realistic graphs possible? Compare your results with those of your neighbour! Finally formulate what you have learnt from this!*

The teacher writes down some preliminary considerations for the lesson:

- *fine warming up exercise for "interpreting graphs" (surprise!)*
- *connection of real world situations with mathematics*
- *first feeling for properties like linear/non linear, increasing/decreasing, gradient, curvature, etc. (without defining these concepts in that situation)*
- *with a system of co-ordinates we can describe connections between quantities (here: time and height).*
- *these graphs intentionally contain no numbers, qualitative considerations are in the foreground. It is possible to describe things without numbers as the following short characterization of the graphs A - F shows:*
  A: *Constant hoisting (possibly automatism)*
  B: *Quick beginning but the forces wane*
  C: *Succession of hoisting and breaking off (possibly alternating hoisting with the right and the left hand)*
  D: *Slow beginning, then steady increase of strength*
  E: *Slow beginning, then increase of strength, but gingerly at the end*
  F: *Quick beginning followed by a short sinking of the flag, finally managing it without problems.*

The math-teacher knows that this task is only one building-stone towards an encouraging introduction to "functions". He decides to construct two further tasks, both interconnected with the former one.

Concerning the "inner logic" of this system of tasks he primarily pays attention to three aspects:

- Development from task 1 to task 3: interpreting (given) graphs - sketching and interpreting graphs - solving problems with the help of a graph (function).
- First only qualitative considerations, then more and more quantitative ones.
- Varying variables (not only length and time).

For a second task the teacher uses a further graph ("Using petrol") out of the script from SHELL CENTRE (p. 11), which he modifies. The pupils have to think of a car ride and to sketch a graph which shows the connection between the driven kilometers and the amount of petrol left in the car. Then they have to find interpretations for their neighbours' graphs and to discuss their results in pairs.

It is not possible to mention the variety of interesting questions which may arise out of this task. Let's only show one possible graph (fig. 3) sketched by pupils (try to find a suitable story!).

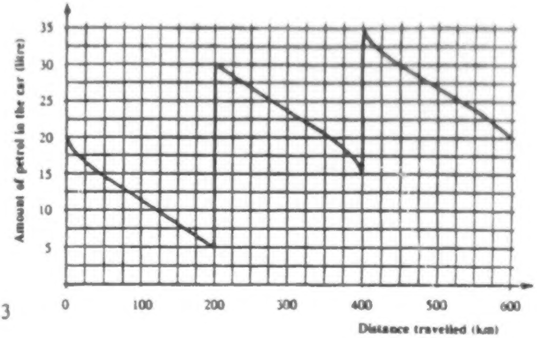

fig. 3

The third task is one of the favourite tasks of the teacher: he often uses it when introducing "functions". He has made good experiences with it, especially because of the big scope for pupils' acting, reflecting, discussing, etc. The task "How long is the shadow?" deals with the connection between the angle of inclination of a rod and the respective shadow caused by an electric torch (fig. 4).

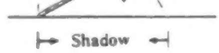

fig. 4

The teacher likes experiments in his lessons and therefore he only wants to tell the pupils about the idea and they should make considerations about to build up such an experiment (materials etc.) and what questions would be of interest. Maybe a "mini-project" will start: different groups of pupils work on (different) problems they have described before.

The teacher thinks that the experiences of the pupils with these three tasks will be a good feedback for planning the next lessons.

## 3. Properties of powerful tasks

All three tasks of the preceding chapter embody two conflicting aspects: on the one hand each task is suited for interrelation to other tasks and contributes with them to specific and general learning objectives. On the other hand each task contains possibilities for deepening, many of which could develop into a kind of mini-project for (groups of) pupils. Furthermore, each task is designed in such a way that much self-acting of pupils is initiated, the attention not only directed to acting but also to reflecting on one's own and the other's acting.
Such tasks with a certain richness and quality we will name *powerful tasks*, or - because of their project-like charakter - *project-tasks*.
They should be seen as more general than *problems* (non-routine tasks). *Powerful tasks* ...

- ... do not necessarily deal with *solving* something (in many cases it will be sufficient to describe or to discuss a situation etc.).
- ... contain reflexions upon possible processes (actions etc.) which might happen. They get their full *power* when the teacher integrally identifies himself with the aims of the task (often modifying the text for his purpose).
- ... are *open*, meaning that *the pupils* ask new questions and discuss them.

*Powerful tasks* do not only take into account the subject-matter but also social dimensions in instruction. We specify *two pairs of properties* as characteristic aspects of powerful tasks:

1) - *Team-spirit*: This property means that tasks should be well interconnected with other tasks. This "horizontal" connection of tasks can be seen as a contribution to the security of mathematics courses.
   - *Self-dynamics*: This property means that tasks facilitate the generation of further interesting questions. This "vertical" extension of tasks to open situations can be seen as a contribution to the insecurity of mathematics courses.

   Powerful tasks therefore embody the dilemma *security* - *insecurity* as a constituting element.
2) - *High level of acting*: This property refers to the initiation of active processes of concept formation which are accompanied by relevant ("concept generating") actions.
   - *High level of reflecting*: This property implies that acting and reflecting should always be seen as closely linked. An important aspect of reflection refers to further questions from the learners (which in their turn could lead to new actions).

   These two properties express the philosophy that learners should not only be seen as consumers but as producers of knowledge. The teachers' task is to organize an active confrontation of the pupils with mathematics.

The above considerations on the properties of powerful tasks are supported by various studies (although there are no assertions about adequate types of instruction), for example:

- Polya, for whom the most important goal of mathematics instruction is the support of the learners' faculty of thought views this realized optimally by solving nonroutine-tasks. For Polya reflection is of great relevance: the last of the four phases of his scheme is termed *looking back* (see e.g. Polya, 1945).
- Clark and Yinger (1987) consider *teaching as a design profession* and stress the importance of reflection in action as one essential activity of teachers (seen as reflective professionals) in mathematics instruction.
- Dörfler (1989) stresses, like Aebli, the complementarity of actions and relations and discusses the idea of *protocols of actions* as a cognitive tool for knowledge construction which can be seen as a suitable means for reflecting.

- Constructivistic positions (Piaget, von Glasersfeld, Maturana etc.) make it clear that simple transfer of teachers' knowledge into pupils' knowledge is an unjustified conceptualisation of learning. Their position challenges the conception of a strict partition of roles in mathematics instruction - the teacher renders theories and the pupils apply this theory (maybe in form of tasks). If pupils are seen as researchers and discoverers this seperation cannot longer be accepted, and acting and reflecting should be seen as an inseparable pair of cognitive and social (further) development.
- Bell (1991) stresses a *set of principles for designing teaching* which have many common aspects with the above mentioned properties, for example: *Richly connected bodies of knowledge are well retained, isolated elements are quickly lost* is an argument for constructing well interconnected tasks; *Scope for pupil choice and creative productions can provide both motivation and challenge at the pupil's own level* is an argument for constructing tasks with a high level of self-dynamics.

## 4. Summary and outlook

Two pairs of properties of *powerful tasks* (rich problems, etc.) have been pointed out:

1) *team-spirit* and *self-dynamics* characterizing the dilemma *security - insecurity* of mathematics courses
2) *high level of acting* and *high level of reflecting* as complementary properties guaranteeing a meaningful link between learners' actions and mathematical concepts.

There is no magic formula for solving the described *dilemma*. Depending on the decision to stress either more *team-spirit* or more *self-dynamics*, we can construct *systems of tasks* or *mini-projects*. This article presents a "little" system of tasks with some possibilities to begin mini-projects. Krainer (1991b) describes a system of 69 powerful tasks for the concept of angle and two mini-projects (concerning discovering and proving in geometry) for pupils to work on computers with 2-D-graphics.

Additionally, I want to point out some experiences with powerful tasks refering to my own teaching in school. The following remarks are not based on empirical studies and they are therefore not systematical but they can show some aspects of concrete working with powerful tasks in mathematics instruction:

- Constructing powerful tasks is no difficult task: textbooks contain a wide range of interesting tasks. In many cases it is easy to modify them in such a manner that they fit the goals the teacher wants to reach or fit the (open) activities the teacher wants to initiate.
- Reflecting upon learning objectives, possible pupils activities and possible pupils difficulties with regard to tasks gives the teacher more freedom of action: he is more flexible in reacting to pupils questions and proposals.
- Powerful tasks are valuable means for group work in mathematics instruction. On the one hand it is possible to construct interconnected tasks which cover a special part of a mathematical field as shown with the *Grand Prix of Integralopolis* (Krainer, 1987) in the case of Calculus. On the other hand powerful tasks as mini-projects give pupils the opportunity to work on different levels, to compare their problem solving strategies and (provisional) results.
- A very important thing with regard to working with powerful tasks is adequate communication among the pupils. Possible activities may be: mutual asking and explaining, common reflecting upon the learning process, making competitions (e.g. inventing new tasks which the others have to resolve), thinking of difficult questions for the teacher, etc.

It is assumed that increasing importance of *powerful tasks* in mathematics education gives teachers more encouragement to commit themselves to didactical research. This also is a valuable contribution to getting theory and practice of mathematics education closer to each other. But this is no easy *task*: we should think in *systems* as well as in *projects*!

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# MEASURING ATTITUDES TO MATHEMATICS

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## Summary

*Various measures of attitudes to mathematics are described in this paper. The results obtained from different self-report instruments administered to students in grade 7 are examined for consistency and compared with overt behavioral indicators.*

## Introduction

Increasingly, formal documents concerned with the teaching and learning of mathematics refer to the importance of student attitudes to the subject.

> An important aim of mathematics education is to develop in students positive attitudes towards mathematics and their involvement in it.... The notion of having a positive attitude towards mathematics encompasses both liking mathematics and feeling good about one's own capacity to deal with situations in which mathematics is involved. (Australian Education Council, 1991, p. 31)

Whether student behavior during mathematics lessons is reflected in different measures of their attitudes to mathematics is examined in this paper. As a first step, what is meant by attitudes needs to be discussed.

## Definitions of attitudes

Almost 60 years ago, Allport (1935) claimed that "attitudes today are measured more successfully than they are defined". Four decades later Fishbein and Ajzen (1975) identified more than 500 different methods of measuring attitudes when they reviewed research published between 1968 and 1970. They argued that the theoretical orientation of the investigator, as well as practical constraints, largely determined how attitudes were operationally defined and measured.

A careful reading of the various definitions allows a number of commonalities to be identified. Attitudes are learnt (the cognitive component); attitudes predispose to action (the behavioral component); these actions may be either favorable or unfavorable (the evaluative component); and there is a response consistency. Triandis (1971) captures the different nuances particularly well:

> Attitudes involve what people think about, feel about, and how they would like to behave toward an attitude object. Behavior is not only determined by what people would like to do but also by what they think they should do, that is, social norms, by what they have usually done, that is, habits, and the expected consequences of behavior. (Triandis, 1971, p.14)

## Measurement of attitudes

A large variety of different instruments has been used to measure attitudes in general, and attitudes to mathematics in particular. In recent years the unitary scales initially used to tap attitude to mathematics have typically been replaced by multidimensional measures. This development reflects the reconceptualisation among mathematics educators of attitude(s) to mathematics from a single dimensional to a multidimensional construct (Leder, 1985).

Many of the approaches used to measure attitudes to mathematics rely on paper-and-pencil, and often self-report, instruments. These do not make use of overt behavior. Commonly used scales of this type include Thurstone scales, Likert scales (used particularly frequently to tap attitudes to mathematics), Semantic differential scales, inventories and checklists, preference rankings, projective techniques, and enrollment data. Clinical and anthropological observations and physiological measures are instances of alternate measures. Examples of each approach for measuring attitudes to mathematics are given in Leder (1985; 1987).

The main aim of the present study is to examine whether student responses on various measures of attitudes to mathematics are consistent and reflect their behavior during mathematics lessons.

## The setting

As part of a larger study, data were gathered during 1991 in seven mathematics lessons - spread over eight schooldays - in a grade 7 class in a coeducational school situated in the metropolitan area of Melbourne, Australia. The students needed that time to complete the task they had been set: a project to determine the feasibility of a new tuckshop, with a group report to be submitted at the end of the time allotted.

## The sample

Particular attention was focused on one of the groups. Like most of the others, it comprised five students. Because of illness and school commitments (e.g., music lessons) not all the students were present for each of the lessons observed.

In this paper, detailed information is presented for two of the students, Chris and Carol.

## The instruments

### Self-report measures

The following instruments were administered to each student in the class before the main data collection period:

ABOUT YOU: a 25 item, Semantic Differential Scale. The instructions to the students indicated that the instrument asked 'what kind of person you think you are'. Bipolar adjectives included competitive/not competitive, messy/neat, and boring/interesting. An additional item asked if 'there are other ways in which you would describe yourself?'

ABOUT YOU AND MATHS I: this scale contained two open ended items in which students were asked whether or not they liked mathematics and to explain why they felt the way they did.

ABOUT YOU AND MATHS II: students were asked to indicate with a cross, on a line marked 'excellent', 'average', and 'poor', where they believed they fitted in. 'How good are you at mathematics?', 'How good at mathematics would you like to be?', 'How good at mathematics does your teacher believe you are?', and 'How good at mathematics would your parents like you to be?' were among the eight items in this instrument.

ABOUT MATHS LESSONS I: students were asked to 'write a description of the kind of mathematics lesson you usually have in year 7. If you like, you can include a drawing.'

ABOUT MATHS LESSONS II: this scale consisted of two parallel parts. In the first, students were asked to indicate how often (often, sometimes, rarely, never) certain activities occurred during mathematics lessons. The second part of the instrument repeated the earlier (19) items, but this time students were asked to indicate how much they would like (like, unsure, dislike) the various activities mentioned. Items included 'Teacher explains to whole class about a topic', 'having tests', and 'conducting research on your own'.

ABOUT GRADING IN MATHS: this scale contained five open ended items which asked students' opinions about various grading approaches.

At the completion of each of the lessons observed, all the students in the class were asked to complete a sheet labelled:

TODAY'S MATHS LESSON: this instrument contained seven items, including 'Circle the face which shows how you felt about your understanding of today's mathematics lesson', 'Explain briefly why you felt this way', and 'What would have helped you to understand better?'

Towards the end of the school year students were asked to complete several other instruments. Two are relevant here:

MORE ABOUT MATHS: this Likert scale contained 26 items such as 'I like doing mathematics problems which make me think hard', 'I like mathematics lessons when we can help each other work things out', and 'Luck is important for a student to be good at mathematics'.

HOW GOOD ARE YOU?: this scale was similar to ABOUT YOU AND MATHS II.

Finally, students were interviewed individually and were asked various prompt questions which encouraged them to reflect on the year's mathematics activities.

## Observational data

For each of the lessons observed, a video camera, supplemented with an additional microphone, was placed near the group of interest. This allowed the students' behaviors and conversations to be captured. These tapes were subsequently transcribed. Field notes were also kept and any incidents of interest recorded.

## Results

Results are presented separately for the different instruments. Because of space constraints, only selected findings are given.

### ABOUT YOU

Chris indicated that she saw herself as very active, independent, not very interested in what others thought of her, yet aware of other people's feelings, able to make decisions very easily, self-confident, likeable, friendly, interesting, neat, clever, ambitious, and rather outgoing. Though she indicated that neatness was not a sufficient condition to do well in mathematics she chose the following additional ways to describe herself:

> Like to always make my work neat & tidy & look good.
> Am quite patient.

For the same open ended item Carol wrote:

> I enjoy being with people.

Carol's other responses showed her to be less positive about herself than Chris. She considered herself rather shy, but believed that she was very likeable, friendly, and neat.

### ABOUT YOU AND MATHS I

Chris indicated that she liked mathematics because

> it is not something that is always easy or always hard. It is more like a challenge. I also like maths because it is a subject that you will definitely (sic) need when you go out in the real world.

For the same item Carol wrote

> I like maths because I know if I don't like it I won't be able to have a good future.

### ABOUT YOU AND MATHS II

Chris believed that she was excellent at mathematics and that others thought so too. Her teacher rated her 5 (excellent) on a scale of 1 to 5. Carol thought she was a bit above average in mathematics, but that others (teachers, parents, classmates) underestimated her performance. Her teacher in fact gave her a rating of 4.

### ABOUT MATHS LESSONS I and II

While both students wrote similar descriptions of their usual mathematics lessons, they differed in their reactions to some of the more structured items. Chris considered that she often talked about mathematics in class - with the teacher, to a friend, or in small groups - and often worked with a partner. She liked working cooperatively with others. Carol indicated that she would like to spend time working with others but claimed that currently there were few in-class opportunities for such personal collaborations.

### ABOUT GRADING IN MATHS

Chris stated that she was quite satisfied with the methods for grading (described as tests and observations of students' work) currently being used. Carol believed that there were better ways to assess students' understanding of the work:

> I don't think [the teacher] can really tell what we know from what he gives us to do.

### TODAY'S MATHS LESSON

Chris consistently indicated that she felt good during the lessons we monitored. At the end of the first lesson she described herself as not feeling sure if she had understood the work 'because it is a new project'. 'More discussion' would have been helpful. By the end of the second lesson she had 'no great worries' about the work 'because I understood the task to be done'. This feeling persisted throughout the remainder of the sequence of lessons. Carol showed herself to be more ambivalent. She circled the 'neutral' face for four of the lessons. Yet she claimed to be generally happy about her understanding of the work because 'I understood what we had to do', 'the work wasn't hard', 'we all cooperated'.

## MORE ABOUT MATHS

Chris's self-confessed enjoyment of solving hard problems, doing really well, finding new ways to solve problems, and feeling that she knew more maths than others was not shared by Carol. Chris disagreed, while Carol agreed, that mathematics lessons were enjoyable when all the work was easy.

### Observational data

The lesson transcripts and our field notes were generally consistent with the overall impressions obtained through the self-report instruments. Two sets of excerpts illustrate this.

1. *Chris's pride in doing well in mathematics and making her work "neat & tidy & look good"*

How to record the information took on increasing importance as the data the students collected accumulated. After various trials, Brian, another member of the group asked: 'Who's got the neatest writing here?'

The others agreed that Chris undoubtedly did.

Brian: (to Chris) Then you can write them all up.

Jodie: But we'll help to write the rough copies.

Chris: (to Carol) Will you go and sharpen this for us? (The video shows Carol taking the pencil and moving off.)

Throughout this and the subsequent lessons Chris not only contributed substantively to the content of the project but also clearly continued to be the main recorder.

2. *Carol's volunteered comment that she enjoyed being with people and that mathematics lessons would be more enjoyable if she could spend more time working in class with others.*

From our field notes for lesson 4:

> Today was spent preparing bar graphs to present data. Jodie appeared to be 'directing the traffic'.... She, Carol and Chris drew, then colored in the graphs. For a long period, Carol appeared to observe and do nothing. Finally, after getting up to sharpen some pencils, she began (lethargically) coloring as well. The two boys were on task the entire time discussing the survey questions and the figures involved. The girls were engaged in social chit chat as they colored.

Excerpt from lesson 5:

> Our group was split into two. The girls worked on the report. Chris and Jodie were writing, Carol was simply observing. When Carol threw in the occasional comment she tended to be ignored completely or dismissed with a short and somewhat curt retort.

Perusal of the transcript for this lesson confirmed that there were relatively few exchanges between Carol and the other four students. The following describes a 'dismissal' episode.

On the tape Jodie can be seen dictating to Chris while Carol looks on:

Carol: (to Jodie) You can't use pros and cons....

Jodie: How can... Why not?

Carol: Well, they won't know what it means....

Jodie: ...Everybody here knows what pros and cons means.... Basically, good and bad.

Carol: Oh, yes

Confirmation of Carol's limited involvement in solving the task also emerged during the interview held with Chris at the end of the year. Chris confided:

> Well, I think all of our group, um felt that, Carol ... she didn't really help as much, and we'd sort of say, oh, you know, I wish Carol would give us a bit more help. She'd just sit there and watch us doing it all. There wasn't much we could do ...

## Concluding comments

The self-report data, gathered mostly before our observations began, provided generally coherent pictures of the students' attitudes and indicated various differences in the ways Chris and Carol thought about mathematics and themselves as learners of mathematics. The lessons we monitored revealed consistencies between in-class behaviors and images evoked by the self-report measures. For example, Chris's pride in neatly presented work was acknowledged and valued by the group. During the lessons, we observed her in a variety of activities compatible with her positive attitudes about herself and mathematics. Carol's marginal position within the group was consistent with, and likely to reinforce further, her perception that others underestimated her mathematical capacities and her wish for more personal contacts in class. While we observed the group, Carol's opportunities to engage in constructive mathematical activities were severely limited by others and by herself.

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# THE FUNCTION OF LANGUAGE IN RADICAL CONSTRUCTIVISM: A VYGOTSKIAN PERSPECTIVE

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**ABSTRACT** Some critics of radical constructivism suggest that the theory seems to lead to solipsism, that is the inability of people to be able to have knowledge other than of the self, and to step outside their own minds to a social domain. In this paper, I argue that the problem is centred around the role that language plays in thought, and I suggest that since radical constructivism is rooted in Piagetian thought, language is seen as one of the external phenomena by which one achieves a 'fit'. In contrast, a Vygotskian position on language and thought places social interaction as the precursor of higher thought, and I argue that such a perspective provides more powerful interpretations of events in the classroom. In particular, I suggest that it contributes a missing dimension to research studies of the mathematics classroom interpreted in the radical constructivist paradigm.

It has been claimed that, in contrast to behaviourist or cognitive learning theories, the radical constructivist paradigm offers multiple interpretations of learning events (e.g. Lerman and Scott-Hodgetts 1991) and of teachers' beliefs (e.g. Scott-Hodgetts and Lerman 1990) and rich alternative research perspectives for mathematics education (von Glasersfeld (Ed.) 1991b). At the same time, debates around the theory have, in my view, enriched ideas and developed discussion in the mathematics education community. The concern of this paper is the criticism that radical constructivism does not deal adequately with the nature of communication and language, and social aspects of conceptual development. In last year's PME proceedings, Ernest (1991) suggested:

> "To regard the social as secondary to the pre-constituted cognizing subject is again problematic, and this difficulty is not adequately resolved by radical constructivism." (p. 32)

At the same time, some researchers working within the radical constructivist approach appear to recognise the source of the difficulty. The following quote from Cobb et al (1991) demonstrates the way in which the problem of the nature of communication is embedded in radical constructivism:

> "Constructivism, at least as it has been applied to mathematics education, has focused almost exclusively on the processes by which individual students actively construct their own mathematical realities.
>
> However, far less attention has been given to the interpersonal or social aspects of mathematics learning and teaching. how does mathematics as cultural knowledge become "interwoven" with individual

> children's cognitive achievements? In other words, how is it that the teacher and the children manage to achieve at least temporary states of intersubjectivity when they talk about mathematics?" (p. 162)

The notion that pupils construct their own meanings from what is offered in the classroom, and that the teacher constructs her own interpretations of the child's constructions which are then relayed back to the child to gain feedback for corroboration or refutation, and negotiation, seems to lead to a continuous process that at best results in a 'fit' of meanings. This does not appear to be the same as communication in any sense that is external to the mind of the individual and thus reaching out to other minds. As for the role of social interaction, von Glasersfeld (1991a) interprets Piagetian theories as offering the following:

> "The experiential environment in which an individual's constructs and schemes must prove viable is always a *social* environment as well as a *physical* one. Though one's concepts, one's ways of operating, and one's knowledge cannot be constructed by any other subject than oneself, it is their viability, their adequate functioning in one's physical and social environment, that furnishes the key to the solidification of the individual's experiential reality." (p. 20-21)

In relation to the function of language, some writers (e.g. Ernest 1991) recognise its essential role in conceptual development, but given the interpretation of social interactions above, there appears to be some confusion about how language can play that essential role.

In an earlier paper (Lerman 1989) I briefly suggested that Wittgenstein's discussion against the notion of private languages highlights the problem and answers it, but as pointed out by Burt (1990), this was not sufficiently developed. In this paper I will attempt to pinpoint the source of what is seen by some as a major difficulty of radical constructivism. I will suggest that whilst there is a problem with communication for radical constructivism its resolution is to be found in an analysis of the social nature of language and consequently of concepts. I will argue that a Piagetian radical constructivism sees language as one of the social phenomena by which one measures the adequacy of one's private theories. For Vygotsky, language is primary, and socially embedded. Consequently a Vygotskian radical constructivism (if one can allow such a name) would focus on how language and concepts are constructed, both for the individual and the group, in the interactions of, for instance, the mathematics classroom. I will also argue that the latter perspective takes on board the post-structuralist critiques of individualistic, including Piagetian, psychology. Although the discussion will relate to language and concepts in general. I will make some connections with mathematics education explicit at the end, with reference to the work of Cobb, Wood and Yackel.

## Language and Thought

Two inter-related and major themes of Piaget's work are constructivism and structuralism. The former has been re-evaluated and elaborated as radical constructivism, and the latter has been subjected to what are termed post-structuralist critiques. Whilst radical constructivism is firmly within the Piagetian paradigm, post-structuralism is a radical shift away from the structuralist underpinnings of Piaget and away from an individualistic paradigm for psychology towards a social paradigm. However the following two quotes appear to indicate that these two theories or groups of theories take the same ontological position with regard to reality.

> "Language is not transparent   it is not expressive and does not label a 'real' world. Meanings do not exist prior to their articulation in language and language is not an abstract system, but is always socially and historically situated in discourses." (Weedon 1987 p. 41)
>
> "The world we live in, from the vantage point of this new perspective, is always and necessarily the world as we conceptualize it. "Facts", as Vico saw long ago, are made by us and our way of experiencing, rather than given by an independently existing objective world." (von Glasersfeld 1983 p. 51)

Thus it seems that Piagetian thought is pulled in two divergent directions that nevertheless have fundamental ideas in common. I am going to argue here that the dichotomy is apparent in the manner in which the relationship of language to thought and reality is conceived by these two theories. (I do not concur with Ernest's reification of a theory, the status of a category of ideas to which in his view radical constructivism does not achieve - see Ernest 1991 p. 25.)

For radical constructivism, language, as other external phenomena, is a medium through which the individual measures and compares her/his thoughts and ideas. As a consequence of comparison of those theories with other people, through language, or indeed interaction with objects (including books), the individual modifies her/his theories through accommodation, in order to achieve a 'fit', the criteria for which, namely adequate functioning and viability, are of course individually relative also.

For post-structuralism, thoughts and ideas are expressed in language and are limited by that language. Words do not capture the essential essence or character of concepts, be they abstract or 'real', because one cannot speak of such a character. What they do capture, because it cannot be otherwise, is the social construction of language. I cannot speak of one object being green and another red in a private way, which I then compare with your private meaning. I can only speak of those ideas because there is a social convention which allows 'green' to be applied in one case and 'red' in the other. Actually I cannot speak of 'green' and 'red' at all, as I am colour blind. When I do use these words my daughters laugh. They do not merely note that their concepts, or mine, must be in need of accommodation. My rather unimportant physical incapacity results in my not being able to share the social constructions of colour. Similarly, to use the word 'equator' is to know its use and occurrence. In fact the equator does not exist in any observable sense; it is as 'real' a concept as the centre of the universe which, for the Greeks in ancient times, was the centre of the Earth and in other places for other peoples at other times. To be able to speak of the Northern hemisphere and the Southern hemisphere, or to calculate distances between points on the surface of the Earth depends on my sharing the current (Western?) language of measurement on the surface of the Earth.

To take a further instance, one that I have used often but that nevertheless captures many of the points I wish to make here, consider the concept 'hat'. How is this acquired? What would be implied by saying that someone has acquired the concept 'hat'? Certainly it is not an innate platonic concept, nor does the concept identify some fixed signified, such that a definition or single occurrence leads to a permanent and complete 'understanding'. I suggest that one engages with 'hat' as one acquires the language of clothes, parts of the body, notions of temperature, perhaps religious significations, and uses to which objects are put. After all, a hat is identified by its use and public interpretation. To talk of my individual construction of the notion 'hat' which I then compare with other people's ignores the issue of the tools I use to make that construction: public signs embedded in language, such as clothes, parts of the body etc.

I am suggesting that to speak of individual constructions in the radical constructivist sense is entirely appropriate, but they are not in a 'private language', inaccessible to outsiders, including in particular teachers, precisely because the tools of thought are necessarily public. It is not surprising, from this point of view, that there appears to be intersubjectivity in the classroom, or that any individual can 'interweave' with culture, in our case mathematical culture. This is not, however, Piaget's notion of how thought and language are connected, as may be seen from the following (1969):

> " . . . it is apparent that one can legitimately consider language as playing a central role in the formation of thinking only in so far as language is one of the manifestations of the symbolic function. The development of the symbolic function in turn is dominated by intelligence in its total functioning." (p. 126)

As Zepp says of Piaget's theory of language (1989): "It states that thought exists prior to, and ouside of, language." (p. 27)

Vygotsky's ideas offer a different interpretation of the connection between public social interaction and the formation of concepts and conceptual development in the individual. I briefly review some of Vygotsky's work in the next section.

Vygotsky on Language and Thought

For Vygotsky the process of thinking, beyond the elementary, pre-intellectual stage, cannot be understood without understanding the social relations in which the individual exists.

> " . . . the social dimension of consciousness is primary in time and in fact. The individual dimension of consciousness is derivative and secondary" (quoted in Wertsch 1985 p. 58)

This is not to say that the individual dimension is reducible to the social, nor indeed the reverse. The connection between external social interactions and individual consciousness is through a complex process that is a major concern for Vygotsky, that he calls internalization. The issue of how this takes place, and how the teacher can set up the social norms in the classroom in order to facilitate internalization is clearly a central one for education. The work of the Purdue project (Cobb et al 1991) on developing those social relations in their research classrooms is, in my view, an excellent example.

Concerning the role of language, Vygotsky says (1962):

> "The structure of speech does not simply mirror the structure of thought; that is why words cannot be put on by thought like a ready-made garment. Thought undergoes many changes as it turns into speech. It does not merely find expression in speech; it finds reality and form." (p. 126)

What is clear in Vygotsky's ideas is that language is more than an external source for adaptation in order to achieve a 'fit', it has a primary role.

> "Real concepts are impossible without words, and thinking in concepts does not exist beyond verbal thinking." (Vygotsky 1986 p. 107)

What is also clear is that Vygotsky's view is very different from Piaget's:

> "This attempt to derive the logical thinking of a child and his entire development from the pure dialogue of consciousness, which is divorced from practical activity and which disregards social practice, is the central point of Piaget's theory." (Vygotsky 1986 p. 52)

A Vygotskian form of radical constructivism retains the notion that reality is a construction, not a received phenomenon, but sees reality as a social construction through language rather than the individual constructions of pre-existing cognizing subjects whose communications are ultimately incommensurable. Whilst thought is internal and there are differences at least in function between the inner language of the individual and the external public language, they are both expressed in and, in a sense, bounded by publicly accessible language. This does not imply that one can read another's mind, as the individual consciousness and the social domain are not reducible one to the other. In focusing on the constructions of individual students, research needs to focus also on analysis of the discourses and the social practices of the mathematics classroom, as these are the means by which the intersubjective is met.

Research in the Mathematics Classroom

Much research in the radical constructivist paradigm seems, I suggest, to be searching for explanatory theories for the role that discourse in the classroom plays, a role that reaches beyond achieving a 'fit' to concept formation itself. For instance, in their work on second grade learners (Cobb 1991, Wood 1991, Yackel 1991, Cobb et al 1991), the researchers refer to sociology and social interaction as being required to explain and guide the activities in the classroom:

> "... we might say that the children learned as they participated in the interactive constitution of the situations as they learned." (Cobb 1991 p. 235)
>
> "Neither a psychological nor a sociological perspective provides an adequate description of the complexity of the process (children's mathematical learning)." (Wood 1991 p. 356)

> "... through peer questioning . . . they develop sophisticated forms of explanation and argumentation which are influenced by the social interaction of the participants." (Yackel 1991 p. 370)

Referring to the concerns expressed by Cobb et al above, namely how individuals participate in mathematics as cultural knowledge, and how teachers and children communicate, I suggest that the answer may be found in the primacy of the social over the individual, that is to say the intersubjective over the intrasubjective. In the interaction of the teacher with the students and the students with each other, one sees the multiple and complex interactions of the discourse of the community of mathematicians with that of the mathematics classoom, as well as with those of the students as peer group and sub-groups of the peer group etc.

In reading the transcripts of the Purdue project one can, I suggest, observe the construction of knowledge, not in the privacy of individual minds alone but in the social domain of the classroom (e.g. Cobb et al 1991 p. 167, concerning the meaning of the fraction 1/1). Through discussion, dispute, cognitive conflict, sharing and so on, the intersubjective becomes internalized as the intrasubjective and the intrasubjective is offered to others, becoming intersubjective. In the mathematics education community we like to use the term 'negotiation' to describe how concepts develop through the activities of the mathematics classroom. In a Piagetian framework this term appears to function only as another word for accommodation, because of the absence of any intersubjective communication. In a Vygotskian framework negotiation is a description of that intersubjective communication. It is the meaning that is negotiated because the meaning is constructed in and through the language of social interactions, and cannot be otherwise. 'Meaning' is not invested in the signified, nor does it appear in the mind of the individual **except** through language which is socially rooted. Other aspects of the mathematics classroom also become more significant with the primacy of social interactions. For instance, the power relations that prevail in the classroom, revealed vividly through some of the work on discourse analysis (e.g. Walkerdine 1989), have purchase because sone individuals, by virtue of the calling up of particular discursive practices, are endowed with more authority than others.

In conclusion, I have suggested here that radical constructivism has a great deal to offer to research in mathematics education, but it does not have an adequate interpretation of social interaction precisely because radical constructivism holds to a Piagetian view of the role of language. There is much to be gained from a

Vygotskian view, whereby language and the social domain are primary, and the intrasubjective is the internalization of the intersubjective.

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# TO BE OR NOT TO BE MINIMAL?

## STUDENT TEACHERS' VIEWS ABOUT DEFINITIONS IN GEOMETRY

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*Abstract*

*The authors examine some aspects of mathematical definition as conceived by prospective teachers. The aspects are minimality and arbitrariness. This is done by means of a geometrical questionnaire which appears to be a didactical questionnaire. Most of the students seem to be aware of the minimality aspect. Some accept it and some reject it. On the other hand, it seems that many of the students are not aware of the arbitrariness aspect. In addition, some serious geometrical mistakes are found in the freshmen.*

### §1. The Minimality Aspect of Mathematical Definitions

Mathematics learning is supposed to have some fringe benefits. One of them is a metacognitive view about mathematics as a deductive system. The two most important tools of any deductive system are definition and proof. In high school, the deductive aspects of mathematics are strongly represented by Euclidean geometry. Van Hiele ( for details see Usiskin, 1982 and Senk, 1989 ) suggested a theoretical framework to deal with geometrical thinking. Several levels of geometrimal thinking were distinguished. However, Van Hiele did not give behavioral criteria by means of which a particular Van Hiele level can be assigned to an individual. This was done by Usiskin and Senk as part of an attempt to examine the validity of the Van Hiele theory. It is only reasnable to assume that a high level of geometrical thinking will include some understanding of definitions. This, however, is included only in an implicit way in the Van Hiele test ( Usiskin, 1982 ). In this study we were not interested in assigning a Van Hiele level to individuals (like, for instance, Mayberry, 1983 ). Our interest was to find out whether student teachers, at different stages of their study, are aware of certain aspects of mathematical definitions. The main aspect on which we concentrated was the aspect of minimality. The other one was the aspect of arbitranness. More specificly, there are some mathematical notions that can be defined in more than one way. In these cases, the mathematician, the textbook writer or the mathematics teacher has the freedom to decide which way he or she wants to go. Probably because of this it is common to claim that mathematical definitions are arbitrary. On the other hand, they are not atbitrary at all; there is always a good reason for the choice of any term in the mathematical vocabulary. Therefore, it is not completety coherent to claim about a definition that it is arbitrary and at the same time to be able to justify it. We consider the claim that mathematical definitions are arbitraty as an expression of the view that mathematical definitions are *man made* rather than *made by God*. Thus, the minimality aspect seems to us much more essential to mathematical definition than the arbitrariness aspect. Minimality can be considered a universal principle as well as mathematical. When you are asked about the meaning of a term you should say only the minimum required to understand the term. This principle is not necessarily accepted by everybody outside the domain of mathematics. Neither it is necessarily recommended as a pedagogical appproach. Nevertheless, it is a most crucial structural element of mathematics as a deductive system. As a matter of fact, it shapes the way in which mathematics progresses when it is presented deductively. Namely, after the definition, theorems which give you additional information about the concept, are formulated and proved. If mathematical definitions were not minimal we would have to prove their consistency. For instance, if you define an equilateral triangle as a triangle whose sides are congruent and also all its angles are congruent then you have to proceed by showing that these two properties can "live together." The most appropriate way of doing it in this case is to show that if all the sides of a given triangle are congruent then also all its angles are congruent. Therefore, what is the point of defining an equilateral triangle by means of both its sides and its angles if you should prove exactly the same theorems you would have to prove when going the minimal way? Being minimal is being economical and this is considered a virtue. We even dare to say that in many cultures, not only in the culture of mathematics, being economical is regarded as positive while being wasteful is regarded as negative.In spite of that, there are few cases in geometry where definitions are not minimal. The most famous one, perhaps, is the definiton of congruent triangles. The common definition of congruent triangles is the following: *Two triangles are said to be congruent if all their sides and all their angles are congruent, respectively.* Every geometry student knows that it is sufficient to require less than that for two triangles to be congruent. This is expressed by each of the four congruence theorems. The reasons for this redundency, so we believe, are psychological. The above definition is a special case of the general definition of congruent polygons where one must require congruence of all sides and all angles respectively. When passing from the general case to the special case, the mathematician ( Euclid? ) did not want to lose the message of the general definition. Namely, the concept of congruence involves focussing on both all sides and all angles. It also expresses the idea that if you put one triangle on top of the other then the later will be exactly covered by the first one. Moreover, if you decide, inspite of all,to use a congruence theorem as a definition, you will not find a didactically best candidate. Thus, in some rare cases, the principle of minimality is not observed in mathematics. In these cases, didactical and psychological principles take over.

## §2.Method

We used a two stage written questionnaire in order to expose the student teachers' views. In addition to it we also had a group discussion using the questionnaire as a starting point. Because of space problems we will report here only on the questionnaire. Its first part was the following:

*In a geometry exam the following two questions were given:*

*1. Define an equilateral triangle.*

*2. Define a rectangle.*

*Here are some answers given to these questions by students:*

*1a. An equilateral triangle is a triangle that has three congruent angles.*

*1b. An equilateral triangle is a triangle that has three congruent sides.*

*1c. An equilateral triangle is a triangle in which all the sides and all the angles are congruent.*

*1d. An equilateral triangle is a regular triangle. Namely, all its sides and all its angles are congruent and each median is also an altitude and an angle bisector.*

*2a. A rectangle is a quadrilateral in which all angles are right angles.*

*2b. A rectangle is a parallelogram that has at least one right angle.*

*2c. A rectangle is a quadrilateral that has 3 right angles.*

*2d. A rectangle is a quadrilateral whose opposite sides are congruent and all its angles are right angles.*

*The teacher decided to give 10 points to each answer which is absolutely correct. How many points would you give to each of the above answers if you were the teacher? Assuming that the students will come to you to argue about their marks please explain why you deducted points if you did so.*

After the students had answered this part of the questionnaire we collected their answer-sheets and gave them the second part which was:

*The teacher deducted several points from the answer defining an equilateral triangle as a regular triangle in which all the sides and angles are congruent and each median is also an altitude and an angle bisector. The student who had given this answer came to the teacher to argue about his mark. The teacher explained to the student that she had deducted the points because the answer included unnecessary details. The student said: If in a Biology test I write " unnecessary details" when answering a question like "what is a mammal" I'll get extra points.*

*What will you answer this student in case you agree with the teacher? If you don't, please, explain why you don't.*

Our aim in the second part of the questionnaire was to raise the minimality issue in a completely explicit way. We assumed that in the first part of the questionnaire, there will be some cases where students will deduct points because of reasons different than we had in our minds. Thus, in the second part we forced them to relate to our main issue -- the minimality aspect of mathematical definitions. We were aware of the fact that some students will understand the questionnaire in a didactical context and not in a purely mathematical context as we really meant. ( This was a price that we were ready to pay. The alternative was to formulate a questionnaire which is quite similar to a mathematical exam. We wanted to avoid that by all means. Testing somebody's knowledge is a threat, especially, to teachers and prospective teachers. ) Bearing this in mind, we were very careful with our analysis. We believe that in most cases we were able to distinguish between didactical arguments and mathematical arguments. We assumed that the didactical arguments whould include explicit clues like: *I would not deduct points because for a student at this stage it is a satisfactory answer.* If we were wrong we were wrong, so we believe, only in very few cases. We also believed that the second question in the first part of the questionnaire would be a better stimulus for the arbitrariness aspect, in case it was known to the student. More specificly, the common definition of an equilateral triangle can be considered as "canonical." This is, perhaps, because of the name that implies "equal sides." The case of the rectangle is a different one. There is a definition that might seem more natural at the elementary school ( four right angles ) and there is a definition that might seem more natural at the context of Euclidean geometry ( a parallelogram with a right angle ). Thus, if somebody is aware of the arbitrariness aspect of definition, he or she will be ready to accept other definitions of the rectangle than the one they were used to in their past.

After collecting the second part of the questionnaire we had a group discussion with the students. The entire activity took place in the students' regular classes and it lasted about an hour.The discussion took about half an hour. We led the discussions but we avoided from expressing our opinions. The main issues we raised for the discussions were the minimality and the arbitrariness aspects of mathematical definitions (of course, we did not put it in thesewords ). Our plans to discuss also the reasons for the minimality principle and some cases where it is not observed were not carried out in most of the classes because of the students' insufficient mathematical background. However, there were few exceptions. The group discussions will be reported elsewhere.

## §3. Sample

Our sample included four groups. The first three consisted of students in a teacher college in Jerusalem. It is a four year teacher college which gives the students an academic degree. The college prepares its students for teaching at the junior high level. Hence, their mathematical preparation is less than what a regular mathematics student gets at a regular university. On the other hand, there is a lot of emphasis on didactics and pedagogy at the teacher college. The first group of the above three was a freshman group, the second was a sophomore group and the third included junior and senior students.

The reason for including both junior and senior students in the same group is that there is no diference between the juniors and the seniors as far as geometry is concerned. In addition to the above three groups there was another group of university students participating in a teacher training program. These were regular mathematics students who had had at least two years of mathematical studies at the Hebrew University of Jerusalem. The above four groups will be called groups 1-4 respectively. Group 1 included students who had not yet decided about the discipline they wanted to teach at the junior high school. Some of them would teach mathematics and the rest would teach biology. The students in groups 2-4 would teach mathematics in case they would become teachers. The sizes of groups 1-4 are 37, 17, 19 and 9, respectively.

## §4. Results and analysis

For the total majority of the students the questionnaire was a strong enough stimulus to make them relate to the minimality aspect of definition. This exluded two students of the 3-rd group. The students were classified, therefore, into the following three categories:

Category 1 -- the student understands the minimality principle. He or she believes that a redundant information in a definition should be penalized.

Category 2 -- the student believes that a definition can include "unnecessary details." For some students, the more a definition includes the better it is.

Category 3 --the student does not relate to the minimality principle.

Table 1

Distribution of Students into Categories 1-3 ( the minimality aspect )

| | Category 1 | Category 2 | Category 3 |
|---|---|---|---|
| Group 1 ( N =37 ) | 1 | 36 | 0 |
| Group 2 ( N=17 ) | 10 | 7 | 0 |
| Group 3 ( N=19 ) | 10 | 7 | 2 |
| Group 4 ( N =9 ) | 6 | 3 | 0 |

Here are some typical examples of categories 1&2.

Category 1:

Student A: *A definition shouldn't include anything which can be implied by another part of the* *definition.* ( group 4 )

Student B: *When we come to give a definition in mathematics we try to give the shortest definition which will uniquely charecterize our topic. Anything beyond this is implied by the definition and it is redundant even if it is correct.* ( group 3 )

Category 2:

Student C: *I deducted points* ( 0n 1b ) *because the student should be more specific about the triangle. He is not supposed to let the reader think about all the consequences that can be implied from his definition.* ( group 2 )

Student D: *I deducted points* (on 1b ) *because although it is true that an equilateral triangle has three congruent sides, this also implies that it has three congruent angles and the student should have mentioned it.* ( group 1 )

As to the arbitrariness aspect of definition, the questionnaire was not a strong enough stimulus to majority of the students as it was with the minimality aspect. The students are classified again into three categories.

Category 1 -- the student is aware of the arbitrariness aspect of definition. He or she accepts different definitions for the same concept in case they are equivalent. It is clear to the student that there does not necessarily exist one and ultimate definition for a given concept.

Category 2 -- the student believes that there exists one and ultimate definition for a given concept. He or she rejects any definition which is not the "canonical" definition.

Category 3 -- the student does not relate to the arbitrariness aspect of definition. It is probably because of lack of reflective attention to this aspect. ( This hypothesis is supported by table 2. )

Table 2

Distribution of Students into Categories 1-3 ( the arbitrariness aspect )

| | Category 1 | Category 2 | Category 3 |
|---|---|---|---|
| Group 1 ( N=37 ) | 3 | 4 | 30 |
| Group 2 ( N=17 ) | 8 | 6 | 3 |
| Group 3 ( N=19 ) | 4 | 7 | 8 |
| Group 4 ( N=9 ) | 6 | 3 | 0 |

Here are some typical examples.

Category 1:

Student F: *This is* ( 2b ) *the definition to which I am used but all the other definitions are also correct.* ( group 2 )

Student G: *This answer* ( 2b ) *is also correct. It represents a different perspective.* ( group 3 )

Category 2:

Student H: *This* ( 2b ), *perhaps, will finally be alright, but I don't want the student to define the rectangle by means of the parallelogram. For this I'll deduct 4 points.* ( group 1 )

Student I: *I reject 2a, 2c and 2d because they are properties, not definitions.* ( group 3 )

There were several students who spoke about properties versus definitions. According to their line of thought, properties are implied by definitions. However, from any analytical point of view, there should not be any difference between properties which are implied by definitions and properties which are used in order to define concepts. It seems that for some students, the properties used in the definitions have a very special status. Because of this status they stop, perhaps, to be properties and turn into something else.

Another phenomenon that was observed in the students' questionnaires was the mistakes they made with very simple geometrical concepts. Soometimes, mistakes are not necessarily an obstacle for the understanding of mathematical principles. In such a case they can be ignored from the cognitive point of view and can be excused from the mathematical point of view. For instance, a student can make a mistake when differentiating a given function. This does not necessarily imply that he or she does not understand the basic principles of the differential calculus. However, there are mistakes which prevent the student from acquiring the appropriate understanding of the subject. We believe that the mistakes which were made by some of the students in our sample belong to the last category. Here are some examples:

Student J: *A rectangle is not a parallelogram and it has more than one right angle.* ( group 1, a reaction to 2b )

Student K: *Not always a triangle that has three congruent angles has also three congruent sides.* ( group 1, a reaction to 1a )

Student L: *This answer* ( 2b ) *is incorrect because there exists a parallelogram with one right angle which is not a rectangle.* ( Here the student draws a right angle trapezoid. Group 2. )

Student M: *I deduct 4 points* (2a ) *because from this one can imply that it is also a square. A square also has 4 right angles.* ( group 1 )

Student N: *It is not* ( 2c ) *necessarily a rectangle. It can be a square.* ( group 1 )

Student O: *This definition* ( 2d ) *does not imply that the adjacent sides are not congruent.* ( group 1 )

It is hard to see how students with such mistakes can relate to the really important aspects of geometry as definition and proof. Table 3 below indicates the number of students with such mistakes in the four groups of our sample. One might find some comfort in the fact that only in the first group the majority of the students made these serious mistakes and in the high school prospective teachers nobody made them. If you remember that the first group included students who might finally decide not to teach mathematics then you might be tempted to think that the situation is not so terrible. We disagree with that. Here we are concerned with the most basic facts of geometry. Every junior high school student should know that a square is also a rectangle. Hierarchical classification is a fundamental thought process that every educated person should be capable of. It is required in many domains, not only in mathematics. What analytical abilities do our students acquire at school if they cannot see that a square is a rectangle according to any legitimate definions of the two? It seems that some high school students who continue their higher education are still in the Van Hiele first level of geometrical thought. Their conception of the geometrical figures is global and lacks analytical elements. They relate only to the visual aspects of the figures and not to their properties.

Table 3

Distribution of Students with Serious Geometrical Mistakes

| | Students with serious mistakes |
|---|---|
| Group 1 ( N=37 ) | 27 |
| Group 2 ( N=17 ) | 3 |
| Group 3 ( N=19 ) | 7 |
| Group 4 ( N= 9 ) | 0 |

After all these school years, with all the information they are flooded with, it seems that many students cannot see the forest for the trees.

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# Algebraic and Non-algebraic algebra

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## Abstract

*In this paper, first we propose a theoretical framework within which it is possible to understand what thinking algebraically is; this framework (first introduced in a paper presented to PME 1990) was developed both on the basis of the study of the historical development of algebra and of epistemological considerations. It is then shown, with data from an experimental study, that our framework is useful to distinguish different solutions to problems requiring the determination of a number, and also that it can be used to explain differences in facility levels between problems with the same underlying "algebraic structure". It is also shown that some aspects highlighted by the use of our framework are of central importance if we are to understand what is it that we want our students to learn-understand in order for them to think algebraically.*

## 1.The Theoretical Framework

In a previous paper (Lins, 1990), we presented the sketch of a framework designed to provide an understanding of what thinking algebraically means. The need for such framework had been pointed out by researchers working on the subject (eg, Lee, 1987).

Our framework proposes that *algebraic thinking* is: (i) thinking ***arithmetically***; (ii) thinking ***internally***; and (iii) thinking *analytically* (see Fauvel, 1990; Vieta, 1968). The *arithmetical* aspect is necessary both to characterise the absence of processes involving limits, in the sense of Differential Calculus, and to remind us that—no matter what formalisation may offer to us—the arithmetical operations are a fundamental model for our understanding of algebraic operations: the elements in non-numerical algebras (abstract or otherwise) are in fact treated *as if they were numbers of a different kind*. The concept of number has, in fact, been transformed many times; Euclid would not call $\sqrt{5}$ a number, and yet we do. By ***internalism***, we mean *operating exclusively within a* ***Numerical Semantical Field***, as opposed, for example, to associating numbers—as measures— to line segments, and from this producing *meaning* to the manipulation of arithmetico-algebraic relationships. The ***internalism*** is necessary to allow us to distinguish between *algebraic thinking* and other models that can be used to produce algebra (eg, geometrical models [eg, deducing the square of a sum of two numbers using equality of areas], whole-part models [see Lins, 1990], or contextualised models [eg, a scale-balance model]). The *analyticity* is necessary to characterise *algebraic thinking* as dealing with *relationships*, involving numbers, the arithmetical operations, and the equality relationship. The *analyticity* is particularly important to allow us to understand differences in conceptualisations of the algebraic activity. To *analytical* procedures, one opposes *synthetical* procedures.

Under those three conditions, it is clear that the manipulation of the model can only be guided by the properties of the arithmetical operations and the properties of equality. This means that the arithmetical operations become ***objects***, while in non-algebraic models they are ***tools***. Only as ***objects*** can they have properties and guide the solution process.

A crucial distinction in our framework, is that between *algebra* and *algebraic thinking*; in relation to our framework, the production of an algebraic *result* does not necessarily involve *algebraic thinking*, nor does the use of algebraic symbolism. It is shown, within the context of our framework, that the development and use of algebraic (literal) symbolism is a possible consequence of, and not an *a priori* condition for *algebraic thinking*. Algebraic symbolism is made possible and adequate by *algebraic thinking*. This discussion is also beyond the possibilities of this short paper, and the reader is again referred to Lins (1992).

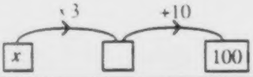
In reference to our theoretical model (i) one speaks of developing algebraic *expertise* and *knowledge*, an approach which seems to be more adequate—in view of the work in, for example, Luria (1976), Davidov (?), and Bishop (1988)—than speaking of the development of *mental (abstract) structures* that enable one to deal with algebra (as in, eg, Garcia and Piaget, 1984). Developing an algebraic mode of thinking involves one's insertion into aspects of a mathematical culture (White, 1956; Freudenthal, 1983), and the acceptance of certain *objects* as the focus of our attention (Lins, 1987, 1990 and 1992). And (ii) our framework is based on the analysis of the use and production of algebra by peoples of different historically situated cultures; however, our historical analysis rejects the notion—characteristic of the Babylonia-Greece-Europe strain (used, eg, in Harper, 1987)—that the central thread to be followed in the history of algebra is that of the development of literal symbolism, and the notion that the history of mathematics is the history of the production of mathematical *results* (Dieudonné, 1987). Instead we examined the ways in which different mathematical cultures produced knowledge that we would identify as "algebra"; we examined the interweaving of the conceptions about number and about mathematics that characterises each culture's approach to algebra and the consequences to their algebraic knowledge, ie, the models they had used to deal with it and what knowledge was possible under those circumstances. It clearly emerged from this study, that although *results* were exchanged between different cultures, those were always reinterpreted in relation to the new mathematical culture into which they had been brouhgt. It is in this sense that one has to understand the reconceptualisation of algebra by Arab mathematicians (Rashed, 1984), and the development of algebra in ancient China (Martzloff, 1988; Yan, 1987); it is also in this sense that we should examine students' development of an algebraic mode of thinking and the development of their algebraic knowledge.

## 2. A "worked example"

Let's examine three possible solutions to the problem of finding a number such that its triple plus ten is equal to 100.

Solution 1: Subtract 10 from both parts of the equality relationship (or add -10 to both parts of the equality relationship) to conclude that the triple of the number is 90. Now divide both parts of the equality relationship by 3, to conclude that the number is 30; or use the fact that $ab = c \Rightarrow b = \frac{c}{a}$

Solution 2: Just "undo" it.



Solution 3: Take a line AB which is 100 units long. Now, take C in AB, such that CB=10, and then partition the remaining segment AC and in 3 equal parts. Each of them is of the sought lenght.

Solution 1 is *arithmetical*, *internal*, and *analytical*. It proceeds by *algebraic thinking*. It is important to notice that the use of symbolism is not *necessary*, although it would closely and concisely represent the process. Solution 2 is *arithmetical* and *internal*, but not *analytical*. The unknown value is produced from the known ones, and the problem is modelled by a sequence of operations to be performed, not by an equality relationship involving the unknown value. Solution 3 is neither *arithmetical* nor *analytical*; as Solution 2, it is *synthetical*. Its *objects* are lines, and thus, it is the properties of lines that guide the solution, in the sense of determining what can and should be done to solve the problem.

The importance of distinguishing the three types of solutions, is that Solution 1 and Solution 2 would apply equally well to a problem like $-3x + 10 = 1$, but not Solution 3, while, almost surprisingly, Solution 2 would not apply to $10 - 3x = 1$. A similar situation would arise from dealing with formulæ such as $(a+b)^2 = a^2 + 2ab + b^2$. We think that some serious didactical misreadings can occur if that differentiation is not taken into account, resulting in inadequate teaching approaches.

### 3. Aims and method of the experimental investigation

The experimental part of our research was designed, then, to investigate the models used by secondary school students when solving problems that required the determination of a numerical value. Some of those problems were of the traditional "algebra word problem" type (**AWP**), some were of the "secret number" (**SN**) type. Our **SN** problems were presented in the form of a numerical equality that the sought "secret number" was said to satisfy. Other types of problems were also used.

In an exploratory study, we had verified that in some cases it may be very difficult—if not impossible—to distinguish or identify the model used. In the case of the "worked example" presented above, for example, if only the calculations had been presented in a script (100-10=90; 90÷3=30), the three models we presented could equally have been used. In order to overcome this difficulty, we devised a set of 6 written test papers, that instead of using "test-items" (in the sense of each problem "corresponding" to one type of equation) consisted of groups of 3, 4, or 5 problems (**AWP** and **SN**) all of which could be solved by solving the same *general* linear equation. Each student was presented with at least two of these groups, the problems being mixed in two test papers, which were solved in two different days within a week. The variations in the problems included different types of number involved (positive and negative, integer and decimal), both in the coefficients and for the unknown and different contexts (including, as we mentioned, the "purely numerical" **SN** problems). Not all combinations were used; instead, we chose, in each group, those aspects that we thought, on the basis of the exploratory study, to provide the most information in relation to our objectives.

We wanted to understand two aspects of the students' solutions. First, the *objects* that had been actually manipulated in order to produce the solution (numbers, arithmetico-algebraic relationships, lines, diagrams, objects of the context). Second, the effect that different variations would have both in the facility levels and in the choice of models used.

In all problems the use of "letters" was avoided, so that difficulties in the problems would not involve difficulties with their interpretation. Whenever available, calculators were used. The students were always told, however, that if they thought a calculation was "too hard", they could only indicate it; solutions with mistakes in computations only were considered correct if the strategy would lead to a correct solution had the calculations been correctly performed.

### 4. The sample

The tests were applied in four secondary schools, two in Nottingham, England and two in São Paulo, Brazil. The groups tested were: three 2nd-year English groups (average age, 13 years, 2 months; 53 students altogether, **FM2**), three 3rd-year English groups (14y3m; 66 st., **FM3**), two 7th-grade Brazilian groups (13y11m; 56 st., **AH7**), and two 8th-grade Brazilian

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**FM3**), two 7th-grade Brazilian groups (13y11m; 56 st., **AH7**), and two 8th-grade Brazilian groups (15y0m; 53 st., **AH8**). The main reason for applying the tests both in Brazil and in England, was to produce a more varied sample of models used, once the teaching programs and methods are quite dissimilar in the two countries. In Lins(1992) we examine some of the consequences of those differences.

## 5.The results

### 5.1 Buckets

**B1**: From a tank filled with 745 litres of water, 17 buckets of water were taken. Now there are only 626 litres of water in the tank.
How many litres does a bucket hold?
(Explain how you solved the problem and why you did it that way)

**B2 and B3**: I am thinking of a "secret" number. I will only tell you that ...181-(12 x secret no.)=97 (**B2**; in **B3** the relationship was 120-(13 x secret no.)=315)
The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

In **Buckets**, the important factors were the presence of a context in **B1** and the negative answer in **B3**.

| | AH7 | | | AH8 | | | FM2 | | | FM3 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **B1** | **B2** | **B3** | **B1** | **B2** | **B3** | **B1** | **B2** | **B3** | **B1** | **B2** | **B3** |
| | 21 | 21 | 21 | 17 | 17 | 17 | 20 | 20 | 20 | 24 | 24 | 24 |
| CREQT | 0.05 | 0.24 | 0.10 | 0.29 | 1.00 | 0.71 | 0.00 | 0.05 | 0.05 | 0.00 | 0.00 | 0.00 |
| CRCALC | 0.90 | 0.29 | 0.05 | 0.59 | 0.00 | 0.00 | 0.60 | 0.45 | 0.10 | 0.88 | 0.50 | 0.17 |
| CRTRE | 0.00 | 0.00 | 0.00 | 0.06 | 0.00 | 0.00 | 0.00 | 0.05 | 0.00 | 0.00 | 0.17 | 0.00 |
| WEQT | | 0.14 | 0.48 | | 0.00 | 0.29 | | 0.00 | 0.00 | | 0.04 | 0.00 |
| WCALC | | 0.24 | 0.19 | | 0.00 | 0.00 | | 0.15 | 0.40 | | 0.13 | 0.71 |
| NATT | 0.00 | 0.10 | 0.19 | 0.06 | 0.00 | 0.00 | 0.20 | 0.30 | 0.45 | 0.04 | 0.17 | 0.13 |
| CR | 0.95 | 0.52 | 0.14 | 0.94 | 1.00 | 0.71 | 0.60 | 0.55 | 0.15 | 0.88 | 0.67 | 0.17 |
| WRONG | 0.05 | 0.38 | 0.67 | 0.00 | 0.00 | 0.29 | 0.20 | 0.15 | 0.40 | 0.08 | 0.17 | 0.71 |
| NATT | 0.00 | 0.10 | 0.19 | 0.06 | 0.00 | 0.00 | 0.20 | 0.30 | 0.45 | 0.04 | 0.17 | 0.13 |

As one would reasonably expect, **B1** was easier than **B2**, and **B2** easier than **B3**. In only two cases (**AH8** and **FM2**) there is no significant difference between **B1** and **B3**. In fact, the only non-correct answer for **B1** in **AH8**, was a *Not attempted*.

Two test-items (question 34: "25-37" and question 35: "20-(-10)") were used to check students' ability to perform subtractions involving negative numbers.The very low facility level for **B3** could be explained in the English groups by difficulties with negative numbers, but not in **AH7**, where only one student answered question 34 incorrectly, and all students answered question 35 correctly. There is also the difference between **B1** and **B2** in **AH7** and **FM3** to be explained.

By far, the typical correct solution to **B1** was a "calculations" solution (76% of all answers)—as opposed to one using an equation. In 64% of the "calculations" solutions, an explanation was given making reference to the fact that to know how much had been taken on the (seventeen) buckets, one had to subtract what was left from the initial amount of water; in none of them a justication for that was explicitly provided (either verbal or diagramatic). We think that the use of a subtraction to ***evaluate*** how much water had been taken was elicited by a part-whole model in the form of, "If from the whole (tank) you *separate* one of two parts (which is known), what is left is the other part (which is yet unknown)."; subtraction is used as a ***tool***, as much as a "building-up" strategy can be used to *evaluate* the remaining part. If this model is to be applied to **B2**, one has to identify, in the given expression, the whole (181) and the parts (12 times the secret number, and 97). The results from **AH7** and **FM3** suggest that this process is not as straight-forward as it might seem. It is possible that the part-whole model used in **B1** was perceived by the students as "belonging" to the context; it is also possible that a subtraction presented as in **B2** does not "translate naturally", to those students, as the ***evaluation*** of a detaching process or as representing "detachinng from the whole". We think that both aspects played a relevant role.

On the other hand, in **AH8**, both **B1** and **B2** had almost the same facility level. In **B2**, as it had happened with the **Ticket & Drive** problems (and in fact with most problems in relation to this group of students), they systematically made use of equations (100% and all correct); in **B2**, however, most of the solutions (60%) did not use an equation, supporting our suggestion that the context strongly suggested a tacit part-whole model. The results for this group indicate, moreover, that the use of an equation represented a change in the model used (which is not always the case); we affirm that it corresponded to a change in the mode of thinking, in the sense proposed by our theoretical framework, ie, a change in the objects that guide the solution process.

In relation to **B3**, a clearer understanding of the picture we have so far suggested, emerges. The only group for which the facility level is high is **AH8** (71%), in which **all** the solutions—correct and incorrect—proceeded by solving "the" equation; in all other groups it stays bellow 20%. As we said before, in the English grops this could be due to the poor understanding (or familiarity) with negative numbers, but not in the case of the Brazilian group **AH7**. Brazilian students of the 7th grade, as those in **AH7**, are well used to solving linear equations where the coefficient of $x$ is positive, and thus, acquainted with equations. This might explain why many of them tried (and failed) to use an equation to solve **B3**. However, it is clear to us that this shift towards an "algebraic attempt" signifies that the model used in **B1** and **B2** was not "available" anymore, that is, a part-whole model (the shift is, in any case, "gradual", as we have that: in **B1**, 91% of the solutions did not employ an equation; in **B2**, 53% did not employed an equation; but in **B3**, only 24% did not); that the part-whole model is not applicable to **B3** is immediately clear, as it is not possible that something is removed from a whole and yet it gets bigger. A few students perceived that the number had to be negative, and treated it as if it had said "+12·secret number" (in which case a part-whole model would easily apply), later correcting the answer's sign.

### 5.2 Ticket & Driving

**Ti1 and Ti2**: Sam and George bought tickets to a concert. Because Sam wanted a better seat, his ticket cost four times (**Ti1**; '2.7 times' in **Ti2**) as much as George's ticket. Altogether they spent 74 pounds on the tickets.
What was the cost of each ticket? (Explain how you solved the problem and why you dit it that way)

**Ti3 and Ti4**: Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds. At a certain point they decided to stop for lunch. After lunch they still had to drive four times (**Ti3**; '2.7 times' in **Ti4**) as much as they had already driven.
How much did they drive before lunch? And after lunch?
(Explain how you solved the problem and how you knew what to do)

**Ti1** and **Ti3** are [4] problems; **Ti2** and **Ti4** are [2.7] problems. In **T&D**, the contexts and the types of numbers were cross-combined to compensate for possible differences due to the context; the main point we wanted to examine was the effect of different types of numbers in the choice of models; by examining the differences in the facility levels, it would also be possible to gather information indirectly.

| | AH7 | | | | AH8 | | | | FM2 | | | | FM3 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **Ti3** | **Ti4** | **Ti1** | **Ti2** | **Ti3** | **Ti4** | **Ti1** | **Ti2** | **Ti3** | **Ti4** | **Ti1** | **Ti2** | **Ti3** | **Ti4** | **Ti1** | **Ti2** |
| | 16 | 40 | 40 | 16 | 19 | 34 | 34 | 19 | 17 | 36 | 36 | 17 | 25 | 41 | 41 | 25 |
| OKEQT | 0.13 | 0.15 | 0.42 | 0.13 | 0.42 | 0.29 | 0.73 | 0.53 | 0.00 | 0.05 | 0.03 | 0.00 | 0.00 | 0.00 | 0.02 | 0.00 |
| OK ÷3.7 or 5 | 0.32 | 0.00 | 0.43 | 0.06 | 0.31 | 0.03 | 0.15 | 0.00 | 0.41 | 0.06 | 0.33 | 0.00 | 0.80 | 0.10 | 0.68 | 0.16 |
| OKT&E | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.06 | 0.04 | 0.00 | 0.02 | 0.20 |
| W ÷2.7 or 4 | 0.13 | 0.23 | 0.02 | 0.25 | 0.00 | 0.09 | 0.06 | 0.11 | 0.06 | 0.14 | 0.20 | 0.00 | 0.08 | 0.32 | 0.07 | 0.20 |
| WOTH | 0.19 | 0.20 | 0.05 | 0.38 | 0.11 | 0.39 | 0.05 | 0.26 | 0.29 | 0.14 | 0.20 | 0.29 | 0.04 | 0.27 | 0.12 | 0.28 |
| NAIT | 0.25 | 0.43 | 0.08 | 0.19 | 0.16 | 0.21 | 0.03 | 0.11 | 0.24 | 0.61 | 0.25 | 0.65 | 0.04 | 0.31 | 0.07 | 0.16 |
| | | | | | | | | | | | | | | | | |
| OK | 0.44 | 0.15 | 0.85 | 0.19 | 0.74 | 0.32 | 0.88 | 0.53 | 0.41 | 0.11 | 0.36 | 0.06 | 0.84 | 0.10 | 0.72 | 0.36 |
| WRONG | 0.31 | 0.43 | 0.07 | 0.63 | 0.11 | 0.48 | 0.11 | 0.37 | 0.35 | 0.28 | 0.40 | 0.29 | 0.12 | 0.59 | 0.19 | 0.48 |
| NAIT | 0.25 | 0.43 | 0.08 | 0.19 | 0.16 | 0.21 | 0.03 | 0.11 | 0.24 | 0.61 | 0.25 | 0.65 | 0.04 | 0.31 | 0.07 | 0.16 |

The main distinction produced here, was between the levels of facility for the [2.7] problems (**Ti2** and **4**) and the [4] problems (**Ti1** and **3**). The differences cannot be accounted for on the basis of difficulties with calculating with decimals, because, as we said before, mistakes with calculations only were disregarded.

The examination of the scripts revealed that in many of the cases in which an explanation had been provided, the model used in [4] problems was "1 section (or lot) and 4 sections" or "1 ticket and 4 tickets". This model is evidently difficult to apply to [2.7] problems. In the older Brazilian group, in which the use of equations was much more frequent than in any of the other groups (practically nil in the English groups), the facility levels for [2.7] problems are significantly higher; even in this group, however, [4] problems were easier than [2.7] problems. Our hypothesis is that although those older Brazilian students used equations, many of them might have relied on the "sections" model to set the equation, resulting again in a greater difficulty in [2.7] problems. The alternative would be to set the equation ***numerically***, ie. "a number plus 2.7 times this number is equal to 261". Another interesting result, is that the one type of error that can be singled out is dividing 261 by 2.7 (or 4); the fact that it has happened more frequently in [2.7] (**Ti2**: 20%; **Ti4**: 14%) than in [4] problems (**Ti1**: 6%; **Ti3**: 9%), seems to indicate that because the "2.7" is less ***suggestive*** of the "parts" model, the solution becomes, in the absence of other models, a "play with numbers"; in such a situation, it is reasonable to suppose that the choice of a division is made on the basis of the perception that each part of the journey is shorter than the whole journey. One further difficulty is hidden. In [4] problems, the reason to add 1 and 4 would be to know "how many parts altogether", so one can share the total into that number of parts; in [2.7] problems, the very notion of "sharing into a decimal number of parts" is not clear at all, and we suggest that this step not being meaningful, it renders the "1+2.7" step also meaningless, and it is for this reason that the "1" is left aside in so many cases. We think that on those cases, it is the inappropriateness of the "sharing" model that is responsible for the failure of many students in solving [2.7] problems, and not—as suggested by Fischbein et al (1985)—that the "primitive model" of division being that of "sharing", division is not perceived as useful; our research suggests that a model is chosen first, and only then a process to produce the required ***evaluations*** is chosen. Had a student noticed that in fact the problem involved searching for a number that multiplied by 3.7 would give 261, she or he could use, for example, trial and error (as in 60% of all correct answers to Ti2 in the English groups), instead of division; division could be "inadequate" because of its "primitive model", yet a solution could be correctly produced.

## 6. Conclusions

Algebra, and here we mean "school-algebra" for the sake of brevity in the discussion, is concerned with the *manipulation* of ***generic*** numerical relationships. Those relationships, however, can be modelled in a number of different ways: geometrical models, "real-life" models, or algebraic models, for example. Correspondently, algebra can be produced *geometrically*, in a *contextualised* way, or *algebraically*. There are other other possibilities. In each case, however, the ***objects*** that the reasoning deals with and is guided by are different, belonging to different ***Semantical Fields***, ie, ***different ways of making sense and producing meaning are involved***. Different levels of *expertise* and different bodies of *algebraic knowledge* can be produced. *Algebraic thinking* may equally happen in the context of solving a simple linear equation or in the context of proving the fundamental theorem of algebra, and the mode of thinking will be the same, although not the ***complexity*** of the mathematical situation. Nevertheless, thinking algebraically, by its own nature, allows the development of a much deeper and complete algebraic knowledge.

Our research shows that our students dealt with problems involving the determination of a sought number using various models, it also showed that some of those models have limitations that might hinder the students' progress in algebra. The ability to solve the equation $100-5x = 40$ does not imply—even when there are no difficulties with negative numbers—the ability to solve $100-5x = 200$, although it might appear that it *should*. Also, the solution of an equation like $181-12x=97$, which ***algebraically*** involves the manipulation of the variable, can be done ***synthetically*** if a part-whole model is used. Our research suggests that the difficulty with this and other types of equations (eg, $181-12n=128-7n$, which underlies one group of problems in our study), might be in finding a suitable model to guide the solution process (ie, to interpret the problem within a suitable *Semantical Field*) rather than something?. It is necessary, thus, not only to distinguish algebra from algebraic thinking, but also to understand the different *Semantical Fields* within which algebra can be produced; operating within a *Numerical Semantical Field*, the arithmetical operations and the equality become *objects* and it is possible and adequate, thus, to use a compact notation that borrows from the symbolism of arithmetic, as the arithmetical operations also retain their role as *tools* (in a renewed and expanded sense).

In view of our analysis and of the results that emerged from the experimental study, we think that the teaching of algebra should rely less on "simple" examples to build an algebraic understanding, as those are the ones more likely to be successfully modelled by non-algebraic models, and as a consequence, obstacles, rather than support for the development of *algebraic thinking* will probably emerge. Although our study focused on the process of solving problems requiring the determination of a numerical value, two groups of problems were designed to investigate whether the students would use the manipulation of arithmetico-algebraic relationships in order do derive new information about a given situation; the answer was overwhelmingly negative, and this finding may—in the light of our framework—be related to the refusal, by subjects with a substantial expertise in algebra, to use it to prove statements in "generalised arithmetic" (as documented, for example, in Lee and Wheeler, 1987).

Finally, our research have led us to believe that any successful effort to study what people can or cannot learn-understand under different conditions, depends inevitably of a clear understanding of what is it that we want people to learn-understand, and this is what our theoretical framework provides in relation to *algebraic thinking*.

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# WHAT'S IN A PROBLEM?
# EXPLORING SLOPE USING COMPUTER GRAPHING SOFTWARE

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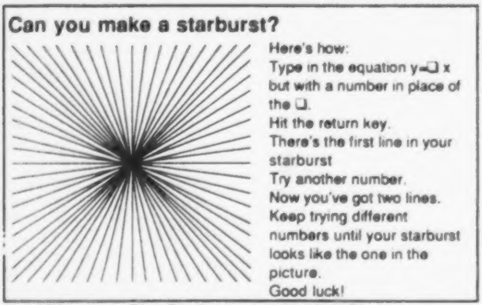
*This paper explores student interpretations of a non-standard problem designed to provide an entry into the domain of linear graphing and motivate an exploration of slope. Students work with computer graphing software to reproduce a picture. In doing so, they explore the concept of slope in a structured way. I use microanalysis to characterize students' conceptions of the domain and specifically of this exploratory problem. The data point to the importance of viewing problems from the problem solver's perspective: seemingly arbitrary features of the problem and the computer environment profoundly influence the students' experiences.*

## Introduction

In the past few years, several innovative curriculum projects have used exploratory problems to introduce new concepts or to frame curricular units. The tray problem, for example, in which students cut identical squares out of each corner of a cardboard square with the goal of folding up the sides to make a tray of the greatest volume, appears in the beginning of several recent innovative U.S. algebra texts (for instance, Stein & Crabill, 1984). Such problems give students a way to begin to explore a new domain in a structured yet open ended way: the goal is clear but the method is not. In this paper, I describe how twelve pairs of students interpreted an exploratory problem. The analysis demonstrates that a problem is not a static entity but rather is redefined by those who solve it -- by the knowledge they bring to it, by the goals they set, and by the assumptions they make.

## The Problem, as Presented to the Students

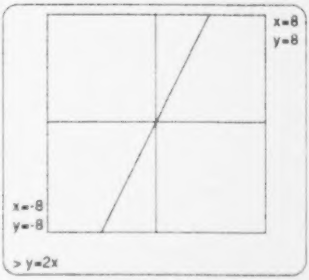
The problem was designed as an introduction to slope for beginning algebra students. The instruction sheet is shown in Figure 1. The pedagogical goal was for students to develop an intuitive sense of slope including general properties of order, symmetry, and direction, benchmarks of slopes of magnitude 0, ±1, and ±∞, and an understanding that evenly spaced numbers do not produce evenly spaced lines where $1 \leq |m| \leq \infty$



**Can you make a starburst?**

Here's how:
Type in the equation y=☐ x but with a number in place of the ☐.
Hit the return key.
There's the first line in your starburst
Try another number.
Now you've got two lines.
Keep trying different numbers until your starburst looks like the one in the picture.
Good luck!

Figure 1: The Starburst Instruction Sheet



> y=2x

Figure 2: A Sample Green Globs Screen

## Theoretical Frame

While problem solving is a well-represented area of research, exploratory problems like the starburst problem are of a slightly different genre than most of the problems previously considered. Greeno and Simon, in their 1988 review of the research on problem solving, discuss several dichotomous ways to classify problems, including well-structured or ill-structured and novel or domain specific. Exploratory problems like the starburst problem tend to bridge these categories. They have well-structured objectives but ill-structured procedures. Further, they require students to draw on domain-specific knowledge as well as general problem solving skills. The main difference, however, lies in the instructional goal. The goal is not for students to apply old knowledge in new ways to solve problems (as in many domain specific problems) nor for students to develop and refine general problem solving techniques (as with many novel problems). Rather, the goal of problems like the starburst is for students to explore a new domain by discovering new information and applying it to solve the problem. Pedagogically, what is important is not the solution, but the discoveries made along the way.

A third dichotomy, expert v. novice, is relevant because it focuses on the problem solver. Larkin (1980) maintains that experts and novices approach and solve the same problems differently because they see a different problem. Once again, the difference is the goal: instruction based on expert/novice research focuses on problems in which the goal is to improve the novice's performance so that the novice will learn to solve these problems more like the expert. In contrast, exploratory problems like the starburst are designed for novices and capitalize on students exploring an unfamiliar domain. Exploratory problems would be trivial and non-productive for experts because experts already possess the domain knowledge in question (in this case, that slope = rise/run). This knowledge defeats the purpose of the problem -- for students to develop beginning intuitions about the domain.

In his seminal work on mathematical problem solving, Schoenfeld (1985) looks at students solving problems that, like the starburst, are non-standard and open-ended, draw on domain knowledge, and serve as springboards into new domains. The distinction between exploratory problems such as the starburst and Schoenfeld's problems is that Schoenfeld assumes that students have adequate resources (mathematical facts, procedures, and understandings) to solve the problem. He is interested in students' higher-order skills for managing these resources on difficult problems. Exploratory problems like the starburst problem ask students to develop new resources in order to solve an unfamiliar problem. In addition to drawing on previously compiled knowledge, students are asked to search for new knowledge and then use it to solve the problem. Furthermore Schoenfeld's primary instructional goal for the problems he chose was to teach students to become better problem solvers. The pedagogical goals for the starburst problem are to motivate students' exploration of a new domain and to help them develop intuitions about it.

## Method

Twelve pairs of students (culturally and socioeconomically diverse) from local public and private junior high and high schools were videotaped working on the starburst problem. Students had previously plotted points on coordinate axes, worked with integers and rational numbers, and practiced simple manipulations of linear equations. They had not studied slope or graphed lines by plotting points.

The software used was a commercially available program called Green Globs written by Dugdale (1982). A facility called Equation Plotter graphs equations typed in by students. More than one equation can be graphed on the same grid. It is not possible to erase a single line; it is only possible to clear the entire screen. Lines are drawn from left to right. If the students type in the same equation more than once, they will not be able to see the computer graph it again. See Figure 2 for a sample screen.

In addition to videotaping students in a lab setting, I also observed six different first-year algebra classes working on the problem as part of their math curriculum.

## Results and Analysis

Students found the problem to be challenging and engaging, and all of the pairs succeeded in reproducing the starburst picture to their satisfaction. There was a wide range of student insights and discoveries which are analyzed in detail in Magidson (in preparation). In this paper, however, I focus on the surprises: on the instances in which students set unexpected goals or in which seemingly arbitrary aspects of the problem or computer program had a profound effect on the resulting activity.

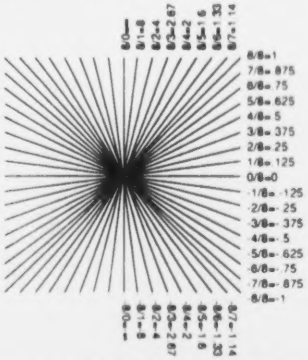


Figure 3: A mathematically correct solution

Although I did not expect students to produce a mathematically precise starburst, it was useful to examine the exact slope numbers involved. Since students were working with an 8x8 grid, and since the starburst on the instruction sheet was constructed by connecting border grid points [(0,8), (1,8), (2,8), etc.] to the origin, the slopes of these thirty two lines can be found by dividing each y-coordinate of a border grid point by its corresponding x-coordinate.[1] Figure 3 shows the slope numbers (in fractions and decimal equivalents) that produce the picture on the instruction sheet.

The solution shown in Figure 3 is based on the number eight because the Green Globs square grid default is 8x8. From my perspective as a problem designer familiar with the underlying mathematics, the dimensions of the grid were arbitrary to the problem: finding the exact value for the slope takes the same procedure regardless of the dimensions. Similarly, the general behavior of slope is not affected by gridsize (providing the scales are symmetric): larger slopes still produce steeper lines, positive and negative slopes of the same magnitude are still symmetric about the x- and y- axes, and the benchmarks of slopes of 0, ±1, and ±∞ remain unchanged. From the students' perspectives, however, the choice of an 8x8 grid significantly influenced their interpretations of the problem.

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[1] Note that the end points of the lines are evenly spaced. Since the grid is square and not circular, the angles between the lines vary slightly: angles between adjacent lines near an axis are greater than angles between adjacent lines near a corner. Since students tend to focus on where lines begin and end rather than on the angles between them, this does not appear to complicate the problem; indeed, were the angles between adjacent lines congruent, the task would become significantly more difficult.

To begin with, most students focused on the numbers in the corners of the screen labelling the dimensions of the grid (x=8; y=8; x=-8; y=-8) as potentially relevant (see Figure 2). Some assumed that the number eight had something to do with the solution. Further, some students inferred that the slope could not be greater than eight. Two students went so far as to figure out how to change the grid size to 10x10 because they convinced themselves that they could not solve the problem with an 8x8 grid. When students asked about these numbers, I explained that they referred to the grid size, but assumed they had little to do with the solution because of my conception of the problem and my expectations of what the students' goals and strategies would be. I expected that students would interact with the problem by spending most of their time trying to define the domain of slope numbers (positive *and* negative; integral *and* rational) and trying to create an approximate picture. Given this scenario, the number eight is not particularly relevant: students would choose approximate values between 0 and 10 or 12 and their negative counterparts. What happened instead was that many pairs set themselves the goal of matching lines exactly (even after I suggested they relax their goal and approximate), in which case slopes of ±8 produce the two lines nearest the y-axis, thus producing lines that are part of the picture and delimiting the range of slope numbers in this problem. Given this goal, the number eight is critical to the problem solution.

Further, one of the most compelling problems for most pairs of students was trying to space the lines evenly. Lines with slopes between one and infinity are particularly tricky given a decimal representation.[2] As can be seen in Figure 3, lines with slopes between zero and one are more accessible once students realize that they can use non-integral numbers. All of the pairs tried a decimal slope at some point, found that it produced a line in the appropriate region, and tried another. Students whose goal was to evenly space the lines (but not to match them exactly) discovered fairly easily that tenths (.1, .2, .3) produced evenly spaced lines that closely approximated the starburst picture. In contrast, students whose goal was to match each line exactly had a much more difficult task since the actual slopes on an 8x8 grid are consecutive eighths. Recognizing and expressing eighths as decimals (.125, .25, .375, .5) can be a complicated task for a first year algebra student. In a related study conducted in our lab, students who worked the starburst problem on a 10x10 grid had little trouble finding the slopes for these lines.

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[2] The computer program supports slopes expressed as fractions, but all of the students in the study chose decimals as a more comfortable representation of rational numbers.

This example may seem to suggest an advantage to presenting the starburst on a 10x10 grid since the goal is not to obscure the problem solution with sophisticated decimal representations. Had I done so, however, at least one pair of students (C&D) would have been denied the opportunity to make some sophisticated discoveries about the behavior of slope that surfaced *because* the grid was 8x8 and *because* their goal was to match grid points exactly. In an 8x8 grid, lines that pass through border grid points include slopes of 1, 2, 4,and 8. By graphing those lines on a clean grid, C&D noticed that each time they doubled the slope number, the region between the previous line and the y-axis was halved. They inferred from this that they would never be able to create a vertical line because the line would keep getting closer and closer to the y-axis but never get there[3]. Similarly, they found that going in the opposite direction, halving the slope halved the distance between the previous line and the y-axis. A transcript excerpt follows:

> D: One, right. One makes the x. One times two splits it in half right?
> C: Uh huh
> D: And point five splits it in half. Two splits this in half. Point five splits this in half.
> C: Okay.
> D: Four splits this into four, point two five splits this into four. So over here you keep multiplying by two -- one times two is two, two times two is four, four times two is eight. Over here you keep dividing by two -- one divided by two is point five divided by two is point two five divided by two is point one two five.

This discovery is mathematically valid and something C&D would later be able to prove when they learned about rise and run (i.e. doubling the rise is the same as halving the run) but would not have appeared had the scale not been a multiple of two. A seemingly arbitrary decision in problem design had significant entailments for these students.

Another unintended aspect of the problem was the density of lines in the starburst picture I gave students as a guide. In contrast to my videotapes where most pairs spend most of their time grappling with how to evenly space the lines, I observed two algebra classes solving a starburst with half the number of lines, and was surprised to find that the spacing issue never came up: there were simply not enough lines to make this a salient feature.

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[3] An application of one of Zeno's paradoxes of motion, slightly altered. Zeno, a Greek mathematician, argued that before an object could travel a certain distance, it must first travel half of that distance, and before that a half of that distance, and so on, so that the object would never be able to begin moving. (Boyer, 1968.) In math class earlier in the year these students had talked about "JJ's nose" -- an example where a student was standing near a metal pipe: before his nose could reach the pipe, he would need to first go half the distance, then half the remaining distance, then half the remaining distance again and would thus get closer and closer to the pipe while never reaching it.

My choices of grid size and density of lines influenced the students' interpretations of the starburst problem in ways I had not anticipated. In addition, a feature of the software seemed to significantly influence students' problem solving strategies, namely, the inability of the user to erase a single line at a time. It is possible to clear the entire grid, but not a specific line. Most students asked about this capability the first time they graphed an incorrect line, and found this restriction a continual source of frustration. Their response to this limitation may have worked in their favor, however: many pairs responded by taking notes. Then, when their screen became too confusing to look at, they could clear the screen and reproduce the "correct" lines with minimal effort. Note-taking led to other organizational strategies (ordering their findings, only recording the positive or negative values, using $m=|1|$ as a referent) that seemed to help the students with pattern hunting and sense making. Furthermore, when they went to recreate their partial starburst on a clean grid, many did so systematically, starting with $|m|=1$ and then either typing in positive/negative pairs ($y=1x$, $y=-1x$; $y=2x$ ,$y=-2x$; etc.) or typing an ordered set of positive slopes followed by the corresponding set of negative slopes. If students had been able to select the lines they wanted to erase, they would not have needed to create more than one starburst: every time they graphed a line they didn't like, they could simply have removed it from the screen. It is unclear whether they would have had to develop the same organizing strategies, strategies which often led to important discoveries about the domain.

### Discussion

These examples illustrate the unexpected consequences of students working in an unfamiliar domain to solve an exploratory problem. What can we learn from their experiences? It would be easy enough to alter the problem and computing environment based on these examples and to bring in some more students to test it out. Undoubtedly this would result in new student constructions and conceptions of the domain, some predictable, some not. The point, however, is not that one computer program or one grid size or a certain density of lines would necessarily be better than another. Rather, what is important is to recognize that the problem, while seeming to be general, is actually extremely context-dependent from the point of view of the problem solver regardless of how we alter it. In an unfamiliar domain, problem solvers have no way to know, a priori, what is significant and what is arbitrary. Further, through exploring seemingly arbitrary aspects of the problem, they may find fruitful avenues to pursue, as did C&D with their doubling and halving strategies.

## Conclusions

In analyzing problem solving activity it is important to put aside our preconceptions of the problem so that we can look at the problem through the problem solvers' eyes. What do they see? What are their goals? What constraints do they impose on the problem? It is too easy to speak of "the starburst problem" or "the tray problem" and to assume certain conceptual and procedural entailments. Particularly in the case of exploratory problems, we must learn to expect the unexpected. If we want to understand what students take from an exploratory problem, we must look carefully at their experiences without preconceptions of our own. We must not assume that students see what we see. If we are to take constructivism seriously, we must remember that there is no one static interpretation of a problem. Rather, the problem lies in the eyes of the beholder.

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# INDIVIDUAL THINKING AND THE INTEGRATION OF THE IDEAS OF OTHERS IN PROBLEM SOLVING SITUATIONS

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*This paper describes the problem-solving behavior of a seven-year old student, Michael, who is observed over a five-month period in two small-group situations in which he initially does his own individual thinking, appearing to ignore his classmates in the sessions, and then eventually by integrating the strategies of other group members modifying his initial representation of the problem. An analysis of the development of his Cartesian product representation of multiplication, first in grade 2 and then in grade 3 is given, focusing on how Michael first built-up his initial representation, and how he revised and modified it over time.*

This research is part of a four-year longitudinal study of how children build-up their mathematical ideas as they are engaged in problem tasks with other students. It takes place in a classroom setting where children, usually working in pairs, are encouraged to justify their solutions, and provide explanations to the teacher and other classmates, as they share ideas. The theoretical perspective for this research comes from a constructivist orientation in which problem situations provide students with the opportunity to build ideas and support or revise their thinking as original ideas become modified and refined.

Earlier findings suggest that although children work in a social context - small group or whole class - individual ways of representing a problem and the methods for solution are initially invented and personally constructed by each individual problem-solver. After an initial representation has been built, the ideas of others may be considered and integrated into the original construction. In fact, we are beginning to see patterns which suggest an initial attempt to build a representation of the problem, accompanied by a deliberate rejection of the ideas of others. Only after there is some ownership of an original idea will new ideas begin to be integrated into a more refined representation of the problem (Davis, Maher, & Martino, in press; Davis & Maher, 1991; Maher & Martino, 1991; Martino & Maher, 1991).

To illustrate this phenomenon, we present an analysis of the development of the Cartesian product representation of multiplication of Michael, a seven year old, who was engaged in solving the same problem on two occasions, first in grade two and then in grade three. This paper traces the development of Michael's mathematical ideas.

**Classroom Settings.** In both the second and third grade mathematics classrooms the expectation was for considerable student initiative in devising ways to solve mathematical problems. Instruction was frequently organized to encourage individuals (working in small groups) to solve problems at their own pace and without teacher intervention. Following the problem-solving sessions, children had opportunity to share their ideas in a whole class discussion. After the sharing of solutions, several children were individually interviewed about the problem activity.

In our classroom work in schools, we find that students may not be ready to agree on a solution to a particular problem task. When this occurs, our preference is to allow for student disequilibration. Rather than reach closure at the time and push for or present a solution, we choose to revisit the same problem later. In our analysis of the problem-solving behavior of Michael, the same activity was presented on two occasions, five months apart. In the first presentation, the problem was not solved. What we observed was the development of strategies and notations for solving the problem. Although small groups of children appeared to have agreed among themselves on a solution, no consensus was reached by the entire class.

**Data Source.** Data for this study came from children's written work and analyses of videotape transcripts from the following three sources: (1) classroom small-group working sessions (one grade two triad-group and one grade three pair-group); (2) each group sharing their solution strategy with the rest of the class; (3) interviews with individual children following the problem activity. This paper specifically focuses on the problem-solving behavior of Michael (M) who was a member of a triad-group in grade two [with Stephanie (S) and Dana (D)] and a pair-group [with Jaime (Ja)] in grade three. The ideas of other group members also will be discussed as they relate to Michael's building of his mathematical ideas. (For a detailed analysis of Stephanie and Dana's problem solving, see Maher & Martino, 1991).

**The Problem.** The problem-solving activity was presented as part of a regular mathematics lesson. Students were not told in advance any method for solution. The problem presented was:

> **Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?**

**Grade Two - Stephanie, Michael and Dana (May 30, 1990).**

The group of children began the problem by focusing on information that dealt with the color and type of clothing in order to build up a representation of the problem situation. Michael's partners, Stephanie and Dana, shared how they planned to draw three shirts, and place one letter inside each to represent color of clothing. Michael was observed silently reading the problem with a puzzled facial expression. Then Michael began to build his own solution; while doing so, he appeared to be detached from the interaction of his two partners. After working independently for about one minute, he then listened to his partners' understanding of the problem situation and compared it with his own.

M: Wait a minute...[*He looked at the problem again.*] Yeah, white shirt, white pants. [*He drew a shirt and placed a letter "W" inside the outline.*]

Stephanie, Michael and Dana each decided to draw a picture to represent the problem data. Initially, students spoke aloud, but did not seem to respond to the comments made by others.

S: Ok, blue and then a yellow shirt. [*Stephanie drew blue and yellow shirts.*] He has a pair of blue jeans... [*She reread the second part of the problem.*]

M: This is simple.

S: And a pair of white jeans. [*She drew two pairs of jeans.*] How many different outfits can he make? Well..[*Dana looked at Stephanie's paper and drew blue and white jeans.*]

M: [*Michael looked up as he spoke.*] He can make only two outfits. [*He drew a pair of jeans and placed a letter "W" inside.*].

Michael's partners quickly drew three shirts and two pairs of jeans placing a one letter code to represent color inside each article of clothing. Michael drew one complete outfit consisting of a white shirt and white jeans. Stephanie, having built an initial representation of the problem situation, decided to respond to Michael's last statement. His suggestion that there were two possible outfits had stimulated Stephanie's curiosity, and she responded by rereading the problem and checking that the input data representation was consistent with her knowledge representation. She then disagreed with Michael's observation and reported that Stephen "could make a lot of different outfits".

When Dana completed her drawing of three shirts and two pairs of jeans, she indicated that three outfits can be made by matching one pair of jeans with each of the three shirts. From her explanation we might infer that Dana had the key idea for exhausting all possible combinations. Simultaneously, Stephanie (Figure 1b) used her diagram to develop a coding strategy to make her combinations of outfits. She then began to illustrate each distinct outfit with a pair of letters, (the first letter designated the shirt, and the second letter the jeans), and monitored her work by numbering each combination that she generated. Figures 1a, 1b and 1c show the final written work of Dana (1a), Stephanie (1b) and Michael (1c).

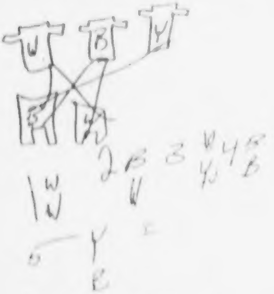


Figure 1a - Dana's solution

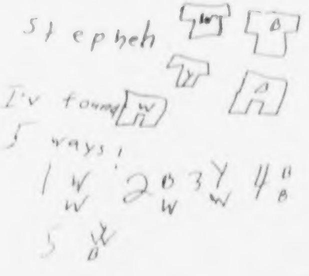


Figure 1b - Stephanie's solution

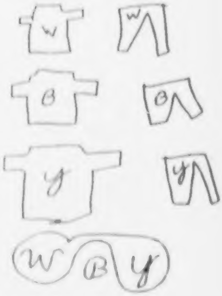


Figure 1c - Michael's solution

As the girls verbally shared their plans for solution, Michael was silent and appeared to be inactive. We observed him gazing in the direction of Dana for approximately four seconds, then he continued to add to his drawing of a single outfit. As Stephanie was recording her first outfit, she said:

S: You can make it different ways too. You can make white and white, that's one...W and W. [*She drew a 1 and W over W. Michael viewed Dana's work.*]
M: That's what I'm doing. [*He erased and redesigned the white pants.*]

As Stephanie and Michael compared their strategies, Dana found a notation that enabled her to make use of her original idea of matching a pair of jeans to all possible shirts by spontaneously drawing lines that connected each of her white and blue shirts to each of her blue and white jeans and her yellow shirt to her blue jeans. She concluded, as indicated in Figure 1a, that there were a total of five different outfits. Stephanie continued to list pairs of letters for her outfits (Figure 1b) and concluded, also, that there were a total of five outfits. During this time, Michael, quietly worked alone, occasionally stopping to listen to his classmates, or to talk aloud about his combinations.

M: [*Michael spoke aloud, but the girls were engaged in an argument about the need for the outfits to match.*] I'm doing white pants and white shirt, blue shirt and blue pants. [*Michael continued to draw.*]
S: [*She addressed Dana.*] No...how many outfits can it make? It doesn't matter if it doesn't match as long as it can make outfits.

Michael continued to perfect his drawing without speaking (occasionally erasing a piece of an article of clothing). Recall that early in the session, Michael reported two outfits, and Stephanie tried to convince him that more combinations of outfits could be made. However, Michael continued to engage in drawing his own picture (Figure 1c) which consisted of a second set of shirt and jeans labeled with the letter "B" inside. Although he seemed to be aware of Stephanie's coding strategy, he appeared to reject it. As Stephanie read her fifth combination aloud, Michael looked up from his drawing at Stephanie, he then returned to his own paper and replied:

M: I don't think...I don't want to do it that way...I want to do it this way. [*Michael referred to his own picture, explicitly rejecting Stephanie's system of coding.*]
D: Well, do it the way you want.
S: Do you know what? There's five combinations...there's only five combinations. Cause look you can do a white shirt with white pants...
M: [*He stared upward, then spoke.*] That's what I did. A white shirt with white pants, a

blue shirt with blue pants...

**S:** [*Michael looked directly at Stephanie as she spoke.*] You can do this...listen Michael! Michael will you listen for once. You can do five combinations...you can do...number one white and white, number two blue and white, number three yellow and white, number four blue and blue and number five yellow and blue.

**M:** [*He holds up his paper with his drawing of two outfits.*] I got these two so far.

**S:** You can do four (sic) combinations Michael! I'm sure of it!

Although Michael refused to change his method of solution, he seemed to be considering his classmate's strategies as he continued to build his model. As the girls shared their solution with the teacher, Michael continued to draw a third set of shirt and jeans with the letter "Y" inside them. He looked in the direction of the girls for several seconds and continued to draw. After drawing a third combination, he looked at all three papers, and drew an "eye-glass" shaped diagram beneath his three outfits (Figure 1c). Stephanie, having completed the activity, tried to collect Michael's paper before he finished writing.

**M:** Wait a second! Ok, I'm done.

**S:** Michael did you find the five ways?

**M:** I don't need to I found six...three ways. It's no big deal. [*Stephanie attempted to change Michael's solution, and he became visibly angry.*] Don't mess up my paper!

**S:** Michael, don't worry!

**M:** You touch my paper and guess who I'm going to chase!

Occasionally, as indicated by the videotape, Michael glanced at Dana and Stephanie's work, but these glances were brief and might easily have gone unnoticed. Michael's written solution (Figure 1c) displayed a diagram with three shirts (B, W, Y) vertically aligned with three pairs of jeans (B, W, Y). At the bottom of the page he wrote the three letters W, B and Y with the letters W and Y enclosed within his "eye-glass" shaped diagram. His solution bore no resemblance to either the connecting line strategy of Dana or the two letter coding strategy of Stephanie; it was clearly different. Michael's first mental representation may not have been sufficiently developed to let him incorporate all of the relevant information. His display of yellow jeans suggests that he may have lost track of which colors were available. Another interpretation is that he redefined the problem by expanding the jeans possibilities. One could also conjecture that his placement of letters in the enclosed region of the "eye-glass" shaped drawing might be a method for rapidly matching up the white and yellow shirts with the blue jeans. However, this seems unlikely since Michael announced that he had found three outfits.

Michael's final solution showed no influence of the strategies used by his partners. In fact, what was particularly interesting about this classroom episode was that each student produced an independent solution, and seemed to be satisfied with his or her own strategy.

Notice that Michael agreed with Jaime's suggestion to draw a picture, but his diagram differed from that of his partner's. He did not use connecting lines in the manner which Jaime did (Figure 2a). Instead, Michael indicated his combinations by moving his finger between the respective articles of clothing in the written problem statement, and later drew these connecting lines with his pencil to establish his combinations of outfits (Figure 2b). To describe and record the outfits, Michael wrote a letter for shirt color over a letter for the color of the jeans. Michael used a notation, similar to the coding system used by Stephanie in the second grade (Figure 1b), to complete his solution of six combinations.

The videotape indicated that Jaime and Michael worked individually, occasionally listening to each other, and pursuing their own solutions. In grade two, Michael's written solution (Figure 1c) displayed a diagram with three shirts (B,W,Y) and three pairs of jeans (B,W,Y), an "eye-glass" shape containing letters and no numerical answer. In grade three, Michael developed a variation on Dana's connecting line strategy (Figure 1a) without drawing pictures of shirts and pants. He drew lines between the colors in the stated problem to generate his combinations, and recorded these combinations with the two letter coding strategy similar to that used by Stephanie the previous year (Figure 1b). Although it wasn't evident from Michael's behavior and grade two solution that he was attending to the representations of his classmates and in the process gaining from the experience, his third grade solution strategy provided strong evidence that he was conscious of the ideas of others.

Following the third grade activity, the children were individually interviewed about the problem task. Analysis of the videotaped transcripts gave us further insight into the children's awareness of their problem-solving activity. Michael and his partners each responded that they had used their own method to obtain the same solution in both second and third grade. This is interesting in light of the fact that each student showed some modification in his/her strategy and found six outfits in grade three. Specifically, Michael's third grade written work (Figure 2b) indicated an integration of elements from both Stephanie and Dana's second grade strategies. He retained these ideas to create a solution which was his own.

## Conclusions and Implications

Initially, Michael worked independently to build his own personal representation of the problem. In the second grade activity he appeared, frequently, to be detached from the discussions of his partners. He also seemed to reject the suggestions of his partners, apparently satisfied with his own effort and progress. Neither Michael nor his partners provided a complete solution to the problem in grade two. Yet we see, in grade three, evidence of the influence of Michael's second grade partners' notation and approach in Michael's third-grade representation of the problem solution. Michael worked with a new partner in grade three who also had a partial solution in grade two that was unlike the representation of either of Michael's former partners, Stephanie and Dana. We found in an earlier analysis of the work of Stephanie and Dana over a five month period

## Grade Three - Michael and Jaime (October 11, 1990)

In the second month of grade three, Michael worked with Jaime on the shirts and jeans problem. Immediately after reading the problem, Michael offered four as the number of possible outfits.

**M:** You can make four outfits.

**Ja:** How do you know?

**M:** I did it. I put blue with blue...[*Michael began to form his combinations by tapping with his pencil to connect shirt colors to jean colors in the written problem statement. Meanwhile, Jaime returned to reading the problem.*]

Jaime pointed to each color in the stated problem, and counted articles of clothing rather than outfits. Michael, having chosen a path of solution, listened as Jaime counted, and responded to her findings.

**Ja:** 5, 6, 7, 8, 9, 10, 11, 12 outfits. [*Jaime counted each article of clothing.* ]

**M:** You put 12 outfits?

**Ja:** No, no, you see the white..pretend this is a shirt and this is jeans. "W" on the blue, "W" again and the "W", white and white, and then blue.."B" with the "B"..How many?

**M:** I don't know. [*Michael continued to work on his solution.*]

Jaime, who in grade two had recorded her outfits as descriptive phrases, now began to draw a diagram (Figure 2a) with six geometric figures for shirts and six more for jeans, each containing a one letter code for color. She connected each shirt and pair of jeans with a line to define each outfit as distinct, and used a pattern to generate her combinations matching each color shirt to both colors of jeans.

**Ja:** I'm drawing a picture.

**M:** Me too.

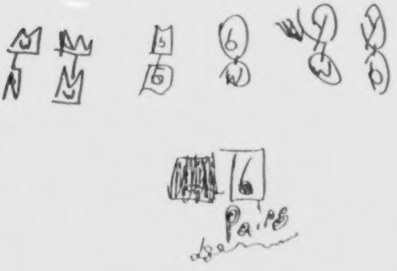
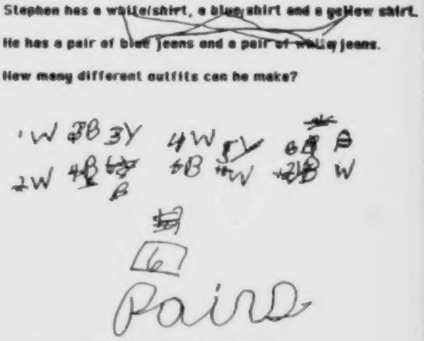


Figure 2a - Jaime's grade three solution



Stephen has a white shirt, a blue shirt and a yellow shirt.
He has a pair of blue jeans and a pair of white jeans.
How many different outfits can he make?

1W 2B 3Y 4W 5Y 6B B
2W 4B B 6B 4W 2B W
6
Pairs

Figure 2b - Michael's grade three solution

(Davis, Maher & Martino, in press) the influence of Dana's second-grade thinking on Stephanie's third-grade work. We see, here, the influence of Stephanie and Dana's second-grade thinking on Michael's third grade work. [Recall that no second grade child obtained a complete solution; no closure was reached in the grade two synthesis; no intervening work in school dealing with this idea occurred in the interim.] Yet we see the influence on Michael from others *following* his initial individual effort. Michael succeeded eventually in expanding, refining, and developing a more powerful representation of the problem solution. One might suggest that although Michael *appeared* detached from group discussions and later verbally rejected his partners' suggestions in grade two, he might well have been attentive to their ideas, although yet unwilling to include them.

Teachers who have observed the grade-two tape of Michael and his grade-two drawing suggest that he appears to be excluded from the group activity, that he probably was not understanding the problem, that he was not benefiting from the activity, and that he was mostly disengaged or "off-task". Some teachers have expressed concern about the appropriateness of the activity until after viewing the third-grade tape in which they expressed surprise and puzzlement.

It seems to us that the episode and the response of others to it, raise some potentially important questions about expectations regarding how students learn and build-up their ideas. Also in question is the issue of expecting immediate cooperation and team-work from members of small groups engaged in a problem-solving task and the necessity of reaching immediate closure to a problem-task after students have had an opportunity to work on it. If the grade-two lesson were brought to closure after the class discussion [ordinarily teachers view it as their responsibility to point out the missing outfit(s) or to point out strategies for finding it (them)] would Michael (or Stephanie) have been better off? We suspect not.

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# MATHEMATICS PRACTICE IN CARPET LAYING

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*Building upon previous research concerning mathematics practice in everyday situations, the mathematical concepts and processes used by carpet layers were studied, using an ethnographic approach, over a period of seven weeks. Mathematics practice in carpet laying is characterized through a discussion of the mathematics used by carpet layers in estimation and installation activities in an effort to describe and detail how people actively give meaning to and use mathematics in the midst of ongoing activities in relevant settings.*

## Introduction

Learning and doing mathematics is an act of sense-making and comprises both cultural and cognitive phenomena which cannot be separated (Schoenfeld, 1989). Part of every culture are the everyday happenings of the people belonging to that culture. As such, mathematical thinking and learning occurs in this everyday practice. Research on cognition in everyday practice (e.g., Carraher, Carraher & Schliemann, 1985; Lave, 1988; Saxe, 1988; Scribner, 1988) points toward the need for studying cognition within a cultural context. My interest, specifically, is to close the gap between doing mathematics in out-of-school situations and doing mathematics in school. It is my contention that the gap between out-of-school and in-school mathematics practice will only be narrowed when the ways in which mathematics is meaningful in the contexts of everyday life are determined. To this end I chose to examine mathematics practice in the workplace—in particular, mathematics practice in carpet laying. Note that I will often use the more general term, floor covering work, instead of carpet laying since I observed the estimating and installation procedures for tile and hardwood, as well as carpet.

## Research Design

For this study, I used an ethnographic approach to examine the mathematical concepts and processes used in carpet laying. The study was conducted over a period of seven weeks in June, July, and August 1991 in a midwestern city in the United States. This study is best categorized by what Wolcott has called a "microethnography." A microethnography "zeroes in on *particular* settings . . ., drawing on the ways that a cultural ethos is reflected in microcosm in selected aspects of everyday life, but giving emphasis to particular behaviors in particular settings rather than attempting to portray a whole cultural system" (Wolcott, 1990, p. 64). This research focused on the use of mathematical concepts and processes in the context of floor covering work. Four methods of data collection were used: participant observation, ethnographic interviewing, artifact examination, and researcher introspection (methods described by Eisenhart, 1988). The data were analyzed using activity theory (Leont'ev, 1981; Wertsch, 1985) as a guiding framework and a process of inductive data analysis to develop a theory grounded in the data that describes the mathematics practice of carpet layers.

## In the Carpet Laying Context

The mathematics practice of the carpet layers I observed is characterized through discussions of two areas: (1) the mathematical concepts, and (2) the mathematical processes used by floor covering workers. Following a brief description of these two areas, a carpet estimating situation is presented and discussed in order to provide a glimpse into the mathematics practice of carpet layers.

### Mathematical Concepts

I observed four categories of mathematical concepts used by floor covering estimators and/or installers: measurement, computational algorithms, geometry, and ratio and proportion. Measurement concepts and skills were involved in most of the work done by the estimators and installers. In particular, I observed four different categories of measurement usage: finding the perimeter of a region, finding the area of a region, drawing and cutting 45° angles, and drawing and cutting 90° angles.

Although algorithms are processes rather than concepts, I mention computational algorithms in this section because I am interested in the mathematical concept of measurement underlying these algorithms. I observed the following computational algorithms used by estimators in measurement situations to determine the quantity of materials needed for an installation job: estimating the amount of carpet, estimating the amount of tile, estimating the amount of hardwood, estimating the amount of base, and converting square feet to square yards.

In addition to the use of measurement concepts and algorithms, I also observed the use of the geometry concepts of (1) a 3 - 4 - 5 right triangle, and (2) constructing a point of tangency on a line and drawing an arc tangent to the line. Floor covering estimators also used ratios and proportion concepts when working with blueprints and drawing sketches detailing the installation work to be done.

**Mathematical Processes**

Besides the use of mathematical concepts, the estimators and installers made use of two mathematical processes: measuring and problem solving. As would be expected, the process of measuring is widespread in the work done by floor covering estimators and installers. Although being able to read a tape measure is vital, other aspects are equally as important in the measuring process: estimating, visualizing spatial arrangements, knowing what to measure, and using non-standard methods of measuring.

The mathematical process of problem solving is used by floor covering workers every day as they make decisions about estimations and installations. However, the problem solving that occurred in this context is slightly different from how problem solving is typically defined. Problem solving is commonly thought of as the process of coordinating previous experiences, knowledge, and intuition in an effort to determine an outcome of a situation for which a procedure for determining the outcome is not known (Charles, Lester & O'Daffer, 1987). Problem solving in the floor covering context deviated from this definition in that procedures for determining outcomes were usually known. However, unfamiliar constraints (e.g., a post in the middle of a room) and irregular shapes of rooms forced floor covering workers to coordinate their previous experiences, knowledge, and intuition to determine outcomes of situations they faced.

The problems that estimators and installers encountered required various degrees of problem-solving expertise. As the shape of the space being measured moved away from a basic rectangular shape, the problem-solving level increased. To solve problems occurring on the job, I observed estimators and installers use four types of problem-solving strategies: using a tool, using an algorithm, using a picture, and checking the possibilities.

**On the Job: An Estimating Situation**

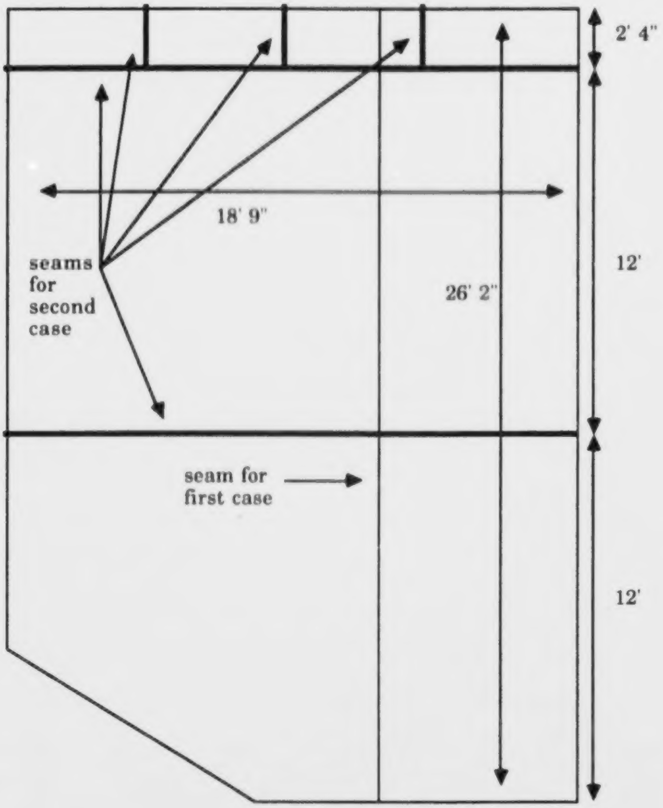
One of the situations in which estimators or installers used the strategy of checking the possibilities was in deciding on the best estimate for a carpet job. This often involved weighing cost efficiency against seam placement. Sometimes a customer specifies that he or she wants to see all the possible installation situations sketched out and then he or she will make a decision. However, in the majority of situations that I observed, the estimator provided a single suggestion on how the carpet could be installed and presented this suggestion to the customer. In almost all cases, the customer concurred with the estimator's suggestion.

The preparation of an estimate for a carpet job is constrained by a number of factors: (1) most carpet is 12' wide, (2) carpet pieces are rectangular, (3) all carpet pieces should have the nap running in the same direction, (4) consideration of seam placement is very important to accommodate traffic patterns and the type of carpet being installed, and (5) some carpets have patterns that must match at the seams.

One carpet estimate situation I observed involved a pentagonal-shaped room in a basement. I accompanied the estimator, Dean, as he took field measurements and figured the estimate. The maximum length of the room was 26' 2" and the maximum width was 18' 9" (see the figure below). Since carpet pieces are rectangular, every region to be carpeted must be partitioned into rectangular regions. The areas of these regions are then computed by multiplying the length and width. Thus, this room had to be treated as a rectangle rather than a pentagon. Dean figured how much carpet would be needed by checking two possibilities: (1) running the carpet nap in the direction of the maximum length, and (2) turning the carpet 90° so that the carpet nap ran in the direction of the maximum width.

In the first case, two pieces of carpet each 12' x 26' 4" would need to be ordered. Note that two inches are always added to the measurements to allow for trimming. After one piece of carpet 12' x 26' 4" was installed, a piece of carpet 6' 11" x 26' 4" would be needed for the remaining area. Since only one piece 6' 11" wide could be cut from 12' wide carpet, multiple fill pieces could not used in this situation. Thus, a second piece of carpet 12' x 26' 4" would need to be ordered for a total of 70.22 square yards. The seam for this case is shown by a thin line in the figure.

Turning the carpet 90° would require two pieces 12' x 18' 11" and a piece 12' x 4' 9" for fill. The 12' x 4' 9" piece would be cut into four pieces, each 2' 4" x 4' 9". The seams for this case are shown by thick lines in the figure. The total amount of carpet needed for this case would be 56.78 square yards. This second case has more seams than the first, but the fill piece seams are against the back wall, out of the way of the normal traffic pattern. Thus, these seams do not present a large problem. In both cases there would be a seam in the middle of the room. The carpet in the first case would cost at least $200 more than the carpet in the second case. Dean weighed the cost efficiency against the seam placement and decided that the carpet should be installed as described in the second case.


2' 4"
18' 9"
seams for second case
26' 2"
12'
seam for first case
12'

## Conclusion

Schoenfeld (1987) has suggested that what is needed in mathematics classrooms is to "engender a culture of schooling that reflects the use of mathematical knowledge outside the school context" (p. 214). Consider the following exercise, taken from a pre-algebra book: "Find the cost of carpeting a floor that is 15 feet by 12 feet at $24.95 per square yard" (Shulte & Peterson, 1986, p. 527). Although this problem uses an everyday context, the absence of actual constraints makes the problem artificial; the textbook example is simply a computational exercise placed in the context of an everyday situation. The effort required by a student to find an answer to this exercise does not reflect the way mathematical knowledge is used in carpet laying. In particular, the carpet estimate described above involves using measurement concepts in a problem-solving situation compounded by real-life constraints.

Schoenfeld has argued that the reason for the lack of success of some curriculum reforms, such as the movements for relevance, new math, basic skills (and he predicts for problem solving) may be partially explained by the lack of connection between the school culture and out-of-school culture: "Each of these curriculum reforms reflects an attempt to embed a selected aspect of mathematical thinking into what is an essentially alien culture, that of the traditional classroom" (Schoenfeld, 1987, p. 214). Since, at the present, these two cultures are so different, this attempted embedding is almost assured of failing. This study built upon previous research in an effort to add to the expanding knowledge base concerning mathematics practice in everyday situations. Further work needs to be done to examine the implications of this research for school mathematics.

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# Solving procedures and type of rationality in problems involving cartesian graphics, at the high school level (9th grade)

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Abstract

*One of the aim of the teaching of mathematics at the high school level, in France, is to bring the students reaching the 9th grade to what may be called "mathematical rationality". Cartesian graphics, which can be viewed as proceeding from both mathematical rationality and "common" or "everyday" rationality, are often used with intent to favour this perspective. A survey of about 150 students ninth graders shows that such a view may be too simplistic. In fact many children, when they have to deal with cartesian graphs, are distracted from the use of algebra, although it is necessary for the relevant solution of the problem they are given, and they tend to rely on notions and procedures which are inspired by everyday rationality.*

*Note : this study is part of a larger program of investigations on didactical status of graphics in the first years of the high school by researchers from ERES * and DEACT** Some references are given here ((1), (2), (5), (7), (8) and (9)).*

## Problématique

La représentation graphique cartésienne se présente comme un registre, au sens où R. DUVAL (1988) emploie ce terme, c'est-à-dire comme un ensemble de signifiants structuré de façon interne par des liens de configuration. Mais il est également structuré de façon externe par deux cadres d'interprétation que nous désignons respectivement par les expressions "cadre de la rationalité du quotidien" et "cadre de la rationalité mathématique".

Par cadre de la rationalité du quotidien, nous entendons l'ensemble des objets, des concepts, de la rationalité et des signifiants dont dispose le sujet pour traiter une représentation cartésienne, en rapport direct avec son expérience familière. Les règles opératoires sont celles de la rationalité du quotidien (M.

LEGRAND, 1989) et les concepts et objets peuvent être qualifiés de "quotidien", au sens où l'entend VYGOTSKI (1934, p.283-285).

Le cadre de la rationalité mathématique adapté à la résolution d'items à support graphique cartésien est associé au "cadre algébrique", sur lequel fonctionnent les règles de la rationalité mathématique. Dans cette dernière expression, le terme "cadre" est utilisé au sens de R. DOUADY (1984).

Le registre graphique est un ensemble de signifiants communs aux deux cadres d'interprétation, lesquels utilisent d'ailleurs d'autres registres, communs ou spécifiques : par exemple, le langage naturel est un registre commun, la formulation algébrique est spécifique du cadre de la rationalité mathématique alors que le registre kinesthésique est, lui, spécifique du cadre de la rationalité du quotidien.

En définitive, une représentation graphique cartésienne peut être traitée de diverses manières. Un élève peut :

* la traiter en terme de "registre" : il analysera alors l'information du seul point de vue de la structuration interne du registre de signifiants,

* se référer au cadre du quotidien : il analysera alors l'information en rapport avec l'expérience immédiate qu'il a de la situation dans la vie courante,

* se référer au cadre mathématique : il s'orientera alors, dans les items qui nous intéressent ici, vers un traitement algébrique de la situation.

L'un des buts de l'enseignement des mathématiques en France, en fin de collège (9ème grade) est de provoquer le changement de statut de certains concepts et du mode de rationalité, autrement dit, de permettre aux élèves d'accéder au cadre de la rationalité mathématique. Se situant à la charnière entre les deux cadres de rationalité précédemment définis, les représentations graphiques cartésiennes sont souvent utilisées, dans la pratique, dans l'intention de faciliter cet accès. Cependant, l'analyse précédente permet aussi bien de retenir l'hypothèse selon laquelle, dans certaines situations, l'utilisation de graphiques cartésiens peut favoriser un traitement dans le cadre de la rationalité du quotidien. C'est cette hypothèse, trop souvent négligée, que nous allons tester ici.

## Présentation de l'expérience

### *Population et méthode*

L'expérience concerne 144 élèves de 3ème (9th grade) issus de 5 classes d'un même collège. Chaque élève doit répondre à un questionnaire "semi-ouvert" comportant 3 items. Les questions de chaque item sont formulées de façon à ce que l'élève fournisse une réponse brute mais aussi un commentaire expliquant sa réponse. Nous appuyant sur un travail préalable d'identification des procédures de résolution, nous étions ainsi, en général, en mesure de repérer la (ou les) procédures mises en oeuvre par l'élève.

La passation s'est faite en classe, pendant l'horaire de mathématiques, sous la forme d'une épreuve "papier-crayon" d'une durée de 50 minutes environ. Le questionnaire se présentait sous deux modalités distribuées aléatoirement dans chacune des classes. Une moitié des élèves de chaque classe environ ont donc reçu la modalité 1, les autres la modalité 2. Au total, 70 élèves ont répondu à la modalité 1 et 74 à la modalité 2.

### *Le questionnaire*

Les deux modalités sont présentées, dans un format réduit, en annexe 1. En fait lors de la passation, l'élève disposait d'une page par item (le texte occupant la moitié de la page) et était invité à expliquer sa réponse à la suite du texte.

Les trois items à résoudre portent respectivement sur le parallélisme, l'alignement et l'orthogonalité. Les procédures algébriques adaptées à la résolution font l'objet d'un enseignement systématique en classe de 3ème. Les situations choisies sont telles que la lecture rapide du graphique invite a priori à apporter une réponse incorrecte (les points paraissent alignés et ne le sont pas, les droites paraissent parallèles ou orthogonales et ne le sont pas). En procédant ainsi, nous souhaitions éviter le réinvestissement automatique de procédures routinières et tester en quelque sorte la stabilité des acquis.

Par ailleurs, nous avons voulu observer l'effet de deux variables : la variable à deux modalités "présence de repère/absence de repère" et la variable à deux modalités "support quadrillé/support non quadrillé". L'effectif d'élèves ne permettait pas de tester pour chaque item l'effet croisé des deux variables. Nous avons donc choisi de ne faire jouer qu'une variable par item, l'autre étant maintenue fixée. Cela nous a conduit à retenir les deux modalités présentées en annexe.

**Analyse**

*Procédures*

Nous avons classé les principales procédures observées selon 3 catégories :

* procédures algébriques : elles mettent en oeuvre des formules de calcul algébrique (équation de droite, calcul de pentes, ...),

* procédures géométriques : elles mettent en oeuvre une validation à caractère géométrique indépendante du registre graphique cartésien,

* procédures graphiques : elles font intervenir la structure interne du registre graphique. Par exemple, on y trouve l'utilisation de l'unité du quadrillage, le dessin de "triangles de pente" dont les côtés sont parallèles aux axes, etc.

*Indicateurs*

Pour analyser les réponses, nous avons utilisé deux indicateurs :

* un indicateur de fréquence d'un type de procédure : il s'agit du pourcentage des occurences de ce type de procédure par rapport au total des réponses,

* un indicateur de rationalité, noté r. La valeur de r pour un ensemble de réponses est égale au rapport des réponses dans lesquelles l'élève effectue une véritable démonstration (cadre de la rationalité mathématique) au nombre total des réponses.

*Tableaux*

Pour chacun des trois items, les résultats de l'analyse ont été respectivement portés dans les 3 tableaux de l'annexe 2. Dans un tableau, on trouve pour chacune des 2 modalités de la question :

* l'indicateur de fréquence de chaque type de procédure (le cumul des pourcentages par ligne est supérieur à 100% car certains élèves ont produit plusieurs procédures dans une même réponse).

* La valeur de r associée à chaque type de procédure. A titre d'exemple, r = .96 dans la colonne des procédures algébriques du premier tableau, signifie que 96% des procédures algébriques mises en oeuvre dans la modalité 1 de l'item 1 relèvent de la rationalité mathémaique.

* la valeur de l'indicateur de rationalité "globale", calculée en prenant pour base l'ensemble des réponses à la modalité concernée.

*Résultats*

L'examen des tableaux de l'annexe 2 fait ressortir les points suivants :

* à l'inverse des procédures graphiques, pour lesquelles r = 0 quels que soient l'item et la modalité, les procédures algébriques relèvent très majoritairement de la rationalité mathématique (r varie de .67 à .96, selon les items et les modalités). Les procédures géométriques sont dans une situation intermédiaire, au moins en ce qui concerne l'item 3 : en effet, certaines, d'entre elles, mettant en oeuvre des théorèmes spécifiques de la géométrie (Thalès, Pythagore, ...) permettent d'établir une démonstration, alors que d'autres, utilisant le graphique comme une figure, ne le permettent pas.

* pour chaque item, la comparaison entre les deux lignes du tableau associé met en évidence la dépendance des procédures sélectionnées et de la rationalité globale à l'égard des variables. De manière plus précise, on peut considérer qu'il y a "régression" de rationalité et recul des procédures algébriques lorsque les élèves travaillent sur support quadrillé (tableaux relatifs aux deux premiers items) ou en présence d'un repère (item 3).

## Conclusion

Dans l'enseignement des mathématiques en fin de collège, tout se passe comme si l'on espérait faire accéder les élèves, sans rupture, par le biais des graphiques, au cadre de la rationalité mathématique. Nous faisions ici, en quelque sorte, l'hypothèse inverse : le registre graphique se situant à la charnière entre les deux cadres d'interprétation "quotidien" et "mathématique", nous pensions que la présence des représentations graphiques cartésiennes pouvait favoriser le traitement des problèmes dans le cadre de la rationalité du quotidien.

Les résultats de l'enquête menée viennent à l'appoint de notre hypothèse : dans les procédures utilisées par les élèves à l'appui d'une démonstration de parallélisme, d'alignement ou d'orthogonalité, en la présence de représentations graphiques cartésiennes, la rationalité du quotidien joue un grand rôle, surtout lorsque le repère est figuré sur le papier ou lorsque le support est quadrillé. Alors que les procédures mathématiques adaptées (il s'agit essentiellement de procédures algébriques) ont fait l'objet d'un enseignement auprès des élèves, ceux-ci sont distraits du recours à l'algèbre.

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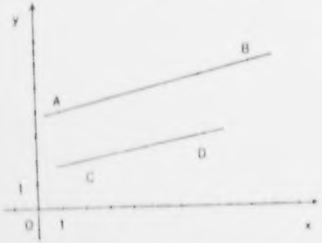
# ANNEXE 1

## Modalité 1 du questionnaire

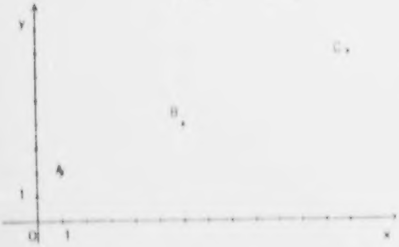
**Item 1** : Les 2 droites (AB) et (BC) sont-elles parallèles ?

OUI ☐ NON ☐



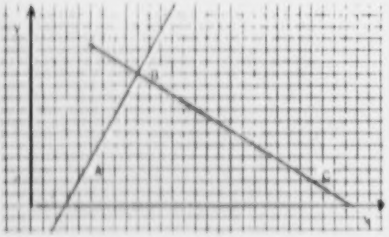
**Item 2** : Les points A, B et C sont-ils alignés ?

OUI ☐ NON ☐



**Item 3** : Les deux droites (AB) et (BC) sont-elles perpendiculaires ?

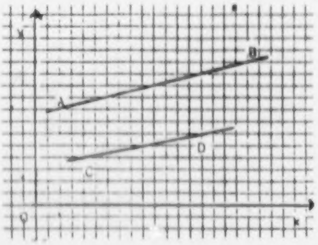
OUI ☐ NON ☐



## Modalité 2 du questionnaire

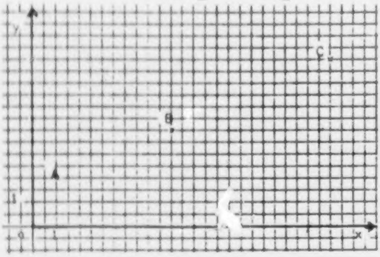
**Item 1** : Les 2 droites (AB) et (BC) sont-elles parallèles ?

OUI ☐ NON ☐



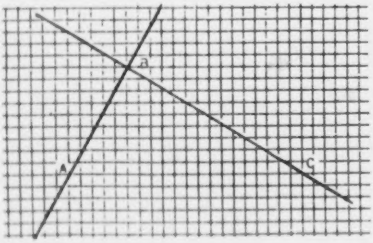
**Item 2** : Les points A, B et C sont-ils alignés ?

OUI ☐ NON ☐



**Item 3** : Les deux droites (AB) et (BC) sont-elles perpendiculaires ?

OUI ☐ NON ☐



## ANNEXE 2

| **ITEM 1** | *Procédure Algébrique* | *Procédure Géométrique* | *Procédure Graphique* | *Rationalité Globale* |
|---|---|---|---|---|
| *Modalité 1 Support non quadrillé* | 83%<br>r = .96 | 25%<br>r = 0 | 1%<br>r = 0 | r = .73 |
| *Modalité 2 Support quadrillé* | 59%<br>r = .93 | 27%<br>r = 0 | 25%<br>r = 0 | r = .49 |

| **ITEM 2** | *Procédure Algébrique* | *Procédure Géométrique* | *Procédure Graphique* | *Rationalité Globale* |
|---|---|---|---|---|
| *Modalité 1 Support non quadrillé* | 83%<br>r = .96 | 65%<br>r = 0 | 6%<br>r = 0 | r = .45 |
| *Modalité 2 Support quadrillé* | 39%<br>r = .74 | 59%<br>r = 44 | 21%<br>r = 0 | r = .46 |

| **ITEM 3** | *Procédure Algébrique* | *Procédure Géométrique* | *Procédure Graphique* | *Rationalité Globale* |
|---|---|---|---|---|
| *Modalité 1 Support non quadrillé* | 83%<br>r = .94 | 28%<br>r = .22 | 6%<br>r = 0 | r = .72 |
| *Modalité 2 Support quadrillé* | 75%<br>r = .74 | 78%<br>r = 44 | 3%<br>r = 0 | r = .32 |

# THE MICROEVOLUTION OF MATHEMATICAL REPRESENTATIONS IN CHILDREN'S ACTIVITY

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## Abstract

*This paper discusses children's design of mathematical representations on paper, asking how displays are constructed and transformed in activity. Based on the detailed analysis of videotaped problem-solving sessions with a pair of eighth graders, I show that: (1) the design of displays during problem-solving shapes one's mathematical activity and sense-making in critical ways; and (2) knowledge of mathematical representations is not simply recalled and applied to problem solving, but emerges (whether constructed anew or not) out of one's interactions with the social and material settings of activity.*

## Introduction

The role of notational systems in mathematical activity is often assumed to be two-fold: (1) as supporting cognitive processing, and (2) as mediating communication (Kaput, 1987). Pimm (1987) detailed the role of notational systems in mathematical activity by listing the following uses that symbols can be put to: "communicating," "recording and retrieving knowledge," "helping to show structure [among ideas]," "allowing routine manipulation to be made automatic," and "making reflection possible" (p. 138). In classical mathematics education, the analysis of symbol use tend to oppose cognitive processing (labeled as "internal") to the actual manipulation of inscriptions ("external"), giving rise to *mapping models* of the kind proposed by Kaput (1989). Kaput's proposal fits a mapping-cognitivist perspective to the extent that it vies meaning as only a by-product of *relations among representations* of the world – primarily in the head, then between mental representations and the physical expressions of a domain. In particular, Kaput's model focuses on the formal-structural aspects of mathematical cognition which are "supported" by features of notational systems, while disregarding issues of *notations-in-use* proper. As a consequence of this mapping view, the research on instructional representations has focused on children's *interpretations of expert-designed materials* (Greeno, 1987; Janvier, Ed., 1987). However, not very many studies can be found in the mathematics education literature that treat children's *production of material representations* (e.g., on paper) as the central issue in any significant way. Pengelly (1990), for example, did study the inscriptions generated on paper by elementary school children while learning about carrying-over procedures in base-ten tasks, but she takes these material representations only as a step in the road to acquiring "abstract" knowledge.

In contrast, I propose here an *activity-oriented view* that takes cultural conventions such as notational systems to shape in fundamental ways the very activities from which they emerge, *at the same time* that their meanings are continuously transformed as learners produce and reproduce them in activity. In accordance with this view of notations-in-use, Pea (1987) suggests that material tools (such as marks on paper) function not only as "amplifiers," supporting cognitive processes and communication, but as *transforming the very ways in which activities evolve*. It is fundamental to this activity view,

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however, that notations do not constitute by themselves the matrix for the emergence of actions and mathematical knowledge. Instead, they are seen as cultural artifacts whose meanings are negotiated and recreated by learners in activity. This perspective was captured in Hall's (1990) study of the making of mathematics on paper which investigated in detail the material designs constructed by algebra teachers and students during problem-solving. Based on an "ecological" hypothesis, Hall suggested that the quantitative inferences that learners elaborate during problem-solving depend critically on the structure of the representations they design on paper (called *representational niches*) and on the way notations are made to "carry" quantities.

This paper investigates children's design of mathematical representations on paper, focusing on the microgenesis of representational niches and on the interplay between mathematical activity and the social and material structuring of settings. The following questions are examined: (1) what characterizes children-designed inscriptions on paper?, and (2) what role do such displays fulfill in the children's activity? Finally, this paper offers an analysis which, instead of describing how students might interpret expert-designed representations, asks how learners build material representations in the first place.

## Background to the analysis

The study described here was part of a larger research project (Meira, 1991), which aimed at investigating children's mathematical sense making and use of physical devices to learn about linear functions. In that larger project, nine pairs of eighth grade students[1] were videotaped while working on problems about one of three physical devices: a winch, a system with springs, or a computerized functions machine. In this paper, I discuss in detail the representational activity of one pair of students who worked with the springs mechanism. This device had two springs placed at the top of a numbered track, and a set of one-pound weights used to stretch the springs. Mathematically, the relation between the length of a spring (L) and the weight suspended on it can be represented as $L_f = KW + L_0$ (where $L_f$ is the spring's final length, K is its coefficient of elasticity, W is the weight on the spring and $L_0$ is its initial length for $W=0$). The springs and weights were identified by color (black or white), and the students were allowed to hang a maximum of three pounds from each spring. Several problems were posed in order to motivate the students to gather and record data, as well as to analyze underlying patterns and functional relationships. For instance, one of the questions on the students' worksheet was: *"Will the black spring ever be 5 inches longer than the white spring?"* The problem-solving sessions took place in the school building after school hours. During these sessions, the students were asked to work together on the tasks and use the device and other material resources at their discretion.

## Analysis and discussion

This section takes up the problem of the genesis of material representations, focusing on the microevolution of student designed tables of values. Tables were extensively produced by all nine pairs of students during the problem-solving sessions. In fact, all but one pair used only tables to solve problems about the devices. All tables designed on paper could be classified as *state displays* because they compared successive states of the functions instantiated by the devices. A case study is discussed below that explores the reconstructive, evolving and transformative nature of tables as inscriptions. The analysis presented is based on a segment of AL and CC's activity, in which they attempted to solve the problem of *"Will the black*

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[1] The sample had eight boys and ten girls, aged thirteen to fourteen years and paired with partners of the same gender.

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*spring ever be twice as long as the white spring?"* In the physical setup corresponding to this problem, the white spring "embodied" the equation $y=x+7$ whereas the black spring embodied the equation $y=2x+8$. (Interestingly, these students did not perceive the white spring as behaving linearly, and concluded that its displacement per pound alternated between 1 and $1\frac{1}{2}$.)

The representation examined below is formed by tables of values of a non-routine nature. Although their format was generally similar to that of textbook designs, creating them was definitely not a matter of simply recalling knowledge resources and applying them during the problem-solving session. Their evolution depended heavily on local aspects of the activity, such as the arrangement of sets of inscriptions (or *representational niches*) already on paper or the communicative intentions of the student-designer. In fact, I will show that the students' mathematical understandings and their material designs mutually constituted each other. In creating tables of values, AL and CC's overall goal is to display and compare intermediate states of each spring's displacement. The students' main strategy is to compute the joint effect of several weights on each spring (calling the resulting displacement a "unit"), and then to calculate the overall outcome (net-displacement) of having a set of "units." This will be instrumental for the task of comparing large values of length, as in the question of *"will the black spring ever be twice as long as the white spring?"* Ultimately, the students are unable to realize that such comparison of displacements does not directly answer the original question. Yet, their strategy organizes the complex behavior of the physical device (e.g., the perceived alternation of displacements between 1 and $1\frac{1}{2}$) by creating a representation that (1) imposes regularities on the data, (2) avoids calculations with fractions; and (3) captures "chunks" of data that make it easy to do large computations. The analysis of AL and CC's activity presented below will also show that (1) the meaning of a mark made on paper shifts over time, e.g., from ticks representing weights to a sequence of numbers representing individual displacements; and (2) individual inscriptions evolve into complex representational niches that facilitate quantitative inferences, but can have no direct "translation" to properties of the physical device. The segment presented below shows the students' recognition of the white spring's irregular behavior and their attempt to build a tabular representation, the evolution of which is shown in Figure 1 (A through G).

1. AL- It's like 1, and 1 and $\frac{1}{2}$, 1, 1 and $\frac{1}{2}$, is that right?
2. CC- Uh-huh.
3. AL- Let's just see... we just have to add... it's like... adding 1, 1, 1 (Figure 1-A), then uh, we know that at the second time it will be 3 (Figure 1-B), so these will be 1, so every 4 times it will be 5 inches (Figure 1-C), so see, it's 1 inch, 1 and $\frac{1}{2}$, 1 inch, 1 and $\frac{1}{2}$ (Figure 1-D)
4. CC- Uh-huh.
5. AL- Add these two it's 2, I mean, it's 3, plus 2 it's 5...

Observe in the segment above that AL starts the design on paper with four marks which initially denote a quantity of weights put on the white spring (Figure 1-A). Then, because of the spring's irregular behavior, AL marks the displacements caused by the second and fourth weights with the parenthesis and the number "3" (for $1\frac{1}{2} + 1\frac{1}{2}$) as in Figure 1-B. Meanwhile, the two other tallies representing weight are announced to cause a displacement of one each (*"so these will be 1"*). Notice at this point that the sequence of displacements represented on paper (1, $1\frac{1}{2}$, 1, $1\frac{1}{2}$) corresponds to the experimental observations

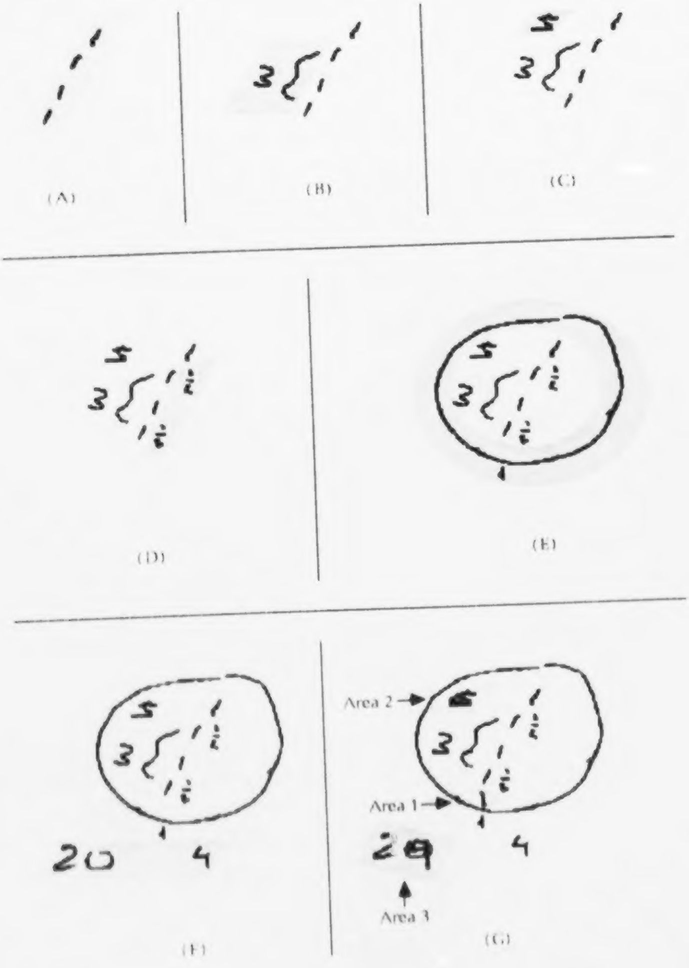
(A)
(B)
(C)
(D)
(E)
(F)
(G)
Area 2
Area 1
Area 3
20
4
29

**Figure 1**
ML and CC's table
(The shaded areas indicate the inscriptions added at each new phase.)

summarized in line 1, while the correspondence itself is represented in a non-routine way (i.e., the half-inch displacements are displayed only indirectly as an attachment to the representation of weights). Then, AL sums up the effect of all four weights on the spring, recording "5" on paper as in Figure 1-C. When communicating his design to CC (line 3: *"so see..."*), however, AL adds the $\frac{1}{2}$ marks shown in Figure 1-D. Thus, the same inscriptions drawn in phase A as a reference to weights, acquire in phase D a new configuration and meaning, as indexes of partial displacements. This episode then shows very clearly that representations are not fixed in meaning, but their signification shifts and evolves over time. Moreover, it is fundamental that the shift between meanings happened as AL attempted to make his inscriptions accessible to CC.

6. AL- So let's just say we went, this is like 1, 1, we'll just call it 1 unit (Figure 1-E)... will, it will go like, 4 more times so that's 20 inches down (Figure 1-F), so that (black) goes only like 2 inches at a time...
7. CC- Uh-huh.
8. AL- So, let's just put one more (Figure 1-G, area 1) so that will be 6 (Figure 1-G, area 2), 6 times 4 is 24 (Figure 1-G, area 3), okay.

In phase E, AL groups all marks made so far inside a circle and assigns to that area a *unit* value (corresponding to a displacement of 5 inches). This very explicitly creates a *representational niche* (Hall, 1990), i.e., a set of related inscriptions which "affords" computations and inferences. One such inference is illustrated in phase F, in which AL takes 4 units (each worth 5 inches) to calculate the spring's net-displacement after 4 times (*"4 more times so that's 20 inches down"*).[2] As suggested before, the creation of a grouping unit is instrumental for working on extrapolation problems, in the sense that it might yield very high output values to be calculated (when combined with appropriate multiplicative factors). Furthermore, the unit itself also functions as an index to previous marks on paper. That is, all the contents of the circle can now be referred to by a single unit tick. Observe also that the set of inscriptions visible in phase F carry relations of a different nature: 4 pounds makes the spring stretch 5 inches, taken as 1 unit; 4 times the unit makes the spring stretch 20 inches. However, none of these quantities are explicitly labeled on paper, illustrating what I call the *minimalist* nature of children-designed representations on paper. The meaning of the inscriptions are instead distributed across spatial locations (e.g., inside *versus* outside the circle that divides the paper in two regions), the conversations between the participants (in which units are verbalized), the device itself (through its visual presence in the setting), and the students' own personal memories of the events and marks.

Still in line 6, AL calculated for the white spring the first of a series of net-displacements (which the students seem to take as the spring's length), and started to work on the black spring by announcing that it *"goes only 2 inches at a time"* (remember that the underlying equation is $y=2x+8$). At that point, he realizes there is a mismatch between the white spring's *displacement per unit* (5) and the black spring's *displacement per pound* (2), perhaps recognizing that these values would not allow a straightforward comparison of spring elongation for the same amount of pounds. He then focuses once again on the representations for the white spring shown in phase F and adds an extra tick to the circle (phase G, area 1), which is immediately followed by other changes in the representation: the unit-displacement is updated from 5 to 6 (phase G, area 2), and the net-displacement from 20 to 24 (phase G, area 3). These actions had the effect of bringing each spring's displacement values to a multiplicative relationship, a sophisticated insight indeed. Finally, the transformations observed above show that a representational niche can be thought of as constituted by multiple *slots* (for unit-displacements, net-displacements, etc.), whose contents and meaning are manipulated according to circumstantial exigencies of the problem-solving situation.

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[2] The analysis that follows uses the terms "unit-displacement" to indicate the total displacement caused by "1 unit" (as defined by the students), and "net-displacement" to indicate the displacement caused by a group of units.

After changing the white spring's unit-displacement (making it a multiple of the black spring's displacement per pound), AL starts working on inscriptions to represent the black spring. As before, AL draws marks corresponding to five weights on the black spring, circles out these marks, indicates the unit-displacement (10) next to the string of "weights" and calculates the net-displacement for *"4 times"*. At this point, the representation carries all the slots and quantities needed to compare displacements between the springs: niches with a slot for unit-displacement, and areas for visually displaying the depended variable. As a consequence, the activity has been organized to a level in which computations should follow in a somewhat unproblematic way. Figure 2 shows the students' representation, after several comparisons between the springs had been made.

Later, as the activity seemed to be reaching an end with the students' apparent recognition that the black spring would not be twice as long as the white spring, AL started a new attempt to make sense of the task-situation by altering the white spring's unit-displacement. However, he erroneously takes the unit tick for the white spring (the "1" right below the circle, as shown in Figure 1-E) to indicate a value of displacement per pound (as if the tick were inside the circle). Then, in order for this value to fit the pattern represented on paper ($1, 1\frac{1}{2}, 1, 1\frac{1}{2}$), he switched it from "1" to "$1\frac{1}{2}$" (Figure 2, area 1). It is not clear whether AL really mistook the unit mark for a displacement one, or if he was simply unsatisfied with his previous attempt to solve the problem and used the "mistake" as a cover to alter the value of unit-displacement. In spite of AL's intentionality, the design on paper was at the very center of the students' problem-solving and could in fact be said to transform their activity in essential ways.

This last action, however, brings the white spring's unit-displacement to an unwelcome fractional value ($S=7\frac{1}{2}$). AL then added an extra "$1\frac{1}{2}$" mark at the very top of the ever-growing list of displacement terms (Figure 2, area 2). This action, I suggest, was aimed at bringing the list to an integer sum ($S=9$; Figure 2, area 4). However, the pattern now represented on paper no longer matched the sequence in which the displacements-per-pound were read from the apparatus ($1, 1\frac{1}{2}, 1$, etc.). How could this event be account for? A mapping-cognitivist explanation would focus on the student's internal processing, claiming that AL was engaged in a mapping procedure which takes events in the world and establishes correspondences to a representational level (first in the head, then on paper). Within this perspective, an account for the mismatch described above could be that a "cognitive overload" (or "system error") occurred as the student worked at a "pure" symbolic level in which sense-making of situations had been suspended. Alternatively, I propose on the basis of an activity view that the student was transforming the representation in tune with the *local circumstances* and *emergent goals* which required the pattern of displacements to alternate between 1 and $1\frac{1}{2}$. The display on paper carried that pattern in a material and flexible way, as the student could use it to make inferences somewhat independently of the empirical observations recorded earlier. As the product of the student's activity, the design on paper provided a critical source for mathematical goals to emerge and structure the activity itself. One advantage of an activity perspective over cognitivist ones is in emphasizing children's productive actions and the prospects for the development of new meanings, rather than assuming a failure in the first place.

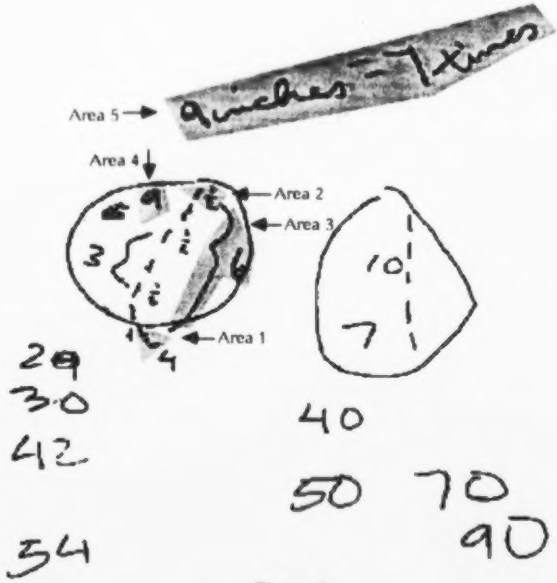

**Figure 2**
AL and CC's table (cont'd)

The segment finally ends with AL's writing of the expression "9 inches = 7 times" (Figure 2, area 5). This expression has an important function in the activity, for it creates a representational niche that explicitly summarizes and "abstracts" computations carried out inside the white spring's unit-circle. In particular, the expression has the quality of being easily reproduced (e.g., on another sheet of paper) and referred to during conversations. AL and CC's expression on paper will indeed function as the basis for regenerating the entire table of displacements for the white spring, as an equivalent design fulfills the same role for the black spring.

## Conclusions

In the segments discussed above, it is clear that constructing the table involved a great deal more than recalled knowledge. While tables of values are familiar tools in classroom practice, these students appropriated and specialized them in very particular ways to solve emergent problems during the interviews. The analysis showed, in particular, that the design of material displays transforms the very ways in which activities evolve, at the same time that it is transformed by the circumstantial organization of the problem-solving setting (which includes social interactions and aspects of the material world). The following is a summary of the specific findings discussed. (1) The meaning of inscriptions in a display shifts and evolves; (2) a design on paper may have several representational niches, each of which may afford different sorts of quantitative inferences; (3) any representational niche contains one to several "variable slots," of which content and meaning are manipulated and recombined (or even abandoned) depending on the requirements of the current setting of activity; (4) displays may have a very minimalist character as "secondary" information, such as labels, are substituted for conversation and gestures among the designers; (5) inscriptions are in many ways continuous with physical devices, but they can also develop a "life" of their own and carry acceptable mathematical information (from an "expert's perspective") that nevertheless contradicts data obtained from empirical observations; (6) material designs can be used to summarize and transport information displayed in yet other designs, creating something like a "cascade of representations"; (7) problem-solving strategies are not simply applied to material displays, but their very emergence may depend on the existence of specific displays in a situation.

Thus, "external" representations constructed by students during problem solving should be taken very seriously in the classroom. They not only reveal aspects of the students' thinking in a mathematical domain, but the making of inscriptions shapes the very activity in which designers are engaged. In the activity of professional mathematicians, displays designed on paper, chalkboards or computer screens do constitute this material basis without which mathematical sense-making becomes seriously handicapped. It is often the case, however, that mathematics instruction not only restricts students' design of unconventional and specialized notational systems (e.g., when doing arithmetic on paper), but also aims at suppressing the students' "dependency" on material representations altogether (viewed only as a means to acquire mental competences). Tallies and diagrams on paper (as well as finger counting and the use of hand calculators) are not lesser means of doing mathematics, but the very material basis of sense-making. As teachers, we should promote and not restrict students' design of material representations. More importantly, we should carefully investigate the processes of their emergence in the classroom and the ways in which material designs can afford the production of more designs and mathematical argumentation.

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# LES TYPES D'APPREHENSION EN GEOMETRIE SPATIALE : UNE ETUDE CLINIQUE SUR LE DEVELOPPEMENT-PLAN DU CUBE*

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*In previous studies in 2D-geometry, we have identified and described three kinds of algorithm apprehension - figural, functional, structural. In this paper, we analyse the possibility of extension to 3D-geometry ; our clinical approach, centered on the plane representation of the cube, seems to confirm this possibility.*

*En études antérieures, en géométrie plane, nous avons repéré trois types d'appréhension d'une construction algorithmique -figurale, fonctionnelle, structurale. Ici, nous analysons la possibilité d'extension à trois dimensions de ces résultats. L'étude clinique que nous avons conduite, en nous centrant sur le cas du développement-plan du cube, semble confirmer cette possibilité.*

Parmi les nombreux obstacles rencontrés dans l'apprentissage de la géométrie, celui que nous avons appelé le **double statut des objets géométriques** (A.L.Mesquita, 1989a ; 1991) semble être un des primitifs (puisque issu de l'origine même de la géométrie) et d'où résultent, par conséquent, d'autres obstacles. Le double statut résulte du fait suivant : en géométrie, tout concept, bien que distinct de ses représentations externes, risque d'être difficilement dissociable de celles-ci. D'où cette ambiguïté fondamentale : en s'appuyant sur des objets idéaux et abstraits, la représentation externe d'un concept ne peut être exprimée que par une configuration spécifique mettant en jeu des objets concrets et particuliers ; des propriétés inhérentes à celle-ci peuvent alors être associées au concept lui-même.

Nos recherches antérieures montrent que les types d'appréhension en géométrie plane sont liés, en dernière analyse, au double statut des objets géométriques (A.L.Mesquita, 1989a ; 1991). En effet, ces recherches montrent que le fait d'avoir, ou de ne pas avoir, dépassé le conflit provoqué par le double statut, a une influence déterminante dans le type d'appréhension d'une construction algorithmique qu'ont les élèves. Nous prétendons que les résultats que nous avons trouvés dans des situations particulières en géométrie plane, relevant d'obstacles plus généraux et de même nature, peuvent être étendus à trois dimensions. C'est celle-ci notre hypothèse.

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Dans la suite, dans une première partie, après avoir sommairement caractérisé les types d'appréhension identifiés dans ces études, nous nous interrogeons brièvement sur les questions soulevées par l'extension à trois dimensions, en nous centrant sur un cas spécifique, celui du développement-plan du cube[1]. Dans une dernière partie, nous présentons quelques premiers résultats concernant une étude clinique que nous avons menée, avec des élèves de 12 à 16 ans.

## Les trois types d'appréhension en géométrie plane

Dans l'appréhension **figurale**, les contraintes associées à la construction sont appréhendées d'une façon rigide et la construction est faite comme s'il s'agissait d'un dessin. Les élèves ne distinguent pas les contraintes ***figurales***, au sens de R. Duval (1988), c'est-à-dire, celles qui sont associées à une configuration particulière utilisée pour réaliser la construction, des contraintes ***algorithmiques***, c'est-à-dire, celles qui sont associées à la procédure algorithmique abstraite, et qui ne relèvent pas de la configuration particulière. C'est le cas des élèves qui reproduisent la construction algorithmique[2], dans le cas d'une configuration perceptive proche de la situation donnée, mais qui ne la font plus dans les autres cas. Notons que ces élèves utilisent, dans la description, les particularités de la configuration donnée : par exemple, "le carré sous le triangle". La désignation employée dans ce cas est une désignation spatiale *nominale*, de type nom propre, ne servant pas à établir des relations entre les objets, mais seulement à les nommer ; elle est liée à un emplacement absolu, tel un marquage sur un terrain. Nous retrouvons ici le type de repérage ***géographique*** mentionné par C. Laborde (1982) : A signifie ici "point situé en A". A ce niveau d'appréhension correspond une vision physique de l'espace, un espace perçu comme matériel et immobile. L'espace de travail associé est en conformité avec les autres caractéristiques : il est perçu comme un plan de dessin, avec une existence physique, matérielle et fixe, où la superposition de figures géométriques est inadmissible, par exemple.

Une deuxième forme d'appréhension est celle que nous avons qualifiée de **fonctionnelle**, dont les caractéristiques de transition recouvrent celles des deux autres types d'appréhension, extrêmes. Les contraintes, algorithmiques ou figurales, sont prises en compte, mais des confusions peuvent subsister (par exemple, une position relative, liée à une configuration perceptive, peut être perçue comme une contrainte algorithmique). La construction apparaît ici comme une espèce de schéma, traduisant quelques relations, mais en omettant d'autres. Liée à la distinction des deux types de contraintes, il y a une prise en compte de certaines adaptations considérées nécessaires, par les élèves, à la réalisation de la construction ; par exemple, dans la situation algorithmique utilisée, la construction est faite en mentionnant qu'il "fallait tourner la feuille". L'espace de travail est ainsi perçu comme un espace en transition : il apparaît encore comme un espace physique, mais mobile. Des obstacles résultants de la superposition de formes géométriques ou encore du dédoublement d'un côté commun au triangle et au carré ont été observés. Ce type d'appréhension se reflète dans la désignation utilisée, laquelle est *fonctionnelle*, liée à un emplacement relatif (et non plus absolu), permettant un repérage, où A signifie "point A". Les désignations apparaissent comme des invariants par rapport à certaines transformations, comme par exemple la rotation.

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[1] Nous remercions R. Pallascio et C. Janvier de cette suggestion.

[2] L'algorithme proposé, basé sur l'homothétie, concernait l'inscription d'un carré dans un triangle.

Un troisième type d'appréhension est celui que nous avons appelé **structural**. Ici, la distinction entre contraintes algorithmiques et figurales est bien nette pour les élèves, qui utilisent effectivement l'algorithme. L'inscription du carré est faite indépendamment de la configuration perceptive. Nous avons aussi noté que les obstacles propres aux autres types d'appréhension n'apparaissent plus : ni la superposition de formes, ni le dédoublement du côté commun. L'espace de travail apparaît comme un espace projectif, mathématique, mobile, sans existence matérielle. La désignation est *relationnelle*, permettant d'établir des relations entre les éléments ; elle n'est plus fixée à un emplacement physique. Les lettres apparaissent ainsi comme des variables "muettes" : cette désignation correspond ainsi à A, tout court, dépouillé de son sens matériel [3].

### L'appréhension en géométrie spatiale

Ces résultats, concernant une situation spécifique en géométrie, celle de la construction, relèvent d'obstacles plus généraux et en particulier du double statut des objets géométriques, obstacles qui ne sont pas exclusifs de la géométrie plane. D'où notre hypothèse : *en géométrie spatiale, on peut également identifier trois types d'appréension, dont les caractéristiques essentiels seraient de même nature*. Pour pouvoir étudier les types d'appréhension en géométrie spatiale, nous nous sommes centrés sur le cas du développement-plan du cube[4]. Les questions suivantes sont à la base de notre analyse :

- quelles sont les difficultés cognitives suscitées par l'utilisation de cette forme de représentation d'un objet à trois dimensions ?
- comment pourraient-on caractériser les types d'appréhension à trois dimensions, en particulier dans le cas retenu, celui du développement-plan du cube ?
- dans ce cas, quelles sont les situations (propriétés, tâches) qui peuvent être des indices révélateurs de ces types d'appréhension ?

***Analyse a priori***

1. Le développement-plan d'un solide est une forme de représentation externe, qui respecte la forme et la grandeur des faces et des arêtes, tout en conservant aussi les relations métriques bidimensionnelles (prises sur les faces ou arêtes). Par contre, l'apparence globale du solide n'est pas conservée ; de même,

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[3] Notons d'ailleurs que les différences que l'on peut trouver dans l'évolution du type de désignation correspondent à l'évolution historique (Freudenthal, 1985, p.147).

[4] Une analyse plus détaillée est décrite en A.L.Mesquita (à paraître).

certaines relations topologiques et les relations métriques, considérées comme relations tridimensionnelles, sont négligées. Etant données les caractéristiques de cette forme de représentation, le développement-plan présente une spécificité par rapport aux difficultés habituelles en géométrie spatiale : **plusieurs** points du plan peuvent être des images d'un **même** point de l'espace (cf. fig.1). C'est le phénomène du dédoublement, au sens de (R. Duval, 1983), où l'on associe à un même objet deux ensembles de propriétés, distinctes. Le dédoublement est un obstacle important dans l'apprentissage des mathématiques, comme le montrent certaines études (R. Duval, 1983 ; A.L.Mesquita, 1989b).

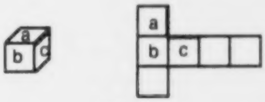


Figure 1

Ce phénomène peut éventuellement contribuer à expliquer pourquoi les enfants (jusqu'à l'âge de neuf ans) ont plus de difficultés avec les développements-plan qu'avec les graphes topologiques (Lunkenbein et al., 1981) ; la même étude suggère toutefois que, chez les adultes et les enfants plus agés, c'est la tendance opposée qui se manifeste.

2. Le développement-plan du cube est une notion qui apparaît dans la plupart des programmes scolaires de l'école primaire, lié à des fonctions spécifiques, celles de la construction de solides et du calcul d'aires; il n'est pas considéré, en général, comme un objet d'étude en soi.

3. Du point de vue des exigences cognitives, certains aspects nous semblent susciter des attitudes différentielles des élèves, et peuvent donc être révélateurs du type d'appréhension des élèves en regard du développement-plan. Ces aspects, qui fonctionnent alors comme des paramètres de notre analyse, sont les suivants :

- la production et la reconnaissance d'un développement-plan ;
- l'identification de critères de reconnaissance d'un développement-plan ;
- l'identification de relations entre les éléments du développement-plan et du cube.

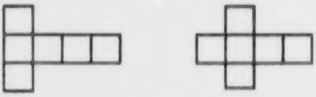
Plusieurs sortes de relations peuvent être considérées :

- *la relation de correspondance*, entre un élément du cube et l'élément(s) correspondant(s) sur son développement-plan ;
- *la relation de dédoublement*, concernant l'identification des éléments du cube qui apparaissent en dédoublement (cf. fig. 1) sur le développement-plan (ce problème se pose pour les sommets et pour les arêtes, ensembles de mesure nulle en $\mathbb{R}^2$ et $\mathbb{R}^3$) ;
- *le "transfert" des relations cube --> **développement-plan**,* ou plus précisement, appréhension des relations sur le cube à travers la représentation externe qu'est le développement-plan :

parallélisme des faces, arêtes, contiguïté des faces, etc.).

Parmi ces relations, le transfert des propriétés nous semble la tâche la plus complexe, spécialement dans le cas des développements-plan autres que **1-4-1**[5] (cf. fig.2). De plus, ce transfert présente la difficulté additionnelle de se réaliser dans un cadre relatif : une association arbitraire préalable entre des éléments du cube et des éléments correspondants du développement-plan est nécessaire ; c'est cette association qui fixe un cadre de référence.

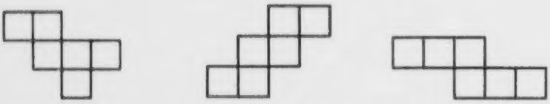
4. D'un point de vue perceptif[6], parmi les onze configurations qu'un développement-plan peut assumer, certaines sont plus faciles à reconnaître que d'autres. Par exemple, celles de la figure 2 sont plus immédiates que celles de la figure 3 (développements du type **2-3-1**, **2-2-2** et **3-3**, respectivement), puisque les transformations nécessaires pour obtenir un cube à partir sont du même genre (fig. 2), ce qui ne se passe dans la figure 3, qui exigent des transformations de genres différents.



type **1-4-1**

version "T" version "croix"

Figure 2



type **2-3-1** type **2-2-2** type **3-3**

Figure 3

***Méthodologie***

Notre étude clinique a été conduite auprès de huit élèves, de 12 à 16 ans, de deux écoles secondaires montréalaises. Le scénario retenu pour les entretiens s'appuie sur des questions déjà mentionnées : reconnaissance du développement-plan, dédoublement, transfert des relations (parallélisme des faces et des arêtes, contiguïté des faces).

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[5] Nous nommons les développements-plan à partir le l'occupation des faces, sur une "matrice" 3 x 4, une face sur une première ligne, quatre faces sur une deuxième ligne, une face sur une troisième ligne.

[6] Nous ne considérons pas ici les questions associées aux habiletés spatiales individuelles.

***Analyse des résultats***

Nos observations suggèrent que, à l'égard du cube et de son développement-plan, il y a certains invariants dans l'appréhension du cube et de son développement-plan. En effet, l'analyse des entrevues[7] montre qu'il y a caractérisation standard des réponses des élèves par rapport aux questions suivantes :

a) Stéréotypie du cube

Dans les dessins des élèves, le cube est tracé d'une forme uniforme. Il est vu en perspective, de type cavalière, où la face fronto-parallèle est systématiquement tracée d'abord. Dans la plupart des cas, les arêtes visibles ne sont pas tracées.

b) Stéréotypie du développement-plan

Dans tous les cas observés, les élèves ont tracé des développements du type **1-4-1**, version "croix" (cf. fig.2) et dans la plupart des cas, la configuration est dessinée sur l'horizontale.

***Les trois types d'appréhension du développement-plan du cube***. Nos observations suggèrent également que, par rapport à certaines tâches, les réponses des élèves se distribuent par trois groupes, avec des caractéristiques bien spécifiques et auxquels nous appelons les **types d'appréhension** d'un développement-plan. Ces tâches sont les suivantes :

- reconnaissance de développements-plan des types **2-2-2**, **3-3** et **2-3-1** ;
- relation de correspondance entre un élément du cube et les éléments correspondants sur le développement-plan ;
- transfert de certaines propriétés des éléments du cube à ses correspondants sur le développement-plan : contiguïté des faces, position relative des faces.

Nous résumons ci-dessous les caractéristiques essentielles des types d'appréhension à trois dimensions.

Une première forme d'appréhension, l'appréhension **figurale**, est dominée par la perception. L'articulation entre le cube et le développement-plan est dominée par la position relative de l'individu face au cube et elle dépend de l'emplacement du cube. Dû à la prégnance de la position relative du cube, le développement-plan apparaît alors comme une construction anisotrope, dépendante de la position relative dans laquelle il est perçu. Dans ce sens, il y a une face privilégiée du cube, appelée "base" par les élèves, une base qui est de nature géographique, unique, déterminée par son emplacement ; cette base est située dans le plan horizontal, (l'horizontale supérieure pour Mat, l'horizontale inférieure pour Gio). Les performances des élèves sont caractérisées par un recours systématique au pliage et aux manipulations, ce qui se reflète dans leurs justifications, de nature exclusivement empirique. En conséquence, le rapport entre le cube - qui est considéré ici comme un objet matériel, concret, (*"réel"*, pour Mat ) - et le développement-plan est ici perçu d'une façon absolue ; une dépendance très marquée existe entre le cube et son développement-plan. Autrement dit, l'association entre les éléments du cube et du développement-plan est bien déterminée par les positions géographiques du cube et du développement-plan, lesquelles déterminent univoquement la correspondance cube --> développement-plan. Le développement de type **1-4-1** est le plus spontanément obtenu. Des difficultés peuvent surgir avec les autres développments et en particulier **2-2-2** et **3-3**. C'est le cas de Mat, par exemple, qui recourt à un modèle en carton pour reconnaître **2-2-2**. En ce qui concerne le dédoublement, il n'apparaît comme obstacle que dans les cas d'éloignement des éléments en question : par exemple, deux arêtes sur le développement-plan peuvent être reconnues comme une même arête du cube à condition d'avoir un sommet commun sur le développement-plan. En résumé, cette forme d'appréhension est caractérisée par des liens géographiques et conjuncturels entre le cube et le développement-plan.

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[7] Des extraits des entrevues seront présentés lors de l'exposé.

L'appréhension **fonctionnelle** est caractérisée par une articulation partielle entre le cube et le développement-plan. Les développements-plans sont reconnus sans difficultés majeures. En ce qui concerne les performances des élèves, et en particulier leurs justifications, elles oscilent entre le recours au pliage et aux manipulations et le recours (non systématique) à des propriétés. Par exemple, dans une tâche, un élève peut mentionner le parallélisme des faces. Mais dans la suite de cette tâche, le même élève peut donner une justification empirique, basée sur le pliage et n'utilise plus le parallélisme mentionné auparavant. En ce qui concerne le statut des faces, l'existence d'une "base" est affirmée par les élèves ; celle-ci est perçue d'une façon relative : en fonction de la tâche, l'élève peut tourner le cube et choisir une "base" convenable. C'est le cas de Jon, par exemple. L'association entre le cube et le développement-plan n'est pas univoquement déterminée par les positions géographiques de l'un et de l'autre, des variations sont admises mais l'étendue et le statut de ces variations ne sont pas bien saisis par l'élève. En résumé, on peut dire que les liens entre le cube et le développement-plan ont perdu leur caractère absolu, ils aparaissent plutôt comme conjunturels, relatifs, dépendant de la tâche donnée.

Enfin, l'appréhension **structurale** est caractérisée par une articulation convenable, abstraite, entre le cube et le développement-plan. Le cube est vu ici d'une façon abstraite, comme représentant d'une classe, et non comme un objet en lui-même. En ce qui concerne le développement-plan, il est vu comme isotrope, indépendant de la position relative du cube et de son emplacement. Le pliage a un caractère relationnel et conventionnel, qui est bien perçu par l'élève. L'association entre un élément du cube et du développement-plan n'est pas perçue d'une façon absolue, mais d'une façon relative, en admettant une identification initiale arbitraire entre des éléments du cube et ceux du développement. Dans ce sens, il n'y a pas de "base" absolue, avec une fonction bien démarquée ; cela n'empêche que dans certaines tâches, il n'existe une face avec une fonction privilégiée, autour de laquelle s'organise le développement. Elle n'est identifiée que dans les cas où le repérage d'une telle face facilite la tâche. Dans ce cas, le choix est fait d'une façon heuristique, en fonction de la tâche. Elle peut ne pas exister, si la tâche ne l'exige pas. C'est le cas de Cat, par exemple, qui identifie une base dans une tâche de correspondance et ne la considère plus dans les tâches de parallélisme. En ce qui concerne les performances des élèves, toutes les configurations de développements sont reconnues sans difficultés. D'autre part, le transfert des propriétés cube --> développement est fait, sans obstacles. On peut observer aussi que les élèves recourent prioritairement aux propriétés, dans leurs justifications ; le recours au pliage apparaît en dernière instance, dans le cas où une autre justification plus rationnelle n'est pas possible. Les élèves qui ont ce type d'appréhension recourent à une stratégie stable de représentation du développement-plan. En résumé, on peut dire que les liens entre le cube et le développement-plan sont réduits à l'essentiel ; l'association se fait dans le contexte d'un cadre de référence minimal, compte tenu des relations de contiguïté.

## Conclusion et discussion

Nos premiers résultats suggèrent que les types d'appréhension peuvent être étendus à trois dimensions. Malgré le nombre réduit d'élèves concernés par notre approche clinique, les comportements des élèves semblent être assez stables : face à des difficultés conceptuelles du même genre, les élèves reagissent d'une façon analogue. Les types d'appréhension en trois dimensions sont liés à des obstacles de même nature que ceux de deux dimensions ; la caractérisation de ces types d'appréhension permet de saisir quelques unes des difficultés majeures en géométrie, et l'indication du type d'appréhension apparaît comme une forme de diagnostic de ces difficultés. Notre hypothèse semble, donc, se confirmer.

Bien entendu, notre analyse concerne une situation particulière en géométrie spatiale, celle du développement-plan ; il resterait à analyser ce qui se passe avec d'autres notions moins familières.

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# RESULTS OF RESEARCHES ON CAUSAL PREDOMINANCE BETWEEN ACHIEVEMENT AND ATTITUDE IN JUNIOR HIGH SCHOOL MATHEMATICS OF JAPAN

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*The purpose of the study is to review and synthesize results of four researches on causal predominance and presumable causal direction between achievement and attitude in school mathematics of junior high school of Japan. The researches are performed using cross-lagged panel correlation method, which is most effective for analyzing causal predominance.*

*Fifty six comparisons of achievement or subcategory of achievement and attitude reveal that there are five statistically significant causal predominance, and including these, 28 comparisons show the direction that attitude might be causally predominant over achievement, other eight show the reverse, and the rest 20 comparisons can not be identified.*

About 20 years ago, Neale (1968) revealed that there had been a modest correlation between achievement of and attitude toward school mathematics and almost all of the coefficients of correlation between them had dropped in the interval (0.2, 0.4). With some exceptions of lower or higher coefficients such as in the research (Minato, 1983), which indicated the existence of higher coefficients, the relation between achievement and attitude in school mathematics has not greatly altered.

In order to clarify the relation between achievement and attitude in school mathematics, there are two main streams: one is estimation of achievement with some plausible variables, for example parental and peer attitudes, as well as student' attitude, and the other is such research as Minato and Yanase (1984) who assume that there might be some differential attitudinal effects to achievement with student differential intelligence levels.

A common and essential problems of the above researches in both streams is to obtain the causal relationship between achievement and attitude in school mathematics. There may be some reciprocal or cyclical effects of achievement and attitude, it is therefore important to deal with not an exact causality, but the causal predominance between them, that is to say, which is true, achievement has more effects on attitude than the reverse, or attitude has more effects on achievement than the reverse.

The purpose of the study is to review and synthesize results of four researches performed by us on causal predominance and presumable causal direction between achievement of and attitude toward school mathematics of junior high schools of Japan. The researches reviewed in the study are as follows: Kamada (1988), Minato and Kamada (1988a, 1988b, 1991), in all of which cross-lagged panel correlation analysis, or simply CLPC, was applied for obtaining causal predominant relationship.

The effectiveness of CLPC in educational research was early indicated by Campbell and Stanley (1963), and Pelz and Andrews (1964), and in the research of mathematics education, it was pointed out by Aiken (1970), and CLPC was used by some researchers: Burek (1975) with some instances of significant causal predominance but inconsistent direction, Quinn (1978) with the result of non-significance, Wolf and Blixt (1981) with the result of non-significant causal predominance but a consistent direction such that attitude might be causally predominant over achievement in elementar school, and Quinn and Jadav (1987) without any significant causal predominance. In spite of the effort of these researchers, clear and consistent conclusive results have little been obtained on causal predominance and the direction of causality in the relation between achievement and attitude in school mathematics

## METHOD

### *Cross-Lagged Panel Correlation*

Cross-lagged panel correlation, or CLPC is a method of analysis in a quasi-experimental design (Campbell and Stanley, 1963) approprite for the setting with difficulty of manipulation of independent variables. It is a correlational procedure, but it can answer such questions as simple correlational analysis can not handle, and reveal causal predominance between two variables.

In the context of the study, CLPC is illustrated in Figure 1. In the figure, AC and AT mean achievement and attitude respectively at each point in time 1 or 2. and r(X,Y) is a coefficient of correlation between the variables X and Y.

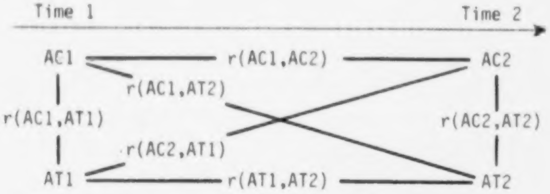

Figure 1. Diagram of Cross-Lagged Panel Correlation

In applying CLPC, following three assumptions are required: positivity of six correlations in the figure because of the requirement of educational research (it is satisfied in the study), synchronicity of measurement, which requires the measurement of achievement and attitude at each two points in time (it is also satisfied), and stationality which means that the structure of the relation between achievement and attitude is unchanged at time 1 and time 2. When the assumptions are all met, then causal predominance between achievement and attitude can be discriminated as follows: if r(AC1,AT2) is significantly larger than r(AC2,AT1), then achievement has more effects on attitude than the reverse, or achievement is considered to be causally predominant over attitude, denoted it as AC ➡ AT (bold face arrow), and if r(AC2,AT1) is significantly larger than r(AC1,AT2), then attitude has more effects on achievement than the reverse, or attitude is considered to be causally predominant over achievement, denoted it as AC ⬅ AT. In these cases, Pearson-Filon z-tests is performed.

When above three assumptions are met, and the difference of the strength of two cross-lagged correlations r(AC1,AT2) and r(AC2,AT1) is not statistically significant, the difference between the effects on both directions is small. In this case, presumable causal direction can be considered as Wolf and Blixt (1981). In the study, if r(AC1,AT2) is at least numerically larger than r(AC2,AT1), then we might think presumable direction as is in the case of statistically significant causal predominance, and denote it as AC ⟶ AT, using light face arrows, and if the presumable direction is the reverse, then we denote it AC ⟵ AT. The statistical significance of stationality is judged using Pearson-Filon z-tests for the cross-lagged correlations r(AC1,AT2) and r(AC2,AT1), which are not independent. When the assumption is not met, we do not deal with any direction and causal predominance, we denote the case as AC ... AT (dotted line) in the study.

### *Subjects and Administration of Instruments*

In the study, we deal with almost common facets of achievement and attitude in our four literatures: A (Kamada,1988), B (Minato and Kamada,1988a), C (Minato and Kamada,1988b), and D (minato and Kamada,1991), neglecting some variables as skill and mathematical thinking in cognitive domain, and anxiety in affective domain. Subjects and administration of instruments are summarized in Table 1.

Table 1. Subjects and Administration of Instruments

| Lit. | No.of Ss (School) | Instrument | Res. | Time 1 (Grade) | Time 2 (Grade) |
|---|---|---|---|---|---|
| A | 150 (College Attached) | CA<br>MSD,MSD(E) | A1 | Winter,1982(7th) | Winter,1983(8th) |
| | | | A2 | Winter,1983(8th) | Winter,1984(9th) |
| | | | A3 | Winter,1982(7th) | Winter,1984(9th) |
| B | 420 (2 public) | CA,NC,FN,GF<br>MSD,MSD(E),FA | B | Winter,1986(8th) | Winter,1987(9th) |
| C | 500 (2 public) | CA,NC,FN,GF<br>MSD,MSD(E),FA | C1 | Winter,1986(7th) | Winter,1987(8th) |
| | | | C2 | Winter,1987(8th) | Autumn,1987(9th) |
| | | | C3 | Winter,1986(7th) | Autumn,1987(9th) |
| D | 230 (Public) | CA<br>MSD,MSD(E) | D | Summer,1990(8th) | Spring,1991(9th) |

Note: School year in Japan begins in April and ends in March of the next year.

In the table, abbreviations are as follows: CA:achievement total, NC:number and calculation, FN:function, GE:geometric figure in cognitive domain, and MSD: an

SD type attitudinal instrument (Minato,1983), MSD(E):the evaluative factor of MSD, and FA:Likert type attitudinal instrument (Minato and Yanase,1984).

## RESULTS

The results of the study are summarized in Table 2, using the arrow diagram stated in the previous chapter.

Table 2. Results Represented by Arrow Diagram

| Research | Causal Predominance and Direction | | |
|---|---|---|---|
| A1 | CA ← MSD | CA ... MSD(E) | |
| A2 | CA ← MSD | CA → MSD(E) | |
| A3 | CA ← MSD | CA ... MSD(E) | |
| B | CA ← MSD | CA ← MSD(E) | CA ← FA |
| | NC ⬅ MSD | NC ← MSD(E) | NC ⬅ FA |
| | FN ← MSD | FN ← MSD(E) | FN ← FA |
| | GF ← MSD | GF ← MSD(E) | GF ← FA |
| C1 | CA ← MSD | CA ← MSD(E) | CA ← FA |
| | NC ← MSD | NC ← MSD(E) | NC ← FA |
| | FN ⬅ MSD | FN ⬅ MSD(E) | FN ⬅ FA |
| | GF → MSD | GF → MSD(E) | GF → FA |
| C2 | CA ... MSD | CA ... MSD(E) | CA → FA |
| | NC → MSD | NC ... MSD(E) | NC → FA |
| | FN ... MSD | FN ... MSD(E) | FN → FA |
| | GF ... MSD | GF ... MSD(E) | GF ← FA |
| C3 | CA ... MSD | CA ... MSD(E) | CA ... FA |
| | NC ... MSD | NC ... MSD(E) | NC ← FA |
| | FN ... MSD | FN ... MSD(E) | FN ... FA |
| | GF ... MSD | GF ... MSD(E) | GF ... FA |
| D | CA ← MSD | CA ← MSD(E) | |

## CONCLUSIONS

Table 2 shows that there are five comparisons of causal predominance for total 56 comparisons of achievement and attitude. Although cases of significant causal predominance is not many, the summarized result of the study is so uncommon that only Burek (1975) exceptionally obtained significant causal pre dominance as we know and make reference to previous reseraches in school mathematics.

Moreover, the direction of the cases of significant causal predominance is uniformly described by arrows from attitude to achievement, and there are same direction, significant, and not significant or presumable, of 28 in 36 comparisons with directions identified.

In the research C1, in which subjects from seventh to eighth grade are involved, and subcategory of achievement is geometry, there are uniform direction of arrows from achievement to attitude: achievement of geometry has more effects on attitudes than the reverse. In Japan, learning of demonstration in geometry starts preliminarily at the late of seventh grade and in thorough manner it begins at eighth grade.

Although there are eight arrows with the direction from achievement to attitude, a plausible direction might be that attitude causes more over achievement than the reverse. This direction coincides with the consistent direction obtained by Wolf and Blixt (1981), who say that it is suggested that attitudes are stronger cause of achievement than achievement is of attitudes. We are going to conduct a research on causal predominance between achievement and attitude in school mathematics, and the results will be appeared two or three years later.

Mathematics education of Japan has embraced a crucial problem of "the lowest scores in affective domain in contrast to the highest scores in cognitive domain (in junior high school)", revealed in the Second International Mathematics Study, as Travers and Westbury (1989), and McKnight and six others (1987) described. Improving students' attitude toward mathematics is one of the not-negligible subjects of mathematics education in Japan.

It is urgent to make teachers and educators deeply understand the importance of attitude toward mathematics as one of the major purpose of mathematics education, and to inform them real situation of the relationships between achievement of and attitude toward mathematics. In our country, some teachers and educators of mathematics seem to believe that increasing of cognitive scores necessarily makes increasing of attitudinal scores, and there may be other teachers and educators who believe that attitude toward mathematics is valuable only when it serves the purpose of high achievement in cognitive domain of school mathematics.

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# CHILDREN'S CONCEPTS OF PERPENDICULARS

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*An attempt was made to teach samples of typical Grade 1 and Grade 4 students the concept of perpendicular lines, without using this word. Testing of Grade 1 students was abandoned when it was found that they were unable to conceptualise perpendiculars as lines in a special angular relationship. Although all of the Grade 4 students had studied right angles in school, only about two-thirds seemed able to understand this conceptualisation of perpendiculars, showing varying degrees of uncertainty in its application. The results are interpreted as confirming the complex nature of the angle concept and the importance of relating static representations to dynamic models.*

Research on the perceptual basis for the formation of geometrical concepts is relatively rare. The mathematics education literature deals almost exclusively with knowledge of the standard terminology, whereas the psychological literature tends to concentrate on accuracy of discrimination and representation. Concerning the concept of angle, an attempt to bring the two research traditions together was made in Mitchelmore (1989), from which it is clear how little we know about how children abstract such concepts from their perceptual environment and the difficulties which they face.

We may consider parallels as lines in a special angular relationship. The perceptual basis for the formation of the parallel concept is now relatively clear (Mitchelmore 1986): Parallelism is a perceptual primitive used in alignment tasks at least as early as 4 years of age, and 6 year olds are well able to apply a verbal label to distinguish parallels from non-parallels. However, young children still experience difficulties locating parallels in complex figures and in drawing parallels on a base line.

The next most obvious special case to look at is the relationship of perpendicularity. Right angles are more accurately copied than other angles and there is some evidence that they may be more easily differentiated by young children, so there is some suggestion they may also be perceptually primitive (Mitchelmore 1986). Perpendicularity is in fact a feature of our "carpentered" environment, and right angles are usually the first angles to be treated in the school curriculum. The purpose of the present study was to examine the perceptual basis for the formation of the perpendicularity concept, following a similar approach to that used previously for parallels.

## METHOD

Two groups of 16 children were selected for individual interview, one group from Grade 1 (average age 7.1 yr) and one from Grade 4 (9.9 yr). Eight children (4 males and 4 females) were selected at random from one class in each grade from two Catholic schools located in predominantly middle class areas.

By analogy with a trick used in previous research, perpendicular lines were called "enemies". Using transparent plastic strips, parallel lines, called "friends", were defined as lines which "go along side by side" and will not "bump into each other" however far they go on. After testing to confirm that this term was understood, the experimenter demonstrated by changing the angle between the strips that some lines were only "a little bit unfriendly" whereas others were "much more unfriendly" because they "bump into each other hard". The subject was then invited to place the strips so that the lines were "as unfriendly as possible". If the subject showed any difficulty with this idea, the experimenter placed the strips perpendicular to each other and explained, again by varying the angle, that this was the "most unfriendly" relation possible. Lines in this relation were then called "enemies".

To further clarify the idea of enemies, the subject was next shown a card with two sets of three angles, each containing one acute, one right and one obtuse angle in similar orientations. (In one set the right angle looked like a rotated T, in the other like a rotated L.) The experimenter helped children to select the enemies from each set, explaining the idea again if necessary and showing them how to lay the plastic strips along the lines to make the relation between them clearer.

To test children's understanding of the term "enemies", they were first shown two cards each containing 8 angles in various orientations, in which 4 angles were right angles, the others being 60, 75, 105 and 120 degrees. The first card showed T-type and the second L-type intersections. Children were asked whether each picture showed enemies, and encouraged to use the plastic strips to check. Errors on the first 8 items were corrected immediately, but no feedback was given on the second 8 items.

As a final step in the initial definition of enemies, a summary card was laid on the table showing two examples each of "friends", "enemies" and "neither" and children were asked to identify the friends and enemies in a further 8 items, 2 of which were right angles and 2 showed angles of 75 degrees. No feedback was given, but the summary card was left on the table for the remainder of the interview.

The second stage of the interview consisted of children identifying the friends and enemies in various figures (see Figure 1). Children were prompted (e.g. "Are there any more enemies?"), if necessary, to ensure that they analysed each figure fully.

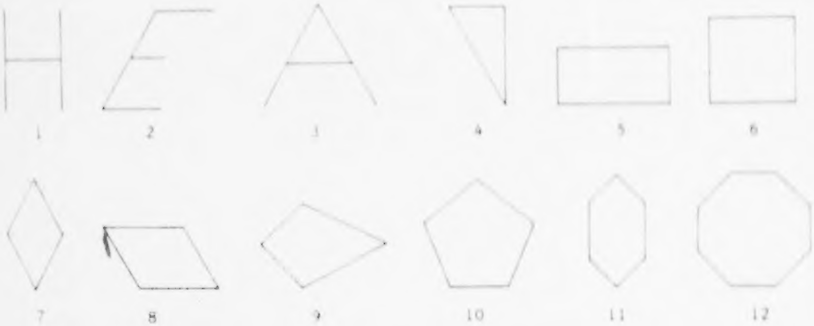

Figure 1: Shapes used for identification of perpendiculars.

In the third stage, children were presented with 9 drawings to complete. For each item, a diagram containing one red line and one or two black lines was presented on a card. The child had a booklet on which each page showed only the black line(s) and completed the figure by drawing a red line to make it look "exactly like" the given diagram. In 3 cases children copied parallels, in 3 cases perpendiculars and in 3 cases neither; the 3 drawings in which the target line was perpendicular to a given line are shown in Figure 2. Children had already done these drawings as a warm-up task, and now it was

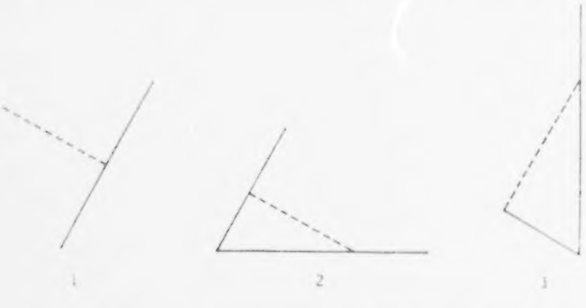

Figure 2: Perpendicular drawing tasks. The broken lines show the target lines (given in red and copied by the subject).

suggested to them that looking for friends and enemies might make such drawing easier. Before making each drawing, children therefore had to identify any friends or enemies in the given diagram; and after drawing, they were asked if there were any friends or enemies in their drawing.

In the final briefing, children were asked if and how identifying friends and enemies helped them do the drawings, whether they had ever thought about or seen friendly or enemy lines, and if they had ever studied anything similar in school.

## RESULTS

We report here only the results on perpendiculars. For the results on parallels, see Mitchelmore (1992).

The interviewing of Grade 1 students had to be abandoned after pilot testing showed that they could not understand "enemies" in the sense intended. The children tested did not seem to see that there could be degrees of "unfriendliness": they could not indicate a "most unfriendly" position for the two strips, and either categorised all non-parallel lines as enemies or gave apparently random responses. This could have been partly a verbal problem (after all, there is little difference between "unfriendly" and "enemy" in normal usage), but it was clear that the children were at a stage when no one way in which lines might be inclined to each other stood out as anything special.

All Grade 4 children were able to indicate or reproduce a "most unfriendly" relation, although some placed one strip on top of the other at their first attempt. Several children offered their own characterisation of enemies: The lines make a T; they make a cross not an X; the line is "in the middle", not sloping; the shape where the strips overlap must be a square not a diamond; the angle must be the same on both sides. However, the transition from concrete to pictorial representation was difficult for many children. Only 6 children (38%) required no feedback on the first 8 test items, and only 6 (including only 4 of those who initially made no errors) made no errors on the next 12 perpendicularity discrimination items. Of the total of 25 errors made by the remaining 10 children on these last 12 items, the great majority (84%) were false positives; in other words, enemies were much less likely to be missed when present than assumed when absent. On the other hand, no child acted as if all intersecting lines were enemies, and only one seemed to be guessing at random (6 errors on 12 items).

Children showed a wide range of success in identifying perpendiculars in the 14 more complex figures (Figure 1:1-12 and Figure 2:2-3). Three groups of children could be identified, depending on their success with four groups of figures.

> Group 1 The 5 children in this group usually identified enemies correctly in figures containing only horizontal and vertical lines (Figure 1:1,5,6); for these figures, the overall success rate was 88-94%. Other figures were usually analysed incorrectly, almost all errors being false positives. Two of these 5 children assessed almost all intersecting lines as enemies, and two did not recognise the corners of the square and rectangle as enemies.
>
> Group 2 The 6 children in this middle group always identified horizontal-vertical perpendiculars correctly and in addition were almost always successful with the pentagon, rhombus and parallelogram (Figure 1:7,8,10), for which the overall success rate was 67-73%. Their success came purely from avoiding the false positive judgements made in Group 1 on this second group of items. These 6 children were successful on no more than 3 of the remaining figures, almost all errors now being false negatives.
>
> Group 3 In addition to the figures already noted, a further four figures (Figure1:2,3,4 and Figure 2:2) were also analysed correctly by the 5 children in this group. The overall success rate on these 4 items was 40-57%, the most difficult item being the "inverted" right-angled triangle (Figure 1:4). None of the 5 children in this group identified the perpendiculars in the remaining 4 figures correctly (Figure1:9,11,12 and Figure 2:3), for which the overall success rate ranged from 0-33%. The octagon (Figure 1:12) was expected to be difficult, but two students in this group recognised two of its 8 pairs of (non-adjacent) perpendicular sides, the only ones to do so. The triangle in Figure 2:3 proved surprisingly difficult, being only analysed correctly by 3 children, all in this upper group. The frequent failure to recognise right angles in the kite and hexagon (Figure 1:9,11), even in the top group, is somehow easier to understand.

There was only a loose relation between success at identifying enemies in simple and complex figures. Of the 6 children who made no errors in identifying enemies in simple figures, one was in Group 1, two were in Group 2 and three were in Group 3 for the complex figures. Of the 6 children who made 3 or more errors on simple figures, three were in Group 1, two were in Group 2 and one was in Group 3.

Accuracy of drawing perpendiculars (recall that children copied each figure in Figure 2 twice) also varied widely between children and across figures. The most accurately copied figure was No.3, where only 12% of errors exceeded 5 degrees. No.1 was the next easiest, with 38% of errors exceeding 5 degrees and a slight general tendency to copy the perpendicular nearer to the vertical. No.2 was clearly the most difficult: 62% of errors exceeded 5 degrees and in all but two of these cases the error was to make the triangle more nearly symmetrical. This bias was expected from previous unpublished research, but why it did not also occur for No.3 is not clear.

Drawing accuracy was hardly related to success at identifying perpendiculars. For example, the one student who made errors of 5 degrees or less on all six drawings was in Group 1 for identifying enemies in complex figures, and the one student who equally consistently made large drawing errors was in Group 3. Also, the perpendicular intersection in Figure 2:3 was the most accurately drawn but the most rarely recognised.

All students said that identifying enemies had helped their drawing, and that whenever they saw an enemy in a drawing they attempted to draw an enemy themselves. In fact, the second set of drawings were slightly more accurate than the first set; for example, the overall percentage of errors exceeding 5 degrees fell from 44% to 31%. No student admitted to ever having thought about enemy lines before, but all stated that they had seen them in school books or lessons and had no difficulty in finding examples in the interview room. (One student even noted that the three lines meeting at each corner of a box were enemies of each other.) They had seen many of the shapes in Figure 1, and most named Nos.5-7 as rectangle (or oblong), square and diamond. All remembered being taught about angles, explained acute and obtuse angles correctly, and stated that they had called enemies right angles. (A few students in one school used the word "perpendicular".) Explanations of what a right angle was included "like the corner of a book", "a square", "an L-shape" and "you can fit a little box into it". Two children added obscurely "you can't fit a little square into an acute angle".

## DISCUSSION

The results of this small study confirm that the angle concept is a very complex one, even for the special case of right angles. It might have been predicted from students' statements that they all understood right angles, at least at a global level, and that this should have been sufficient to allow them to recognise right angles in various figures. And yet this was clearly not the case: Of the 16 Grade 4 students interviewed, approximately one-third could see nothing special about right angles and placed most vertices of polygons in the same category; approximately one-third regarded a right angle as a special angle but were easily misled into errors of judgement; and approximately one-third could recognise right angles in most figures but often overlooked them in more unusual figures.

There are apparently two reasons why students had not yet been able to form a general concept of a right angle. Firstly, definitions in global terms by reference to perpendiculars in the environment or corners of a square tend to limit the concept image to the relation between horizontal and vertical lines. For perpendiculars in the environment are almost always parallel to the sides of a rectangular enclosure, and frequent children's comments differentiating a diamond from a square show that they generally believe a square has to be in the standard orientation. With such a restricted conception of a right angle, learning perpendicularity as a relation between lines in any orientation must be like starting something completely new. This was in fact observed in several students, who initially checked all intersections with the plastic strips and only gradually went over to purely visual checking.

The second characteristic of the right angle which the right-angle-as-square-corner approach does not attack is its special status in the set of all angles. While we can assume that children can visually distinguish a square from a non-rectangular rhombus, no attempt is normally made to ask what it is that makes the square different. Fitting a "square corner" into an angle in order to classify it as acute, right or obtuse does nothing to show what is special about a right angle.

It is not difficult to imagine activities which would make the special nature of right angles explicit. Fitting two squares together to make a rectangle can be contrasted to what happens with non-rectangular rhombuses, and the result extended to tessellations. Right angles can be made by folding a piece of paper, and they can be identified in symmetrical intersections. The critical point is that the two angles of intersection are the same. In this respect, the standard L-shaped representation of a right angle is completely inadequate; right angles need to be seen as parts of T- and X-shaped intersections.

The results show clearly that drawing activities alone are unlikely to develop the concept of a right angle. Drawing itself, without reflection, cannot be expected to draw attention to the special nature of some of the angles drawn. Sighting along a line to test for accuracy of perpendiculars (as several students did) would be one starting point for useful discussion.

The interviews incidentally pointed to general problems children have in analysing complex figures, problems which are quite independent of their conceptions of perpendicularity. Several comments such as "it's like a T if you take away this line" and "those lines would be friends if that line weren't there" show that even for 10 year olds the action of disembedding two lines from a figure is not a natural one. Expressing binary relations was also difficult, as shown by frequent statements like "all of them are enemies" referring to the sides of the rectangle, and "there are two enemies" when they meant two pairs of enemies; none of these students had any problem pointing out the pairs of lines involved, when queried. It was perhaps variations in disembedding and language ability which led to the rather loose relation observed between success at identifying perpendiculars in simple and complex figures.

The general conclusion from this study is that perpendicularity is not perceptually primitive in anything like the same way as parallelism. The special relation between parallel lines (their alignment) is visually obvious and can be easily verbalised, analysed and applied; but the special relation between perpendicular lines needs analysis before it can be generalised. Reflection based on learning activities such as those suggested above is necessary to make the special nature of perpendiculars explicit in a way which the static conceptualisation of the right angle as the corner of certain common shapes cannot. Treating right angles is this more dynamic way would also contribute to the longer term development of the general concept of angle, for which the notion of varying degrees of incidence is critical (Davey & Pegg 1991). In other words, the right angle should be seen as a special kind of angle, even though a complete concept of angle is not yet available. Educators should not be disconcerted by such apparent contradictions. It is becoming clearer and clearer that psychological development does not usually follow the logical development of basic mathematical ideas.

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# Students' Use of the X-Intercept: An Instance of a Transitional Conception

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*This study discusses the nature of students' use of the x-intercept in equations of the form y=mx+b by summarizing the results of written assessments and the case studies for two pairs of students exploring and discussing linear equations and their graphs. I argue that the uses of the x-intercept documented in the assessments and the videotaped discussion sessions are not a superficial error, a simple mismatch with convention, or a misconception. Instead, this student conception is an example of a transitional conception: a conception which is reasonable, useful, applicable in some contexts, and has the potential for refinement.*

## Introduction

In a previous paper (Moschkovich, 1990) I summarized students' interpretations of linear equations and their graphs documented in two algebra classrooms. Written assessments designed to explore these interpretations in greater depth were completed by 18 students from these classrooms. The responses on these pre-tests showed that 13 of the students in this study (72 %) used the x-intercept at least once when working with equations of the form y=mx+b, in place of either the parameter b (12 students) or the parameter m (5 students) in their equations. Six of the students also described lines as moving left to right (or right to left) along the x-axis as a result of changing b in an equation. This conception was also evident during the discussion sessions and on the post-test responses of several students[1].

These uses of the x-intercept could be considered as misconceptions, errors, or even a simple mismatch with the convention that the x-intercept does not appear in equations of this form. On the other hand, the use of the x-intercept could be understood as a reasonable conception which reflects the mathematical complexity of this domain, which is applicable in some contexts, and which can be refined. I will summarize the analysis of the videotaped discussion sessions, focusing on two pairs of students. This data supports the claim that the use of the x-intercept is an example of a transitional conception rather than an error or a misconception.

The data from the discussion sessions show that the use of the x-intercept was more than a superficial mistake from which students could easily recuperate. This conception became a central point of the students' exploration and discussions. Most of the students spent a considerable amount of time discussing the x-intercept and making sense of their responses regarding the x-intercept. Moreover, the post-test results show that many of the students continued to use the x-intercept even after the discussion sessions (50% used the x-intercept at least once in the post-test).

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[1] The results of the written assessments and the analysis of the videotaped discussion sessions are discussed in detail in my doctoral dissertation (Moschkovich, 1992).

## Perspectives of student conceptions

Research on student conceptions in mathematics has documented particular student ideas and described how they are at variance with expert ideas. However, it has neither presented a comprehensive account of student conceptions nor resolved crucial questions regarding the nature and transformation of these conceptions (Smith, diSessa, and Roschelle; n.d.). Analyses of student conceptions describing errors and misconceptions have focused largely on the "mis-" aspect of student ideas and have not considered conceptions that may be useful, applicable in some contexts, or productive for advancement. On the other hand, while the term "alternative interpretations" shows a certain respect for student ideas, its use misses the point that while there may be many alternative ways to conceive of a domain, there is a mathematically accepted way to think about the subject matter.

The problem is not only in the theoretical perspective, but also in what types of student responses or ideas are the objects of analysis. The analysis of errors or bugs emphasizes the procedural aspects of a domain, does not address differences in terms of conceptual content, and usually refers to responses on limited tasks. Some student responses are mistakes or errors which can be addressed directly and erased or replaced by the correct convention (for example a reversal of the coordinates of a point). Some errors are the result of repairs to achieve an answer on a written problem, while others can be related to deeper conceptions or principles. Other student conceptions may, at first glance, look like simple mistakes or misconceptions but may in effect be useful conceptions which are sometimes applicable and have the potential to be refined.

## Transitional conceptions

Understanding learning in a complex domain such as linear functions necessitates a perspective which lies somewhere between the two extremes of seeing student conceptions as simply either wrong or merely alternative. In the following discussion I will attempt to balance the positive and negative aspects of student conceptions, while still acknowledging the tension between mathematically accepted conceptions and those conceptions students generate. I will introduce and use the concept of "transitional conception" to describe students' use of the x-intercept as a way to include important conceptual knowledge which is not always an error, which is sometimes useful (depending on the context), and which can change by refinement.

While initial conceptions are sensible to the learners themselves, they are also different from expert conceptions; different objects may seem relevant, the relationships among objects may be unspecified, and language usage may be different and ambiguous. Students' use of the x-intercept will be described in terms of the objects and the relationships between these objects that students see as relevant. The refinement and transformation of conceptions is assumed to occur through inherently social processes which are mediated through language[2].

### Mathematical complexity of the subject matter

The task for the students in this study was to make sense of the connections between the algebraic and graphical representations of linear functions. Using the x-intercept for equations of the form y=mx+b is a reasonable conception to include in this process. In effect, it reflects the mathematical complexity of this domain[3]. I have thus far used the term "x-intercept" to refer to either the point where a line crosses the x-axis or the number which is the x-coordinate of this point (and which can be used in an equation). For the sake of clarity, during the following discussion I will continue to use the terms "y-intercept", "x-intercept", and "slope" as general terms for the objects in either representation. However, I will distinguish between the graphical and the algebraic objects as follows:

$\mathbf{b_E}$: the algebraic y-intercept
$\mathbf{m_E}$: the algebraic slope
$\mathbf{a_E}$: the algebraic x-intercept

$\mathbf{b_G}$: the graphical y-intercept
$\mathbf{m_G}$: the graphical slope
$\mathbf{a_G}$: the graphical x-intercept

The **m** and **b** correspond to the parameters in the form **y=mx+b**. **a** corresponds to the parameter **a** in the equation **x/a + y/b=1**, where **a** is the the x-coordinate of the x-intercept, and **b** is the y-coordinate of the y-intercept. The subscript E stands for "Equation" and the subscript G stands for "Graph".

There are differences in how perceptually salient the slope, the y-intercept, and the x-intercept are as well as what steps are necessary for obtaining information from and about these objects. Thus, these objects have a different status within each representation and in relation to each other. In the graph of the equation y=mx+b, $\mathbf{b}_G$ is a special point, accessible directly by looking at the graph and locatable by its ordered pairs. On the other hand, $\mathbf{m}_G$ is only accessible graphically after estimating the slope or determining the rise and the run. While $\mathbf{b}_G$ is one point on the line, $\mathbf{m}_G$ is an object which is a characteristic of the line as a whole. When connecting the two representations, while $\mathbf{b}_E$ is the coordinate that pairs up with an x-coordinate of 0, $\mathbf{m}_E$ does not correspond to one coordinate. In contrast, in the equation $\mathbf{m}_E$ and $\mathbf{b}_E$ do have similar status. They are both letters, and although one object is related to the variable x by multiplication, and the other by addition or subtraction, this representation in no way gives one a clue to their different graphical status.

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[2] The transformation of students' descriptive language is discussed in detail in my doctoral dissertation.

[3] Although the following is an analysis of the subject matter from a competent perspective, it was inspired in large part by the observations of students grappling with this domain.

In terms of the two intercepts of a line, on the graph $\mathbf{b}_G$ and $\mathbf{a}_G$ have the same status: they are perceptually salient and directly accessible by their coordinates. However, in any equation not of the form x/a + y/b=1 (and even in that form the coordinates of the intercepts are not exactly easily accessible), the two intercepts do not have the same status in the algebraic form that they have on the graph. In the form y=mx+b, $\mathbf{b}_E$ is directly accessible in the equation, while it is necessary to set y equal to zero and solve for x to arrive at $\mathbf{a}_E$= -b/m.

The conceptual asymmetry between the two representations is that not all the information accessible from the graph shows up in any one equation form. Moreover, not all the information that is available graphically is necessary for determining a line algebraically through any one form of the equation. For any of the available algebraic forms, the mapping between the algebraic and graphical objects is by no means perfect or simple. Thus, this conceptual asymmetry exists regardless of which form of the equation is used. For equations of the form y=mx+b, this conceptual asymmetry is evident in the x-intercept: while the x-intercept has status as a graphical variable, it does not as an algebraic variable in this form of the equation. Since $\mathbf{a}_G$ is as perceptually salient as $\mathbf{b}_G$, it is not surprising, then, that students would expect it to be as easily accessible in the equation as the y-intercept is.

While $\mathbf{a}_G$ and $\mathbf{b}_G$ are points, $\mathbf{m}_G$ is a more global object. For some students the concept of intercept may not include the fact that one of the coordinates is 0, and thus they may regard the intercepts as one-dimensional objects. In the case of the slope, it is necessary to see the line as an object which changes its inclination in a two-dimensional plane. If the choice is between using one two-dimensional object (slope) and one one-dimensional object (y-intercept), or two one-dimensional objects (the intercepts), the choice is clear from a simplicity standpoint. Even if students do not see the intercepts as one-dimensional but as points in a two-dimensional plane, the intercepts are still simpler objects than the slope, both perceptually and algebraically, since slope is neither a point nor an extensive quantity. It is not surprising, then, that students would choose to focus on the x-intercept over the slope. Simply on the basis of the algebraic and graphical status of these six objects, students' attempt to use the x-intercept in the form y=mx+b is a very reasonable notion.

Furthermore, since the x-intercept changes as either the slope or the y-intercept change, it is also reasonable that students would focus on the x-intercept and have difficulties specifying the relationships among these objects. Changing either $\mathbf{m_E}$ or $\mathbf{b_E}$ has a graphical effect not only on $\mathbf{m_G}$ and $\mathbf{b_G}$, but also on $\mathbf{a_G}$. The use of the x-intercept for $\mathbf{m_E}$, then, is also a reflection of the dependence of the x-intercept on the slope. The x-intercept does appear in the form y=mx+b but not in a simple or directly accessible way. The very subtle simplifying assumption that we will not consider the x-intercept for this form of equations is neither explicit nor simple.

Developing a conceptual understanding of linear equations and their graphs means developing a perspective which involves much more than accepting the simple convention that the form y=mx+b involves only the parameters m and b, or that the x-intercept would only be accessible algebraically in other forms of this equation (such as x= y/m - b/m or x/a +y/b=1). Developing this perspective means unpacking how the parameters in the equation are manifested on the graph, how objects on the graph are manifested in the equation, and which objects in each representation are independent. Thus, the use of the x-intercept is not merely the result of choosing or emphasizing the form y=mx+b over other forms but is, instead, a reflection of the mathematical complexity of this domain.

### The Use of the X-Intercept as a Transitional Conception

The use of x-intercept is not always wrong. Sometimes it is not an error at all but a useful concept; other times it is correct, but only within limited contexts. First, using both intercepts to compare lines is an easy and direct way to check that the lines are the same. Many students used $\mathbf{a_G}$ and $\mathbf{b_G}$ to check that a line they had produced on the screen was the same as a line that was graphed on the worksheet. FR and HE (Case Study 1, summarized below), for example, used either the x-intercept or both intercepts to check their lines 7 times for 21 problems.

Furthermore, when generating an equation from a graph, it is possible to use the opposite of the x-coordinate of $a_G$ in the place of $b_E$ when m is 1, since in this case the x-coordinate of $a_G = -b_E$. Initial examples using only slopes of 1, where $|a_E|=|b_E|$, may obscure the difference between the two graphical intercepts and their status in the equation, and thus be another origin for this conception. Students may make conclusions about $b_E$ from these initial examples (for example that $b_E$ corresponds to $a_G$, or to both $a_G$ and $b_G$), and then generalize these conclusions to lines and equations where the slope is not 1.

## Case Study 1: FR and HE

These two students had the highest scores on the pre-test (29/31 and 23/31) and gave the best explanations for the pre-test question targeting the use of the x-intercept (m=1). Even though these two students also answered the problem targeting the x-intercept (m=1) on the discussion worksheet correctly, and did not use the x-intercept for two other problems with slope 1, they did invoke the x-intercept when working on a problem where the slope was not 1. During their discussion of the first worksheet problem with slope 2 they were asked to change the equation for one line on the graph (y=2x) to an equation which would produce the other line on the graph (y=2x+6, where $a_G$ is (-3,0)). Their first attempt was to use the opposite of the x-coordinate of $a_G$ for $m_E$. Their second attempt was to graph the equation y=2x+3, using $-a_G$ for $b_E$, which may have been a use of the x-intercept which had worked for the previous problems where the slope was 1. After several other attempts, they arrived at the equation y=2x+6. When they later worked on the problem targeting the use of the x-intercept, but where the slope was 1, these two students had no difficulty answering that $b_E$ did not correspond to $a_G$, and explaining why this was not the case using the connection between the ordered pairs, the equation, and the line.

The use of the x-intercept is a conception which was invoked even by the most capable students, and thus is not simply related to students' ability or achievement in mathematics. This case study also shows that while students may seem to understand the x-intercept in one context (a problem with slope 1), they may use the x-intercept in other contexts (problems with slopes other than 1).

## Case Study 2: MT and JS

Another pair of students (MT and JS) provided an explanation of why they expected $b_E$ to correspond to the x-coordinate of $a_G$. When predicting what the graph of the equation y=x+4 would look like (the problem targeting the x-intercept mentioned in Case Study 1), MT expected to generate lines of the form y=x+b by starting from the line y=x and moving to the right along the x-axis. MT further specified that he expected the sign of $\mathbf{b}_E$ to correspond to the sign of $\mathbf{a}_G$. After his partner JS suggested that the line for y=x-4 might cross the x axis at (4,0), they graphed this equation. MT was then puzzled to see that a plus 4 in the equation would appear as an x-coordinate of minus 4. They concluded that $\mathbf{b}_E$ corresponded to the opposite of the x-coordinate of $\mathbf{a}_G$, which is correct for lines of slope 1. They went on to use this conclusion in their first attempts at the next problem, where the slope was 2: "Write the equation for the line graphed below (y=2x-6)". After realizing that $\mathbf{b}_E$ did not correspond to the opposite of the x-coordinate of $\mathbf{a}_G$ in this case but to the y-coordinate of $\mathbf{b}_G$, they then tried to use the opposite of the x-coordinate of $\mathbf{a}_G$ for $\mathbf{m}_E$.

These two students went on to explore the relationship between $\mathbf{a}_G$ and $\mathbf{m}_E$ by graphing several equations (y=-3x+6, y=6x-3, and y=3x+6), using either the x-coordinate of $\mathbf{a}_G$ or its opposite for $\mathbf{m}_E$ and for $\mathbf{b}_E$. Using the x-coordinate of $\mathbf{a}_G$ for $\mathbf{m}_E$, while incorrect, reflects the fact that the x-intercept does change as $\mathbf{m}_E$ changes. The fact that changing m changes the x-intercept makes it difficult to separate the two objects or discover that $\mathbf{m}_E$ is not manifested on the line as the x-intercept. Thus, even for the students who recognized that x-intercept does not work in the equation y=mx+b and addressed this issue in their discussions, the relationship between the x-intercept and the equation was complicated and difficult to unravel.

## Conclusions

The use of the x-intercept is a transitional conception which is: 1) between two states; 2) applicable even if in limited contexts; and 3) can be transformed through refinement. The use of the x-intercept reflects the recent understanding that there is some connection between graphs and their equations, that if you change one representation there should be some change in the other. Future states are those in which this connection between the two representations is specified and refined in terms of which and how many parameters are necessary and sufficient to determine a line in two-dimensional space, how these parameters are related, and in which contexts the use of the x-intercept is applicable or not.

Using the x-intercept for the form y=mx+b is not always wrong. For example, it is useful for comparing two lines. Others times it is applicable, though in limited problem contexts. For problems where the slope is 1, the opposite of $\mathbf{b}_E$ will generate the correct x-coordinate for $\mathbf{a}_G$, when going from an equation to a line, or the opposite of the x-coordinate of $\mathbf{a_G}$ will generate the correct $\mathbf{b_E}$, when going from a line to an equation. Other times unpacking how the x-intercept is dependent on either the slope or the y-intercept makes exploring the two representations quite complicated.

This transitional conception can evolve by redefining the relevant objects, refining the relationships between them, and specifying the contexts in which it is applicable. If it is explored as a marker for changes in slope, it can serve as a bridge to the concept of slope. Although I will not describe the refinement of this transitional conception in detail here, the analysis of the videotaped discussions for six pairs of students shows that the use of the x intercept was refined in the following ways:

- The x-intercept was explored as a reflection of the slope.
- The contexts in which the use of the x-intercept is applicable was specified. For example using the opposite of the x-coordinate of the x-intercept ($-\mathbf{a_G}$) for the b in the equation is applicable, but only when $m = 1$; another context in which the use of the x-intercept is applicable is when comparing two lines.
- The use of the x-intercept for b when $m=1$ was refined from using the x-coordinate of the x-intercept ($\mathbf{a_G}$) as the b in the equation, to using the *opposite* of the x-coordinate of the x-intercept ($-\mathbf{a_G}$) as the b in the equation.

Changing our perspective of the nature and transformation of student conceptions has important implications for instruction. Transitional conceptions such as the use of the x-intercept are precisely the kinds of ideas that appear during exploration and open-ended problems. If the instructional goal is to support students making sense of the mathematics (Schoenfeld, in press), "correcting" students' use of the x-intercept is less useful than attending to how this conception makes sense. Once we we begin to interpret some student conceptions as transitional and explore the underlying mathematical complexity they reflect, it becomes possible to explore their potential as a bridge to more competent conceptions.

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# TEACHERS AS RESEARCHERS: DIALECTICS OF ACTION AND REFLECTION

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**Abstract**: *In the University course under review, qualified teachers seek to upgrade qualifications and extend their knowledge of mathematics teaching. The course involves teachers in making changes within maths classrooms, after working with others to decide upon an area of focus. This paper addresses the nature of innovations undertaken, the longevity of those changes and the wider effects within the school. The notion of a dialectical relationship between teachers' theories and their practice is explored. Three aspects of pedagogical change are examined. These are (a) the contrast between symbolic and real change, (b) institutional and social factors which facilitate, inhibit or influence the direction of change within schools, and (c) the conditions necessary for ongoing change in pedagogy. This paper also reports on the development of teachers' perceptions of the nature of mathematics.*

It is quite common for academics to categorise styles of pedagogy when talking about school clasrooms. Voight (1989), for instance, presents the three common perceptions of what it is to be a teacher as (a) a conveyance perspective, with transmission of a body of external knowledge from expert to novice, (b) a developmental perspective, involving the teacher as a facilitator of the development of internal cognitive structures, and (c) negotiation, where teaching is perceived as facilitation of the negotiation of individual and shared meanings through social interaction. This paper presents an investigation of a model of teacher in-service education which would seem to align itself with the latter perspective.

Teachers upgrading qualifications generally enrol in university courses expecting the transfer of a growing body of knowledge, anticipating a one-way process of teaching and learning. Even if teachers have a strong sense of their own strengths and weaknesses, they tend to look to others to provide professional development opportunities and content. The completion of mathematics teacher education courses thus usually involves studying given texts (written and oral), with a view to later implementation of the theories they hold a conveyancing of knowledge from the experienced to the inexperienced participant in the eductional process. On the other hand, tertiary teachers generally think of themselves as supporting the sharing and development of skills, bringing to mind the "gardening" metafore for teaching (Pimm, 1991) where we could be thought of as having primary roles of facilitating professional growth and nurturing teachers' skills.

In the Deakin University subject presently under review, however, the emphasis is on neither transmission nor nurturing - but on critique of individual practice. Teachers undertaking the subject research their own beliefs and clarify their own perceptions of what it is to learn mathematics. This requirement is based on the belief that increased self-awareness is a major component of successful long-term pedagogical change (Smyth, 1984; McQualter, 1986; Southwell, 1986). Teachers work together in making planned series of changes - not to what is taught but to social contexts and methods of teaching. This involves looking back as well as forward: for as Kilpatrick (1985) notes,

> many teachers' apparent reluctance to depart from tried-and-true practices may stem, at least in part, from a lack of commitment to looking back at these practices and making a careful assessment of them. (p.1)

Course participants are thus involved in reflection on concepts of mathematics, teaching and learning, as well as processes of mathematics, teaching and learning. In line with Kilpatrick (1984) they use reflection as a tool, in a cognitive sense, to gain conscious control over these concepts and processes.

In the second and subsequent stages of the course teachers enact a change in classroom behaviours and then monitor, in a systematic way, the results. This leads to a further stage of reflection, action and collection of data. Thus teachers are simultaneously effecting and researching their own professional development. They become active creators of knowledge, choosing pedagogical aspects to study and their own research routes to follow, with - hopefully - a spiraling growth of understanding about the learning of mathematics.

## Teachers as researchers

In recent years, much mathematics education literature has focused on the deficiencies of school mathematics and the need to take more account of how children learn. While research into children's construction of mathematical concepts confirms a need to challenge pedagogical habits, most mathematics teachers are still largely bound by institutionalised tradition. But because real change in teaching usually involves taking on a new philosophy and a set of pedagogical practices contrary to our own educational experiences and training, such implications are rarely acted upon at more than a superficial level.

The Deakin University subject, MATHEMATICS CURRICULA, requires teachers to challenge traditional teaching behaviours, involving participants in a systematic, dialogue-based evaluation of what and how they are teaching. As meanings constructed by participants in this process are both personal and taken-as-shared (Bruner, 1990), this school-based research revolves around negotiation and collaboration with other teachers, offering the collegiate support and continual input of ideas necessary for ongoing professional development. During participatory research, shared meanings arise out of social interactions as theories and practices are discussed, negotiated and evaluated (Voigt, 1985).

A model of professional development that endorses mathematics teachers as knowledgeable about, and capable of accurately researching, their own teaching practice as it stands within the wider contexts of institution and society challenges the assumption that knowledge is separate from and superior to practice. It also calls into question the belief of many practitioners that a teacher's role is to apply, rather than test or create new knowledge. Henry (1990) notes that if major change in classrooms toward less transmissive teaching is to be institutionalised, the force must come from within schools. He writes

> The countervailing force to the hegemony of the 'teacher-as-manager' ethos must have its source within the collective of educators in the school if radical pedagogical change is to be sustained. ... It is clearly contradictory to seek to counter the hegemony of 'teacher-as-manager' ethos through dependancy relationships given shape and meaning by the very same institutional practice that sustaines the hegemony. (p. 213)

### How teachers pursue the course

MATHEMATICS CURRICULA is a whole year (2 unit) fourth-level subject which is undertaken by about 100 pre-school, primary, secondary and tertiary teachers each year, as part of Deakin University's Bachelor of Education or Graduate Diploma in Mathematics Education programs. Teachers from all over Australia and other parts of the world undertake their studies in an off-campus mode: they have phone contact with tutors, but no face-to-face tutorials or lectures. Readings are provided, but these are essentially a stimulus to thought and discussion rather than prescriptive sources of information. A key requirement is that the course participants must work with colleagues within a school or local group of schools.

The course involves teachers working in small groups to research ideas and articulate their own philosophies about what mathematics is and the ways school mathematics should be taught. They then critique their own teaching and taken-for-granted practices within their own institutions and communities, in the light of current trends in mathematics education as well as their espoused theories. The teachers choose one area of their practice change, and use cycles of action research until they feel that the innovation sits comfortably within their routine patterns of teaching. Some participants research several areas over a period of time, others focus on only one.

University staff also use action reserach on their continuing development of the course. In this particular research project, we were interested to collect data on longer-term and wider effects. This paper reports our findings regarding the extent to which MATHEMATICS CURRICULA has led to substantial development of teachers' understandings about the processes of changing mathematics education.

### Methodology

The present study involved contacting a representative sample (60) students from 1988-89 by letter, telephone or by personal interview. These teachers were asked a series of open-ended questions and were asked to describe any longer-term impact that MATHEMATICS CURRICULA had had on their teaching. Questionnaire responses, tape recordings and field notes were then available for the analysis of data. We also had access to some personal-professional journals, as well as final assignments (summarative research reports), completed by previous course participants.

### Results and discussion

Most interviwees willingly discussed the benefits of, and understandings derived from, participation in the course. The majority has continued with autonomous, practice-based research; and many of the innovations made during the course have been retained to this date. What we found was not only some ongoing restructuring of pedagogy, in terms of content, organisation and classroom interaction, but also growth of understandings about (1) the nature of mathematics, (2) the processes of teaching and learning of maths, (3) the power of institutional contexts of teaching and learning, and (4) the processes of pedagogical change. It is on these latter four developments that this report will focus.

*Understanding the nature of what is being taught.* A common comment was that research into the teaching of mathematics raised the issue of what mathematics is and how it is likely to be used by a range of students. The notion of mathematics as a stable body of knowledge and skills to be transmitted and practised became problematic. Questioning traditional classroom practices provided an incentive for teachers to confront given curriculum content.

Some interviewees have continued to work with groups established several years ago. One described in detail a major step in this mutual support, concluding

> *I think that was the first time we actually discussed as a group the purpose of what we were teaching. Before that it was always 'what' and 'how'. Never 'why?' (B.D.)*

Working together to challenge traditional pedagogy also helped some teachers overcome negative attitudes and self-blame for inadequate mathematical skills and knowledge, legacies of their own schooling. Lortie (1975) and Southwell and Khamis (1991) note a propensity of earlier schooling to reproduce existing conceptions and methods. Course participants could see how restrictive teaching practices perpetuate, across generations, a cycle of fear and poor self-concept. Some teachers described how they have continued to articulate problems and theorise about these and how implementing theories and learning from the results, then consequently adapting teaching behaviours and sharing findings with each other has added to the bodies of knowledge about mathematics pedagogy within their schools.

Not all teachers, however, found solutions to dilemmas about the content of mathematics courses. For instance, in talking about her own work one teacher said

> *(Researching my classroom) raised for me a whole lot of questions : like how you can have a local curriculum when there is so much emphasis on common elements of knowledge and skills. The course didn't give any answers to problems like that. You need to give more answers - not just upset people's security then leave them to find their own. (L.K.)*

In contrast, some participants appreciated the fact that we did not attempt to provide ready recipes.

> *Starting from where I was at - that's what made all the difference. We were forced to go back to a theoretical base and question what we were doing. We'd never been in a position to link theory so closely with our own practice, or for me to develop my own theories and knowledge as a result of teaching - rather than reading or listening. (F.R.)*

Many teachers came to the realisation that changing only the content of lessons is not sufficient to bring about different ways of thinking about of mathematics. They described the more fundamental innovations they had undertaken and which they had further developed since completing MATHEMATICS CURRICULA. For instance, one Year 7 teacher wrote

> *Experimenting with changing methods made me realise that how you teach actually teaches kids as much about maths as the content does. I'll never go back to just planning content. What I have been focusing on this year is making algebreic symbols meaningful - through work with recording*

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> *and manipulating patterns or representing equations with concrete materials. The results are fantastic. It's not just algebra - the same principles about bringing abstract ideas into the children's realm of reality apply in geometry and the work we've been doing on introducing calculus concepts. (R.R.)*

*Understanding the processes of teaching and learning.* Collecting data of their own teaching led many course participants to recognise gaps between their espoused theories and "theories in use" (Schön, 1983). An example of this was given by a teacher who has continued to research gender issues in his math and science classes. He said

> *I knew I was a good teacher. But I saw my teaching as going along with any persuasive person or movement. It's so easy to look at a policy and think "I do that, so all is okay." But the course made me ask "But do I?" And often I found I didn't! That took some soul searching. I had to clarify where I really stood as a teacher. I'd read a lot about girls and maths, but I felt I wasn't part of the problem. That is, until I started working with a friend to look at my usual teaching in the light of the reading. That was a shock - to see myself and some of my colleagues as perpetrating the very situations we had been denying. Then when I tried to change, I kept slipping. In fact I still pull myself up occasionally - just to see if I am really doing the right thing by the girls. (S.L.)*

Such useful but painful recognitions that practice may not be representative of either perception or ideology were common: critical enquiry frequently disturbed complacency and illuminated inconsistencies in participants' teaching. As teachers started to discuss the meanings of various phrases in their personal beliefs about the teaching of mathematics, and as they examined closely individual interpretations of their schools' written mathematics policies, the process was not always a comfortable one.

Phrases which continually appeared in teachers' journals, recording changing attitudes to teaching and to mathematics, reflected the notion of different notions of curricula - loosely mirroring the intended, implemented and attained curricula of Robitaille and Dirks (1982) and Ball (1990). What had previously seemed like ordinary experience, such as teachers' habits of controlling discussions by asking repeated questions, came into question - in origin as well as effect. The staff noted, as Feiman-Nemser (1979) claims, that reflection is the means by which ordinary experience may be transformed from that which is privately perceived to that which is understood.

One common focus of change was the belief that it is a teacher's role to control the sorts of mathematical understandings and processes that children develop. Many participants experimentedwith allowing their students to determine their own ways of solving real-world problems, to pose their own problems, and to take a more active role in negotiating the curriculum. However, this commonly led to worries that "the basics" wouldn't be covered adequately: teachers' journal entries and final research reports raised this issue repeatedly. This dilemma is recognised by Cobb (1988) when he writes

> In school, any construction is not as good as any other - anything does not go. Teachers, in order to fulfil their wider societal obligations, attempt to realize institutionally sanctioned agendas in their classrooms. They have in mind things they want students to learn. There is a tension between

> encouraging students to build on their current understandings on the one hand and initiating students into mathematical culture of the wider community on the other. (p. 3)

*Understanding the institutional contexts of teaching and learning*

Many participants questioned, for the first time in their careers, constraints to learning which arise from particular aspects of school organisation. On interviewee, for instance, said

> *I tried to work on integrating the curriculum, and was successful to some extent. I involved the physics, English, phys ed and art teacher in what I was doing and they took up some of the same themes. But what really irked me was that we had to continue to have the bell ring every 50 minutes and the students would all move on to other classrooms and new subjects. The divsion of knowledge is forced on us as well as them. (M.N.)*

It is easy to fall into this trap of believing that pedagogical beliefs and practices are defined by social bodies. However, the work of Lakatos (1976), Foucault (1979) and Ernest (1991) alerts us to the dialectical nature of the individual and the social, i.e. the individual as constituted by social, discursive practices and those practices being re-constructed by individuals. We found that teachers undertaking critical reflection about their own classroom practices by examining the historical (past, present and intentional) factors became more aware of both the social habits and bodies which tend to control teaching; but that they also came to recognise their power to confront those constraints and the fact that their social realities were shaped largely by their own actions. One teacher's journal, for instance, recorded a point of awakening as

> *...why do I always respond with an answer when a child asks a question? Probably because my teacher did - and my colleagues still do. My parents did too. But I don't have to keep maintaining the teacher-child power relationship because of my greater knowledge. What other ways could I respond? - with a question, throw it to another pupil, with "Let's find out together." I must try these starting tomorrow. But the kids won't like it. They are used to people acting the way they have learned to expect. But if they learn that it's possible to seek their own solutions then they will become better teachers and parents. I need to plan how to teach how to learn. (S.R.)*

*Understanding the process of change in mathematics classrooms.* Many interviewees claimed that the information gathering and critical reflection required in the course enabled them to grow in confidence, to the extent that they felt more able to understand the difficult process of changing teaching methods. A relatively typical comment was

> *I was previously fearful at the thought of changing my teaching of maths. It seemed so untouchable. But I discovered that content could be handled and developed in purposeful and controllable ways. I approach change positively now and this comes across to my students and the other staff. They often ask me now about helping them make small steps toward changing their classroom practice. I feel quite confident about leading a review of our maths curriculum. (C.G.)*

However, some teachers participating in the course also developed a realistic awareness of the constraints to implementing change within classrooms and schools.

> *Over the last 25 years, maths has undergone a series of radical changes. When I started teaching, it was all Cuisenaire. Then set theory, then text book programs and worksheets galore. Just when it seemed that the "back to the basics movement" was about to succeed, ministerial guidelines that lead teachers away from chalk and talk teaching styles have become influential. After a while you say, "Hey, wait a minute! What about asking the teachers about all this? What about giving them time to think? What about supporting them? We can't suddenly drop one style of teaching and the philosophy behind it and take up another overnight. And it's not just a matter of changing one classroom – it's got to be the whole school, and then even wider than that. Parents, high schools, ... (P.D.)*

Such statements supported the view that involvement of teachers in planned experimentation and move-testing within classrooms is necessary if educational innovations are to have underlying values which fit with those of the consumers. The above quote demonstrates, for instance, that innovations need support, and that they have to to be seen as manageable and feasible in terms of resources required, time available and the status of individual actors.

The possibility of radical change to mathematics education is largely inhibited by traditional social practices, both within schools and in the wider context of societal sub-cultures. Siemon (1987) writes

> *The principal source of currently negative pressure are the perceived goals of mathematics education: to enable students to pass external examinations which value abstract formal mathematics and adhere to process-deficient, linearly-ordered, topic-based curriculum statements. While these remain unchanged, conflict about the need for change and questions about the practicality of change are inevitable. (p.17)*

Participation in the activities suggested in MATHEMATICS CURRICULA made new demands on teachers and initiated the creation of some changes within the social structure of educational institutions. Some teachers were involved for the first time in communication with teaching peers about the act of teaching, and described the manner in which these discussions helped shape wider decision-making and consequent action. They became more aware of the personal and institutional obstacles to collaboration, but a few found that these could be overcome when talking together was given higher priority.

> *At first it was hard to talk together. When I had time release he didn't. When we asked for the timetable to be changed, the vice-principal said he would do it but he didn't see the point. After school and before it there were always other things to think about and do - parents, record of procedure, preparing work. We were awfully self-conscious about talking in the staff room at lunch time. When one guy said "Don't you two ever knock off?", we felt embarrassed. It was only when we got together and decided that we should be talking about those ideas in front of others - that that's what learning about teaching is all about - that we made a breakthrough. It*

*wasn't long before there were informal debates going on - a stimulating change from obnoxious kids and football.* (L.R.)

## Conclusion

Dewey (1933) claimed that reflectiveness is an essential component of professional growth. The University staff involved in the regular monitoring and development of this subject believe that participatory, experience-based research has the power to emancipate some teachers from taken-for-granted classroom routines which constrain and control mathematical learning. The dialectical interaction of reflection combined with social interaction allowed innovation in the nature of teaching and learning mathematics as well as in curriculum content. It is clear that working together to clarify ideas about what it is to teach mathematics, as well as to plan, implement and evaluate progressive innovations, has led to some lasting changes. We have found that action and reflection have allowed some teachers to further understand not only the complexities of the institutional and socio-cultural factors which affect the construction of meaning, but also their own personal and professional theories of action (Schön, 1983).

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# CHILDREN'S SOLUTIONS TO MULTIPLICATION AND DIVISION WORD PROBLEMS : A LONGITUDINAL STUDY

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*Children's solution strategies to a variety of multiplication and division word problems were analysed at four interview stages in a 2-year longitudinal study. The study followed 70 children from Year 2 into Year 3, from the time where they had received no formal instruction in multiplication and division to the stage where they were being taught basic multiplication facts. Ten problem structures, five for multiplication and five for division, were classified on the basis of differences in semantic structure. The relationship between problem condition (i.e. small or large number combinations,use of counters or pictures), on performance and strategy use was also examined. The results indicated that 75% of the children were able to solve the problems using a wide variety of strategies even though they had not received formal instruction in multiplication or division for most of the 2-year period. Performance level generally increased for each interview stage, but few differences in performance were found between multiplication and division problems except for Cartesian and Factor problems. Solution strategies were classified for both multiplication and division problems at three levels: direct modelling with counting and additive procedures, no direct modelling with counting and additive procedures, and use of known facts. Analysis of underlying intuitive models revealed preference for a repeated addition model of multiplication, and widespread evidence of a 'building-up' model for division.*

Research investigating young children's understanding of number concepts and problem-solving processes has revealed widespread use of informal or intuitive strategies developed prior to formal instruction (Carpenter, Moser, & Romberg, 1982; Hughes, 1986; Steffe, Cobb & Richards, 1988; Steffe & Wood, 1990). Studies investigating solutions to addition and subtraction problems have indicated that children use a wide variety of modelling and counting strategies that reflect the semantic structure of the problem (Carpenter & Moser, 1984; De Corte & Verschaffel, 1987).

Over the past decade, researchers have analysed secondary student's solution processes to multiplication and division word problems based on differences in semantic structure, mathematical structure, size of quantities used, and student's intuitive models (Bell, Fischbein & Greer, 1984; Bell, Greer, Grimison & Mangan, 1989; Brown, 1981; De Corte, Verschaffel & Van Coillie, 1988; Fischbein et al., 1985; Nesher, 1988; Vergnaud, 1983, 1988). More recently, a growing number of studies on young children's solution strategies to multiplication and division problems have emerged (Anghileri, 1985; 1989; Boero, Ferrari & Ferrero, 1989; Kouba, 1989; Mulligan, 1992; Olivier, Murray & Human, 1991; Steffe, 1988). These studies have provided complementary evidence that the semantic structure of the problem, an understanding of the problem context, and the development of counting, grouping and addition strategies may influence solution process.

The 2-year longitudinal study on multiplication and division word problems reported in this paper analyses young children's solution processes prior to formal insrtuction, and investigates the relationship between informal and formal multiplication and division strategies. More specifically, this research (i) develops a broader classification scheme for multiplication and division problem structures for young children, (ii) classifies solution strategies into levels of modelling and abstractness,(iii) analyses the relationship between problem structure, problem condition and strategy use, and, (iv) provides evidence of children's intuitive models for multiplication and division.

## Methodology

The methodology was based essentially on Carpenter and Moser's (1984) longitudinal study of children's solutions to addition and subtraction word problems, and was trialled in a cross-sectional pilot study of 35 children (Mulligan, 1988). The interview sample controlled for sex differences and comprised 72 Year 2 girls ranging from 7 to 8 years in age. These were randomly selected from 8 schools after a postal questionnaire was administered to 47 schools in the region, and the sample retained 60 girls at the final interview.The interview sample controlled for sex differences and comprised Year 2 girls ranging from 7 to 8 years in age. These were randomly selected from 8 Catholic schools in the Sydney Metropolitan area, after a postal questionnaire was administered to 47 schools in the region. Each child in the sample (n=72), was interviewed and tested twice, for reading comprehension and oral comprehension. Two children having very inadequate reading comprehension ability, as indicated by the ACER Primary Reading Survey Test, were eliminated from the sample. Others moved from the region during the interview period, and the sample retained 60 girls at the final interview.

The researcher conducted 261 individual interviews at four stages over the 2-year period. These took place during March/April, and November/December of the school year for 2 years. The first interview took place at a time when children had received no teacher instruction in multiplication and division concepts. At the time of the final interview all children had been instructed in basic multiplication facts but not in division facts.

Subjects were interviewed by the researcher in a room separate from the classroom and an audio-tape was made so that transcripts could be analysed. Each problem was presented for small and large number problem combinations, in written form on cards, with the availability of counters. The problems were re-read to the child if requested. The large number problems were asked only if the child was successful on the small number problems. Each interview lasted from 15 to 55 minutes.

At each interview the child was asked to solve ten different problem types (Table 1). These were developed from previous classification schemes (Anghileri, 1985; Bell et al., 1984; Brown, 1981; Kouba, 1986; Mulligan, 1988; Vergnaud, 1983), but extended the range of multiplication and division problem structures with sub-categories representing a variation in linguistic terms.

Table 1: Word Problems (Small Numbers)

| **Multiplication** | **Division** |
|---|---|
| Repeated Addition<br>(a) There are 2 tables in the classroom and 4 children are seated at each table. How many children are there altogether?<br>(b) Peter had 2 drinks at lunchtime every day for 3 days. How many drinks did he have altogether?<br>(c) I have three 5c pieces. How much money do I have? | 1. Partition (Sharing)<br>(a) There are 8 children and 2 tables in the classroom. How many children are seated at each table?<br>(b) 6 drinks were shared equally between 3 children. How many drinks did they have each? |
| Rate<br>If you need 5c to buy one sticker how much money do you need to buy two stickers? | Rate<br>Peter bought 4 lollies with 20c. If each lolly cost the same price how much did one lolly cost? How much did 2 lollies cost? |
| Factor<br>John has 3 books and Sue has 4 times as many. How many books does Sue have? | Factor<br>Simone has 9 books and this is 3 times as many as Lisa. How many books does Lisa have? |
| Array<br>There are 4 lines of children with 3 children in each line. How many children are there altogether ? | Quotition<br>(a) There are 16 children and 2 children are seated at each table. How many tables are there?<br>(b) 12 toys are shared equally between the children. If they each had 3 toys, how many children were there? |
| Cartesian Product<br>You can buy chicken chips or plain chips in small, medium or large packets. How many different choices can you make? | Sub-division<br>I have 3 apples to be shared evenly between six people. How much apple will each person get? |

All the problems contained numbers representing discrete quantities and were presented using two different groups of number size, e.g. those products and related division facts between 4 and 20 for small numbers (see Table 1) and between 20 and 40 for larger combinations.

## Discussion of Results

Analysis of individual profiles across the four interview stages indicated that 75% of the children were able to solve most of the small number problems at some stage, even though they had not been instructed in multiplication or division for most of the 2-year period. Table 2 indicates that the performance level generally increased for each interview stage but varied according to the difficulty of the problem structure and size of number combinations used. However, there were few differences found in the performance level and solution strategies between multiplication and division problems, except that performance was much lower for Cartesian and Factor problems.

**Table 2: Percentage of Correct Responses for Each Problem Structure: Interviews 1 to 4**

| PROBLEM STRUCTURE | | SMALL NO. | | | | LARGE NO. | | | | PICTURE | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Interviews | | | | Interviews | | | | Interviews | | | |
| **MULTIPLICATION** | | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| **Repeated Addition** | (a) | 50 | 77 | 79 | 92 | 27 | 45 | 54 | 80 | 50 | 20 | 20 | 8 |
| | (b) | 51 | 74 | 84 | 95 | 27 | 52 | 68 | 65 | | | | |
| | (c) | 59 | 74 | 85 | 95 | | | | | | | | |
| **Rate** | | 72 | 82 | 89 | 98 | | | | | | | | |
| **Factor** | | 11 | 29 | 44 | 57 | 0 | 16 | 35 | 47 | | | | |
| **Array** | | 46 | 77 | 84 | 92 | 39 | 70 | 76 | 78 | 39 | 16 | 11 | 5 |
| **Cartesian** | | 1 | 1 | 3 | 18 | 1 | 1 | 2 | 10 | | | | |
| **DIVISION** | | | | | | | | | | | | | |
| **Partition** | (a) | 66 | 69 | 74 | 75 | 23 | 33 | 29 | 55 | | | | |
| **(Sharing)** | (b) | 61 | 80 | 81 | 97 | 34 | 64 | 64 | 83 | 14 | 14 | 14 | 2 |
| **Rate** | | 51 | 54 | 66 | 85 | | | | | | | | |
| **Factor** | | 4 | 3 | 6 | 17 | 0 | 0 | 0 | 10 | | | | |
| **Quotition** | (a) | 34 | 58 | 55 | 85 | 26 | 36 | 44 | 72 | | | | |
| | (b) | 47 | 64 | 69 | 93 | 34 | 45 | 50 | 73 | | | | |
| **Subdivision** | | 41 | 60 | 73 | 82 | 10 | 23 | 35 | 43 | | | | |

The numbers interviewed were 70, 69,62 and 60 respectively.

In comparing small and large number problems, a marked decrease in performance overall was found for large number problems with many children reverting to direct modelling and counting procedures. Further analysis of individual profiles revealed some contrasting evidence that 25% of the children were unable to solve two or more of the easiest 11 small number problems at any interview stage. Many of these children relied on immature strategies, such as using key words, looking at the size of the numbers to choose an operation, or applying number facts incorrectly.

## Primary Strategies Used Across Problem Structures and Interview Stages

There was a wide variety of strategies used to solve the ten different problem structures. These were largely based on grouping, counting and additive procedures, and the increased use of known addition and multiplication facts at Interview Stages 3 and 4. There were few differences found between the solution strategies for multiplication and division problems except for sharing, one-to-many correspondence, and trial-and-error used exclusively for division. Most compelling was the evidence that the solution strategy reflected the semantic structure of the problem and in general, the children tended to model the action or relationship described in the problem.

## Levels of Strategy Use For Multiplication and Division Problems

From the broad range of strategies used to solve the ten problem types across four interview stages, three basic levels of strategy use were formulated by integrating the level of abstractness, and the level of modelling of the solution strategy: (i) direct modelling with counting and additive or subtractive procedures; (ii) no direct modelling with counting and additive or subtractive procedures; and (iii) use of known or derived facts.

At Level (i) grouping and counting strategies were combined where children formed equivalent sets representing the quantity given in the problem and then counted-all (**one-by-one to gain total**), **skip counted** ("3, 6, 9") or **double counted** (two counts made simultaneously for number of groups and number in the group). At Level (ii) strategies were identical to those at Level (i) but were identified by the children verbalising the solution process and describing their **visualisation** of the model of the problem. At Level (iii) use of known `and derived addition and multiplication facts for multiplication and division emerged clearly at Interview Stage 4.

Although the solution strategies for multiplication and division problems were more complex, the levels of modelling, counting, and use of known facts, were found to be analogous to the addition and subtraction study (Carpenter and Moser, 1984), and were consistent with the strategies found in Kouba's (1989) study with multiplication and division problems.

**Table 3: Levels of Strategy Use on Small Number Problems Across the Four Interviews**

| Multiplication | | Division | |
|---|---|---|---|
| **Level (I) Direct Modelling** | | | |
| (1) | Grouping, counting-all | (1) | Grouping, counting-all<br>Sharing one-by-one<br>One-to-many correspondence<br>Trial-and-error grouping |
| (2) | Grouping, double counting | (2) | Grouping, double counting |
| (3) | Grouping, skip counting | (3) | Grouping, skip counting |
| (4) | Additive and subtractive:<br>Repeated addition<br>Doubling<br>Halving<br>Repeated subtraction | (4) | Additive and subtractive:<br>Repeated addition<br>Doubling<br>Halving<br>Repeated subtraction |
| **Level (II) No Direct Modelling** | | | |
| (1) | Grouping, counting-all | (1) | Grouping, counting-all<br>Sharing one-by-one<br>One-to-many correspondence |
| (2) | Grouping, skip counting | (2) | Grouping, skip counting |
| (3) | Grouping, double counting | (3) | Grouping, double counting |
| (4) | Additive and subtractive:<br>Repeated addition<br>Doubling<br>Halving<br>Repeated subtraction | (4) | Additive and subtractive:<br>Repeated addition<br>Doubling<br>Halving<br>Repeated subtraction |
| **Level (III)Known Facts** | | | |
| (1) | Known addition fact | (1) | Known addition fact |
| (2) | Known multiplication fact | (2) | Known multiplication fact |
| (3) | Derived multiplication fact | (3) | Derived multiplication fact |
| | | (4) | Known division fact |

## Intuitive Models

Fischbein et al. (1985) identified a repeated addition model for multiplication and two models for division; partitive and quotative. In the process of analysing solution strategies, underlying intuitive models appeared to the researcher to overarch the method of solution, but it seemed that these models were more complex than those previously described by Fischbein et al. (1985) with older pupils. Further analysis of the solution strategies revealed predominance of the repeated addition model for multiplication, but an 'operate on the set', an array, and cartesian models were also found.

Three underlying intuitive models for division were found: sharing one-by-one, 'building-up' (additive) and 'building-down' (subtractive). These were consistent with recent findings by Kouba (1989), but further analysis of Partition, Quotition and Rate problems across the four interview stages showed **widespread preference for the 'building-up' model** (Mulligan, 1991a) and this was based on counting or additive strategies where the child 'built-up' to the dividend. For example, in the Quotition (a) problem, the child 'built-up' groups of 2 until 16 was reached, and verbalised counting-all, skip counting or double counting "2, 4, 6, 8 ...". 'Building-down' was distinctive because the child always modelled or counted the dividend first such as " 16 take away 2, take away 2".

A change in problem type may have affected the intuitive model used which supports the notion that children can develop more than one intuitive model. Some children who were consistent in their intuitive model across problems tended to be restricted to 'building-down'. Those children who were more successful across problems were more likely to change their model. These findings raise questions for teaching and learning methods that rely on the sharing and repeated subtraction models for division. It is proposed that teaching traditional partitive and quotative models may restrict the solution process, rather than building on children's intuitive understandings. It can be questionned whether the use of additive and estimation strategies, especially efficient use of multiple and group counting might be a more effective way of teaching division.

## Implications for Teaching

This study has provided new evidence that young children can solve a variety of multiplication and division problems prior to instruction in these processes. Counting and additive procedures were found to be essential in the development of multiplication and division processes, with the use of efficient skip counting, and double counting as central to this development. Teaching programs could incorporate the development of informal strategies rather than focussing only on mastering number facts and computational skills that may not relate to the child's level of strategy development. Teachers could facilitate more meaningful learning by establishing links between children's intuitive strategies and the formal teaching of addition, subtraction, multiplication and division. Perhaps the teaching of these processes in an integrated fashion, and based on the child's experience of a range of related situations might best reflect the natural development of these processes. The relative difficulty of different problem structures and number combinations has been more clearly identified and thus, teachers could expose children to these with a bettter understanding of the relative ease or difficulty which children may encounter.

The analysis of intuitive models for multiplication and division indicated that children can develop different underlying models for these processes. The preference for additive strategies in the development of division warrants careful attention as common teaching practice focusses on using sharing and repeated subtraction strategies. Teaching strategies that reflect the informal development and intuitive models of multiplication and division were successfully integrated into a teaching experiment conducted in the later part of the longitudinal study (Mulligan,1991b).

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# The Development of Young Students' Division Strategies

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*This paper traces the development of third graders' strategies for solving division problems involving two- and three-digit numbers. It indicates major differences in the rate of development as well as developmental phases of higher ability and lower ability students.*

## Introduction

A number of teaching experiments are focussing on attempts to elicit and build on young children's conceptualizations of computational problems and the strategies they construct based on these conceptualizations (e.g. Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991; Carpenter, Fennema, Peterson, Chiang & Loef, 1989; Kamii, 1989). Our research group is engaged in an ongoing research and development project on the mathematics curriculum in the first three grades of school, trying to build on children's informal knowledge and studying and facilitating the development of their conceptual and procedural knowledge (Murray & Olivier, 1989; Olivier, Murray & Human, 1990). In a previous report (Murray, Olivier & Human, 1991) we described a number of different strategies (cf. Kouba, 1989), constructed by young students attempting to solve division-type problems in a socially-supportive learning environment where students are required to construct their own solution strategies. The data available at that stage were mostly of the snapshot type and did not capture individual children's development over a period of time, and were gathered from many different classrooms representing many different teaching styles.

For this study, a single third-grade classroom was selected which was observed and videotaped on a regular basis during the 1991 school year, and all written student work on division-related problems during the course of the year was analyzed. In this paper we elaborate on:

- the substantial differences in the rate of development and the development paths of higher ability and lower ability students
- the influence of the semantic structure of the problem type on children's concept of division and solution strategies
- some of the factors involved when students change strategies.

## The Classroom Setting

Our theoretical orientation and research base have been outlined elsewhere (Olivier, Murray & Human, 1990). It is based on a constructivist theory of knowledge in which children actively build up their knowledge based on their own experience. Our approach is further inspired by socio-constructivism: Learning mathematics is a social activity as well as an individual constructive activity (Cobb et al., 1991).

The meanings of operations and solution strategies are not *taught* to students; students are confronted with realistic, meaningful word problems which they are expected to solve using their existing knowledge, and they construct their meanings of the operations and solution strategies *while* solving the problems. The symbols for the operations are introduced only *after* students had already solved problems involving the particular concepts. Students are therefore never required to "identify *the* operation" to solve a word problem; a division problem may be solved by direct modelling, addition, subtraction, or multiplication. They are not required to use number sentences, and are particularly not required to produce a "correct" number sentence for any problem; they nevertheless frequently *elect* to round off their work with a number sentence. The problems were always posed as word problems and of mixed types.

The teacher used the format of grouping the students into three main ability groups as required by the education supervisor in this area; this implies that the teacher takes approximately a third of the students for an informal but very intensive problem solving session (usually on the floor), while the other students are engaged in completing work cards at their tables or playing mathematical games. The main distinction between the ability groups is that the higher ability groups are able to handle larger numbers earlier.

The teacher *consistently* posed word problems in each group's zone of proximal development, requiring each student first to attempt to solve the problem on his own and then encouraging students to consult with one another. One of the main functions of the teacher is to establish a classroom culture in which students engage in interactive negotiation of mathematical meaning (Cobb et al., 1991). Students are expected to explain, discuss, and diagnose each other's mistakes and lend support where needed. The teacher exercised strict control over the social interaction aspect, never helped any student with mathematically-related problems except notation, and never allowed a problem session to be terminated until the *group* had ensured that every student had obtained the correct answer by using his *own* method, and understood fully what he was doing. Students were never required to explain to the group how they themselves had solved a problem (a deviation from our suggested approach). The teacher also continuously affirmed her approval of supportive behaviour and *competently*-given help, showing no interest in who had first obtained the correct solution.

The class consisted of 13 boys and 9 girls from middle- and upper-middleclass homes. For a variety of reasons this class had not been granted the opportunity to construct any meaning for division during their second grade, and all of them resorted to direct representation (i.e. drawing the problem in detail) when presented with their first division-type problem in their third grade.

### A Variety of Strategies

Division problems are broadly of two types: sharing and measurement. Young children's informal solution strategies initially reflect the different semantic structures of these two problem types, illustrating two independent conceptions of division. For a sharing problem such as "There are 18 cookies that you must share equally among three friends. How many cookies must each friend get?", children typically draw three faces or other icons representing the three friends, and then deal the cookies out by drawing them one by one underneath each icon. More advanced numerical methods may still reflect the sharing structure, e.g. the student may use a *repeated estimation* strategy by first trying to give each friend (say) five cookies, and finding it to be wrong, then trying six. Or the student may use an *estimate-and-adjust* strategy by first giving each child (say) four cookies, then another two each, recording the strategy as

$$4 + 4 + 4 = 12$$
$$2 + 2 + 2 = 6$$

To solve a measurement problem such as "How many bags of 3 cookies each can be made up from 18 cookies?" young children typically draw 18 circles, and then bracket off groups of three. A more advanced numerical strategy may involve adding threes until 18 is reached and then counting the number of threes, or subtracting threes until there are no cookies left.

One can deduce that students have constructed an integrated meaning of the concept of division if they are able to transform between problem types, i.e. if the strategy is no longer determined by the semantic structure of the problem, but rather by the particular student's personal preferences.

The first division-type problem posed to the class in April 1991 was a sharing problem. All the students used direct representations (i.e. drew the problem in greater or lesser detail) to solve the problem. Seventeen students kept to the sharing structure of the problem, but the other four employed a strategy which entailed bracketing off groups of circles, and which could be interpreted either as one of "counting the rounds dealt out," or as a transformation of the problem to enable them to count in groups.

The majority of students used a variety of strategies during the course of the year. We now illustrate and trace the development of the strategies used by three particular students who are typical of the three ability groupings in the class. Although all problems were posed as word problems, we will indicate only the problem type (S = sharing, M = measurement) and the numbers involved. Since children's informal strategies frequently reflect an accumulative process, the arrow notation is introduced to students as an additional symbol to try to prevent incorrect use of the equal sign. We have attempted to reproduce the student's original layout as closely as possible.

**Thys** (lower ability group)

| | | |
|---|---|---|
| April | 96 ÷ 12 (S) | direct representation (bracketing off groups of 12 circles, then counting the groups) |
| June | 60 ÷ 5 (M) | direct representation (bracketing off groups of 5 circles, then counting the groups) |
| | 35 ÷ 5 (M) | direct representation (bracketing off groups of 5 circles)<br>35 ÷ 5 = 7 |
| | 36 ÷ 9 (S) | direct representation (bracketing off groups of 9 circles) |
| July | 42 ÷ 3 (M) | first an uncompleted direct representation, then, after listening to Nicol:<br>10 10 10<br>4 4 4 |
| | 72 ÷ 6 (M) | first an uncompleted direct representation, then<br>6 × 6<br>6 × 2, then<br>10 10 10 10 10 10<br>2 2 2 2 2 2<br>12 sheets |
| | 104 ÷ 8 (S) | 10 10 10 10 10 10 10 10 → 80<br>2 2 2 2 2 2 2 2 → 16<br>1 1 1 1 1 1 1 1 → 8<br>80 + 16 + 8 → 104 13 roses |
| August | 92 ÷ 3 (S) | 10 10 10<br>10 10 10 30 2 remain<br>10 10 10 |

| | | |
|---|---|---|
| | 84 ÷ 7 (M) | 7 14 21 28 35 42 49 56 63 70 77 84 = 12<br>1 2 3 4 5 6 7 8 9 10 11 12<br>84 ÷ 7 = 12 |
| | 58 ÷ 4 (S) | 4 8 12 16 20 24 28 32 36 40 44 48 52 56<br>1 2 3 4 5 6 7 8 9 10 11 12 13 14<br>58 ÷ 4 = 14 and 2 remain |
| September | 168 ÷ 7 (M) | 7 14 21 28 35 42 49 56 + 56 → 112 119 126 133 140 147 154 161 168<br>168 ÷ 7 = 24 |
| October | 850 ÷ 130 (M) | 260 + 260 → 520 + 130 → 650 + 130 → 780<br>850 ÷ 130 = 6 rem 70 |
| November | 792 ÷ 36 (S) | 72 + 36 → 108 + 108 → 216 + 216 → 432 + 216 → 648 + 36 →<br>684 + 36 → 720 + 36 → 756 + 36 → 792<br>792 ÷ 36 = 22 |
| | 988 ÷ 38 (M) | 38 + 38 → 76 + 76 → 152 + 152 → 304 + 304 → 608 + 304 →<br>912 + 38 → 950 + 38 → 988<br>988 ÷ 38 = 26 |

It is possible that Thys was able to transform between problem types right from the start; it is, however, equally possible that he was dealing out rounds when he solved 96 ÷ 12 with a strategy that also fits a measurement structure. During July there is a clear movement away from direct representations towards numerical strategies, as the numbers involved also became larger and the question of cognitive economy arises. By August he settled on a general solution strategy for both division problem types which he proceeded to refine during the next few months.

**Gerrit** (middle ability group)

| | | |
|---|---|---|
| April | 42 ÷ 3 (S) | repeated estimation, checked by direct representation, i.e. bracketing off 3 groups of 12, then bracketing off 3 groups of 13, then bracketing off 3 groups of 14. |
| June | 56 ÷ 7 (M) | 56 – 14 → 42 – 14 → 28 – 14 → 14 – 14 → 0<br>8 sweets |
| | 56 ÷ 7 (M) | 28 + 28 = 56 8 days |
| | 153 ÷ 9 (M) | 153 – 36 → 117 – 36 → 81 – 36 → 45 – 36 → 9 – 9 → 0<br>17 players |
| | 192 ÷ 8 (S) | 22 22 22 22 = 176<br>26 26 26 26 = 194<br>24 24 = 192<br>192 ÷ 24 = 8 |
| July | 342 ÷ 18 (M) | 180 + 90 → 270 + 72 → 342<br>342 ÷ 18 = 19 |
| | 390 ÷ 13 (M) | 52 + 52 + 52 → 156 + 52 + 52 + 52 → 312 + 52 → 364 + 26 → 390<br>390 ÷ 13 = 30 |
| | 168 ÷ 7(S) | 25 × 7 = 175<br>23 × 7 = 161<br>1 × 7 = 7<br>168 ÷ 7 = 24 |

| | | |
|---|---|---|
| August | 672 ÷ 24 (M) | 24 × 24 = 496<br>30 × 24 = 720<br>26 × 24 =<br>28 × 24 = 672<br>672 ÷ 24 = 28 |
| | 264 ÷ 6 (S) | 25 25 25 25 25 25 = 150<br>19 = 114     150 + 114 = 264<br>25 + 19 = 44 |
| September | 646 ÷ 19 (S) | 190 × 3 → 570 + 57 → 627 + 19 → 646<br>646 ÷ 19 = 34 |
| November | 3283 ÷ 67 (M) | 670 + 670 → 1340 + 670 → 2010 + 670 → 2680 + 134 → 2814 + 134<br>→ 2948 + 134 → 3082 + 134 → 3216 + 67 → 3283<br>3283 ÷ 67 = 49 |

Gerrit persisted with estimation as strategy for solving sharing problems (first repeated estimation, e.g. for 192 ÷ 8, then estimate-and-adjust, e.g. for 264 ÷ 6) until August. Although he used the same basic strategy, his representation improved from repeatedly writing down the same number, e.g. for solving 192 ÷ 8; to repeated addition, e.g. for solving 390 ÷ 13; to multiplication, e.g. for solving 168 ÷ 7. His strategy for solving measurement problems evolved from subtracting multiples of the divisor to adding larger multiples of the divisor, which later became his general solution strategy for both division problem types.

**Marianne** (upper ability group)

| | | |
|---|---|---|
| April | 42 ÷ 3 (S) | repeated estimation, checked by direct representation (like Gerrit) |
| June | 156 ÷ 13 (M) | 10 × 13 → 130 + 26 → 156     13 + 13 = 26<br>156 ÷ 13 = 12 |
| | 253 ÷ 11 (S) | 21 + 21 → 42 × 2 → 84 × 2 → 164<br>23 × 2 → 46 × 2 → 92 × 2 → 184 + 46 → 230 + 23 → 253<br>253 ÷ 11 = 23 |
| | 224 ÷ 14 (M) | 15 × 10 → 150 + 60 → 210<br>16 × 10 → 160 + 64 → 224<br>16 days |
| | 338 ÷ 13 (M) | 338 – 39 → 299 – 52 → 247 – 39 → 208 *will take too long*<br>13 × 10 → 130 + 130 → 260 + 52 → 312 + 26 → 338<br>10 20 24 26<br>338 ÷ $\underline{26}$ = 13 |
| | 476 ÷ 17 (S) | 17 × 10 → 170 + 170 → 340 + 34 → 374 + 34 → 408 + 34 → 442 + 34 → 476<br>10 20 22 24 26 28<br>17 × $\underline{28}$ = 476 |

Marianne solved all subsequent problems in this way regardless of problem type, for example:

| | | |
|---|---|---|
| November | 6278 ÷ 73 (M) | 73 × 10 → 730 × 2 → 1460 × 2 → 2920 × 2 → 5840 + 730 → 6570 – 292 → 6278<br>10 20 40 80 90 86<br>73 × 86 = 6278 |

Marianne differs from Gerrit and Thys in that she very quickly constructed an economical general solution strategy which she only slightly refined during the course of the next five months.

## Further Discussion

This case study confirms our previous observation that children can solve both measurement and sharing problems in a problem-centered learning approach, i.e. without prior instruction in the meanings of division or in division strategies.

It also confirms that children's solution strategies are initially very much influenced by the semantic structure of the problem, but that they eventually construct an integrated concept of division and one general economical strategy for solving all division problems.

From an analysis of the development of students' strategies in this case study, it seems reasonable to conjecture that their single general economical strategy develops from their measurement strategies, and that sharing strategies are progressively discarded.

Most of the more able students started transforming between the problem structures by the second or third problem presented and after three months they all evolved a general division strategy using multiplication and addition and/or subtraction, like Marianne's final strategy. There was no further development during the last five months except that the size of multiples they handled became bigger and subtraction became more common to adjust the final answer, i.e. they exhibited a confidence and fluency in handling numbers.

The lower ability students also evolved a general division strategy, but their development took place over the full period of eight months and was characterized in general by longer use of direct representation, longer use of the particular problem structure as a cue for a strategy, and slower development towards shorter and more powerful strategies such as multiplication. They consistently preferred to *undershoot* and adjust by adding, i.e. they are less fluent in handling numbers.

It must be noted that the general strategies arrived at by all the students in the class are structurally the same; this an illustration of the social element in the construction of knowledge. The other third-grade class in the same school constructed a different general strategy. (This will be reported elsewhere.)

The students in this classroom constructed an integrated meaning for division more quickly than students in other schools exposed first to measurement and later to sharing problems, although other factors apart from sequencing of problem types must also have had a major influence (e.g. the quality of social interaction).

There is some evidence that increased number sizes encourage students to develop more efficient strategies. For example, Thys abandoned his direct representations when the numbers became larger. To solve 338 ÷ 13, Marianne started subtracting small multiples of 13, decided it would take too long, and added larger multiples of 13 instead. The classroom culture of discussion and argument encourages reflection and evaluation of their own strategies by the children, as shown by Marianne's remark; this must also be regarded as a major factor for the improvement of strategies. These students regard the *process* of solving a problem as interesting in its own right, worth discussing and worth reflecting on.

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# INTERACTIVE DEPICTIONS OF MATHEMATICAL CONSTRAINTS CAN INCREASE STUDENTS' LEVELS OF COMPETENCE FOR WORD ALGEBRA PROBLEM SOLVING

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*An interactive, computer animation-based tutor was developed as part of an ongoing test of a theory of word problem comprehension. Tutor feedback is unobtrusive and interpretive: Unexpected behavior in the equation-driven animated situation highlights equation errors which the student resolves through iterative debugging. Earlier experimental controls show improvement cannot be solely attributed to practice, computer use, or use of the situation-based method. New data from concurrent think aloud protocols over two days uncover specific changes that underlie these improvements. On day 2, subjects spent more time reviewing problem texts and correcting initially flawed expressions. They developed self-directed debugging skills without relying on the interpretive feedback. These changes in behaviors are reminiscent of expert problem solving in many domains.*

Previous research in understanding and solving mathematical story problems has shown that text complexity and the comprehension failures of readers are central to the difficulty of word problems (e.g., Cummins et al, 1988; Kintsch & Greeno, 1985; Lewis & Mayer, 1987). Thus, a model of problem solving that draws on research in reading comprehension may identify specific difficulties and ways to improve performance. A theory of problem comprehension, derived from research in discourse comprehension (van Dijk and Kintsch,1983; Kintsch, 1988), has been posed which focuses on the mental representations produced while reading problem texts and accessing the relevant situational and mathematical knowledge (see Nathan, Kintsch, & Young, in press, for details). An interactive, computer animation-based tutor, ANIMATE, was developed as an indirect test of this model. In this paper empirical results from students' interactions with ANIMATE over a two day period are used to illuminate the cause of pre- to posttest improvements.

A text comprehension-based perspective identifies several factors which make these problems so difficult. Critical information is left unstated, so the student must make certain necessary inferences which taxes processing; problem texts are propositionally very dense texts; the language can be inconsistent with the underlying mathematics or be abstract and ambiguous (Cummins, Kintsch, Reusser, & Weimer, 1988; Lewis & Mayer, 1987; Nathan, 1991). Poor comprehension of these texts in turn hampers students' abilities to access and apply the relevant problem-solving strategies, thus producing degraded problem-solving performances. Additionally, students' understanding of the abstract mathematical expressions they use is often shallow, which further contributes to problem-solving difficulties. Researchers in word problem solving have identified ways to rewrite certain problems so that comprehension and performance is increased substantially (e.g., DeCorte, Verschaffel, & DeWin, 1985). This work has been valuable in identifying some of the linguistic factors of word problem solving difficulty, while controlling for the underlying mathematical content. As a pedagogical tactic, however, this approach is problematic. Word problems come from a variety of sources and generally cannot be expected to be formulated in a manner ideal for students.

An alternative to rewriting problem texts is to impart to students highly robust reading and problem solving skills which can be applied to word problems written in an inconsiderate manner. This is the goal of the research reported here: To provide students with an improved ability to generate a formal set of algebraic expressions from problems presented in story form. The theory which supports this goal is described in the next section along with a brief review of its preliminary successes. Then, new empirical results are presented which identify specific qualitative changes to the solution-generation process that underlie these improvements. One such change is that while students continue to develop expressions containing conceptual errors, they learn to detect and correct them at a later time. This change signals improvement in students' competency in interpreting abstract expressions in a situational manner, and in perceiving the problem-solving process as iterative rather than as a single pass effort. This change comes about from interactions with ANIMATE which provides interpretive feedback. It is the student, not the tutor, who evaluates each solution attempt, by relying on his own understanding of the story situation and mathematics needed to describe it.

## Previous successes of the text comprehension-based theory of word problem solving

Arithmetic word problem solving can be understood within the framework of the general theory of discourse processing of van Dijk and Kintsch (1983; Kintsch, 1988). Kintsch and Greeno (1985) and Cummins et al. (1988) have shown that the successful and erroneous performances of first graders could be modelled by production rules within this framework. The critical features of these models were the representations of the semantics of the problem texts and the underlying problem situation (the *situation model*), which guided selection of the problem-solving strategies.

More recently Nathan (1991; Nathan et al., in press) hypothesized that for robust comprehension to take place a strong correspondence must exist among a problem solver's situation model and his representation of the formal, mathematical knowledge, the so-called *problem model*. The situation model uses cues from the text to access real world knowledge relevant for fully comprehending the story. Often, unstated but crucial information is assumed by the author to be readily accessible to the readers, leaving the reader to make certain necessary inferences and elaborations. Failure to access this information can lead to serious comprehension failures (Kintsch, 1988). Similarly, the problem model uses text-based cues and problem-solving goals to access relevant mathematical knowledge. In a proper, analytical solution, expressions in the problem model capture, mathematically, key relationships of the problem situation. An inadequate situation model fails to properly constrain generation of a complete and correct solution. The establishment of a correspondence between the formal and situational representations increases the likelihood that the student will express mathematical ideas which are consistent with the situation and interpret story events in a mathematical form.

An interactive, computer animation-based tutoring system was developed to test specific predictions of this theory. Students construct equations which drive an animation of the situation (e.g., workers painting a fence at different rates). Unexpected behaviors in the animation -- actions that are inconsistent with the student's situation model -- suggest errors in the mathematics, the resolution of which is highly constrained by the nature of the misbehavior (Figure 1). Students repeatedly debug their expressions and test them by re-running the animation until an acceptable situation is depicted.

Earlier experimental work using a pretest-posttest, control group design (n=96) revealed that this training leads to large performance improvements which persist after the tutor is removed. The methodological details are presented in Nathan (1991, Nathan at al., in press). Experimental controls show this improvement cannot be solely attributed to algebra practice, use of a computer tutor, or use of the situation-based method with no explicit reference to the mathematical formalism. It appears that the coupling of the mathematical expressions to a concrete depiction of the situation is necessary. Nathan et al., (in press) tentatively interpret the results as conveying situational meaning for abstract algebraic expressions which supports more robust mathematical reasoning and the improvements in solution performance.

## How mathematical competency is improved

The experimental work reported above can say little about *how* this improved mathematical reasoning manifests itself as higher test scores. What is it that students are actually learning from this situation-based algebra tutor?

New results are presented which identify some of the specific behavioral changes that underlie these performance improvements. Subjects (n=7) produced "think aloud" reports while solving word problems in two experimental sessions. Throughout the training there was continuous improvement, even after a two-day delay in working with algebra, $t(6)=5.36$, $p<.002$, MS=.5. An analysis of students' verbal and computer protocols revealed that the frequency of animation-driven debugging (changes students made to the mathematics that were triggered by running the animation and finding that it performed unexpected behaviors) did not change significantly over the two days. Students' solution attempts do not contain fewer bugs with continued practice and feedback in this instructional setting (Table 1). However, the amount of student-motivated debugging (solution changes which preceded execution of the animation as a source of feedback) increased substantially from Day 1 to Day 2. Furthermore, whereas the (tutor-free) pretest showed no indications of student-motivated debugging, posttest episodes (again, tutor-free) revealed 10 such instances, indicating that this behavior was learned from tutor exposure.

TABLE 1
Frequency of corrections to algebraic errors made with and without the use of animation-based feedback

| | Number of animation-driven corrections | Number of student-motivated corrections |
|---|---|---|
| Pretest (no tutor) | N/A | 0 |
| Day 1 (w/ tutor) | 22 | 3 |
| Day 2 (w/ tutor) | 21 | 10 |
| Posttest (no tutor) | N/A | 10 |

It appears that having been provided an explicit link between the situation and the mathematics students began to see equations as meaningful. They then evaluated the situational validity of the equations and changed them accordingly with no outside intervention. In further support of this view, differences in how students allocated their time during solution generation were found. It's important to note that the following results are trends, not statistically-reliable differences, because of the small sample size used. While overall solution times did not differ over the two days, subjects on Day 2 spent more time reviewing the problem texts. This was true on an absolute scale and as a proportion of the total solution time. It was also found that the individual reading events became longer on the second day. Subjects on Day 1 tended to re-visit the problem texts in a very focused manner, looking for specific values, for example. On Day 2, the reading episodes were characterized as reviewing more phrases, sentences, or the entire problem description.

Thus, it seems that subjects tended to spend more of their time comparing problem situations to their mathematical descriptions of the situations, and refining their solutions until the situation and the mathematics were seen as mutually consistent. Taken alone, these findings may seem unimportant. But they, along with the emergence of a repair-based solution generation process, help to explain students' overall performance enhancement.

## Conclusion

Changes in the problem-solving processes of ANIMATE tutor users lead to problem-solving reminiscent of expert behavior. Skilled behavior in a variety of domains is often characterized by a substantial number of errors early, but expert practitioners have learned to correct their errors. In writing compositions, for example, experts tend to write their initial ideas down, knowing they have not stated them perfectly the first time, and then they revise them (Scardamalia, Bereiter, & Steinbach, 1984). Novices, in contrast, tend to make a single pass over the information. In computer-based text editing, Card, Moran, and Newell (1983) show that skilled secretaries spent about one-quarter of their task time making and correcting errors. Professional mathematicians also make mistakes while performing routine manipulations in algebra (Lewis, 1981). Flawlessness of the process is not the goal of many of these experts. They know their flaws are easy to detect upon reexamination.

Many conceptual errors were made by subjects at both pretest and posttest time while using the ANIMATE system. Conceptual errors have been characterized as "the most serious" for algebra problem solving (Hall et al., 1989, p. 263), manipulation errors being more recoverable. Conceptual errors were not fatal for ANIMATE users, however. Performance improved, even though it was colored by a fairly constant error frequency because students learned to detect and repair them as part of the problem-solving process.

One cannot characterize students' improved performance as the generation of error-free solutions. Rather, students learned to generate buggy approximations to a solution and refine them over time. This was possible for two reasons. First, the tutor provides interpretive feedback to the students, allowing them to make errors, but also see their ramifications in a situational form. This form is believed to be more meaningful than the abstract equations required for a solution. The situational context taps into students' knowledge of events thereby supporting the reasoning needed for self-assessment and correction. Secondly, the environment encourages students to develop partial solutions and refine them in an iterative manner. There is no penalty for errors of commission or incompleteness. Consequently, students seem to learn that errors can be made and then discovered and corrected. The problem-solving processes of these students and the allocation of their problem-solving time shifted from one pass solution methods to one of self-directed diagnosis and error correction. This shift occurred in a technologically minimal setting by most tutor standards, allowing students to exercise a good deal of freedom in the way in which they derived solutions to problems that were difficult a short time earlier.

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Problem text. Huck and Tom agree to paint a fence. Tom can paint the entire fence in two hours while it takes Huck four hours. If Huck starts one hour earlier than Tom, how long will it take the two boys to complete the job? (Problem text adapted from Hall, 1990).

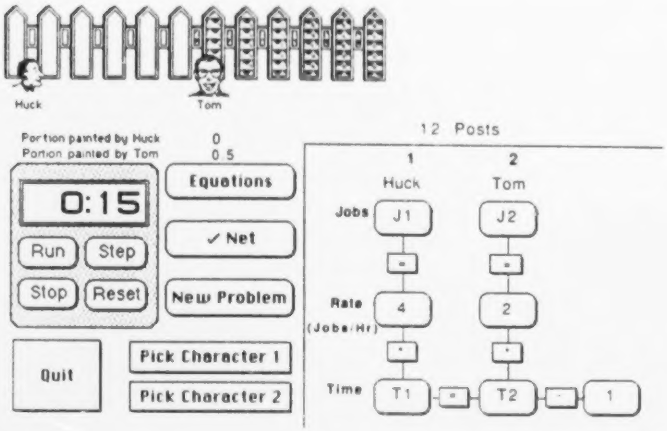
Portion painted by Huck 0
Portion painted by Tom 0.5
Huck
Tom
12 Posts
0:15
Run
Step
Stop
Reset
Equations
✓ Net
New Problem
Quit
Pick Character 1
Pick Character 2
1
2
Huck
Tom
Jobs
J1
J2
=
=
Rate
(Jobs/Hr)
4
2
*
*
Time
T1
=
T2
-
1

Figure 1. Checking a solution attempt for a Work problem by running the equation-driven animation. Several errors in the solution exist and produce behaviors on the screen inconsistent with the given problem text. The time equations causes Tom to start while Huck waits for one hour (the relation implies that Tom's time, T2, is greater than Huck's time so an hour must be subtracted for a balanced equation). Rate values are incorrect (they are the given values which *times*, not rates) so the characters paint much faster than expected. Lastly, with no equation relating Job variables, J1 and J2, (the amount of work done) the animation continues indefinitely, leaving each character to continually cycle over the entire fence.

# The Influence of Numerical Factors in solving simple Subtraction Problems

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*Many investigations have illustrated that it is the structure and context of the problem – not numerical factors – which influence children's choice of problem solving strategies when they subtract. Yet, problems formulated with the intention to test the influence of numerical factors have rarely been given in these studies. In a Swedish study, where 82 school starters were interviewed, it was illustrated that numerical factors sometimes can play an important role. At a certain point in the developmental process children seemed to create a strategy which helped them to subtract in the most comfortable way – independently of the structure of the problem – as long as the numbers were not bigger than 10. In a teaching experiment, aimed at testing the didactical implications of these findings, pupils who were helped to find this strategy when they subtracted within the number range 1–10, learned to add and subtract mentally also within higher number ranges.*

## Introduction

Fischbein et al. (1985) illustrated that the operation which pupils choose in order to multiply or divide are often governed by primitive models which are formed early in their lives, and they point out that it is also the case for addition and subtraction. Subtraction appears to have at least two behavioral interpretations: (a) take away (John has 9 marbles. He gives 7 to Jenny. How many marbles has he kept?) and (b) building up (John has 2 marbles. How many more does he need in order to have 9 marbles?). Carpenter and Moser (1982) examined the implications of the primitive models for addition and subtraction and found that children chose subtractive strategies for the former kind of subtraction, (a), while they chose additive strategies for the latter, (b). A lot of more recent research has confirmed these findings. Thus, structure and context of the problems, not numerical factors, have been considered to influence children's choice of problem solving strategies.

However, Resnick (1983) found, when analysing existing research concerning subtraction, a strategy she has called 'choice'. Children using this strategy choose either additive or subtractive strategies for problems of type (a) depending on which strategy requires fewer steps.

The problems given in the studies illustrating that context and structure govern pupils' choice of problem solving strategy have rarely been of a purely 'discriminating' kind, worded in such a way that they could also test if numerical factors might influence choice of strategy. In a Swedish study (Neuman, 1987, 1989a) all the questions were of a 'discriminating' kind. In this investigation 82 school starters – all the pupils from four classes – were interviewed immediately after school started, before they had any instruction in mathematics. The problems were chosen in such a way that they would first of all test if the strategy 'choice' was used. This strategy had been found in informal observations and in pilot studies related to the main study (which actually started 1982, thus before Resnick's report on the 'choice' strategy). This investigation illustrated that numerical factors sometimes seem to influence children's choice of problem solving strategies, even more than the structure and context of the problem.

The fact that 'discriminating' problems were posed might not be the only cause of the divergent findings in the Swedish study. For example, school starters in Sweden are 7 years old, and have not had any formal teaching of addition and subtraction during their preschool years. Further, the problems given were within the number range 1 – 10 only.

The decision to use numbers in the range 1-10 was part of the design, to make the problems 'discriminating' by making explicit how children see the part-whole relations which constitute the numbers 1-10. If children add two numbers greater than five before they know how to split the added number into two parts at the 10-border, the strategy 'choice' cannot help them overcome the 'keeping track' problem, and thus there is less need to create it. According to Miller (1967) not even adults can operate with more than 7 ± 2 components at the same time, and the capacity is less for children.

From a didactical point of view, the interesting problem is to analyse how the modeling strategies children use for problem solving within the number range 1–10 gradually change into mental calculation strategies and finally into abstract thinking. Problems within higher number ranges are either variants of problems within the number range 1–10, or they involve splitting up the added or subtracted numbers at the 10-border. Thus, it was hypothesized that if teachers can help children to find economic proble.. solving strategies within the number range 1–10 their pupils will have a readiness to add and subtract mentally even within higher number ranges. This hypothesis was later tested in a teaching experiment.

Another aim of the study was to understand apparently strange conceptions of subtraction observed in pilot studies carried out with first and second graders. At PME 1989 a presentation was made of these conceptions, and of the the 'child-logic' inherent in them (Neuman, 1989). Now a review will be made of the strategies used by the children who gave correct answers.

## The phenomenographic approach

The study was carried out as a phenomenographic investigation, where the aim is to map and describe the conceptions of certain phenomena, in this case conceptions of number and of the addition and subtraction strategies through which these conceptions are formed.

Phenomenographic studies assume that conceptions are relations between man and world, created through experiences, akin to phenomenological intentionality. All our conceptions can be said to be all our knowledge. *'Conception'* is a 'key concept' in phenomenographic research. The conception can be said to be the offspring of the thought. The strategies that are observed are mainly interesting because they express – but also form – the conceptions.

Therefore, in phenomenographic studies, concrete as well as abstract strategies are related to one and the same conception. It is the *idea* – the conception – behind the modelling strategy, the mental calculation strategy or the more abstract thought that is of interest in phenomenographic research. Since the same idea is practically always observed at different levels of abstraction (Neuman, 1991) it is possible to find hypotheses about how an idea can be constructed at a concrete, modeling level and after that gradually be transformed into more and more abstract thinking.

## Method

Before the main study started some pilot studies were carried out, in one of which older pupils with 'math difficulties' were interviewed. There it was revealed that those pupils, even at the age of 17 or 18, still solve subtractions of the 'take away' kind through subtractive strategies, and subtractions of the 'building up' kind - often called 'missing addends' – through additive strategies.

Thus problems of the kind: 'You have 2 kronor and want to buy a comic that costs 9 kronor' were solved through counting on from 2, while problems of the kind 'You have 9 kronor and lose 7 kronor' were solved through counting back. Sometimes the older pupils made an estimate, forwards or backwards, which was occasionally wrong. None of them seemed to have found the 'choice' idea, which would have helped them to think backwards in the first and forwards in the second problem, and thus solve the problems in a very easy way.

According to the theories presented by Miller (1967) seven words are too many to list and simultaneously keep track of mentally. The pupils solved this problem by putting up one finger for each counting word they listed forwards or backwards. After that they read off the answer: 5 + 2 = 7: the fingers were used for 'keeping track'. With the help of their hands, the pupils grouped the seven words into two perceivable groups. This strategy was referred to as 'double-counting', a strategy expressing a conception of number as 'Counted numbers'. The pupils with 'math difficulties' rarely used any strategies other than this concrete double-counting for all operations, even for multiplication and division problems (which, however, were mostly impossible to solve in this way).

The realisation developed, related to Miller's theories, that double counting strategies (or the conception of number as counted numbers) which lack the idea of 'choice' cannot lead to abstract thinking. This was confirmed confirmed by the fact that even high-school pupils with math difficulties who had not discovered 'choice' still used these concrete 'double-counting' strategies.

Another observation that was hard to understand in relation to the observations of pupils with math difficulties, was made during informal studies of primary school children. Even as soon as school started, a few of them could solve the kinds of problems presented above, giving an immediate answer: 'Seven missing' or 'Two left'.

The question of how they had arrived at this apparently abstract knowledge, which some people never seem to arrive at, was at the focus of interest in the school starter interviews, where the following four subtraction questions were presented:

1. *If you have 3 kronor and you want to buy a comic for 7 kronor, have you got enough money? – No! – How many more kronor do you need?*
2. *If you have 10 kronor in your purse and lose 7 kronor, how many have you got left?*
3. *If you have only 2 kronor and you want to buy a comic for 9 kronor, how many more kronor do you need?*
4. *If you've got 4 kronor and you want a comic that costs 10 kronor, how many more kronor do you need?*

Beside these problems a *'guessing game'* was played, where the children had five turns to guess how 9 buttons were hidden, distributed between two boxes. Addition tasks were also given, but since the strategy 'choice' is of no interest in addition, these problems are not taken into account here.

Problems 2 and 3 were aimed at testing the existence of the strategy 'choice'. They were the kind of subtraction problems which were so hard to solve for older pupils with math difficulties

Problems 3 and 4 were chosen according to observations made in pilot studies, illustrating that primary children often use 'doubles' in two ways, when they solve specific kinds of problem.

All interviews were audio taped and transcribed word for word.

**Results**

Four conceptions of number were found in the main study:

| | |
|---|---|
| *1. Numbers measured by twos, threes or fours* | *3. Ordinal–cardinal finger numbers* |
| *2. Counted numbers* | *4. Structured and counted numbers* |

*Numbers measured by twos, threes or fours'*

Children expressing this conception often use doubles. After the answer 'Four' to problem 1 the following explanation might be given: 'cause 3 + 3 makes 6 .. That's why I knew it'. This was the most commonly used strategy for problem 1, the problem given in order to test its existence.

*Counted numbers*

This conception is related to the double-counting strategies described above. Only 6 pupils used it for subtraction, 5 of them only in one answer. One of these pupils used something in between the strategies counting all and counting on. He solved question 1, for example, by putting up the three middle fingers of the first hand saying 'Three', and after that four fingers of the second hand which touched his mouth, while he counted '4, 5, 6, 7'.

*Ordinal–cardinal finger numbers*

This conception illustrates how the strategy choice can be created through concrete modelling, and subsequently developed further into abstract thinking. It was the most often expressed conception of number. (Even if it was also the most often expressed conception for addition, the children very often used their fingers in a lot of other ways in addition, e. g. for double-counting).

The significant idea of this conception is that the fingers form a precursor of the 'number line'. Within the 'row of fingers' up to 10 each 'finger number' ends at a certain position. The finger number for seven, for instance, ends at the second forefinger: one finger after the end of the finger number for six and one finger before the finger number for eight. Thus each number can be experienced in an ordinal way. Since all finger numbers which are too big to be subitized are constituted by two well known groups – i.e. the fingers of the two hands – they can also easily be recognised in a cardinal way without counting. They are simultaneously ordinal and cardinal, which is consistent with Piaget's (1969) theories. Piaget states that the sequence of whole numbers is inseparably ordinal and cardinal, a conclusion he stresses that his data has clearly illustrated.

When the finger numbers had begun to be indentified in the children's problem solving, all pupils in two of the four classes (n=43) were asked to put different numbers of fingers on their desks. Around 3/4 of these children could immediately put up all finger numbers without counting, except the finger number for 8, which only 1/4 of them knew in this way. They explained that they knew how to put up all these fingers without counting, because 7 was 2 more than 5, 9 was one less than 10 and so on. Further, many of the children who could not put up their finger numbers without counting, began to put up their first hand directly and after that counted the remaining fingers of the second hand, or set out from ten fingers and fold one or two beginning from the tenth finger.

When the children illustrated 'two' and 'three' any fingers could be used. Two children put up four fingers on each hand for eight and five of them put up one hand and the fingers except the thumb on the other hand for nine. These finger groups were not ordinal: nine made up of 5 + 4 in that way, for example, ended at the tenth finger. However, two of the five children who did this explained that they would have liked to fold the little finger instead of the thumb, but that this was very hard.

Six strategies – gradually more and more abstract ones – were related to this conception of number:

| | |
|---|---|
| *1. 'Estimate the last part'* | *2. 'Count the last part'* |
| *3. 'Don't divide up the first hand'* | *4. 'Transform finger numbers'* |
| *5. 'Think of finger numbers'* | *6. 'Abstract operations'*. |

The strategy *'Estimate the last part'* only ended up with correct answers by chance (and is therefore not included in table 1). This strategy and the strategy *'Count the last part'* of the finger number were rarely used. It seemed that they were very soon exchanged for more economic and adequate strategies. The five children who used them put up two fingers to begin with and went on to put up fingers until the ninth finger was placed on their desk (fig 1). Since the fingers remaining after the two first ones were not structured by the hand, the children could not immediately perceive their number. Four children estimated the number of fingers in the last part of the finger number to six, seven or eight (strategy 1). One child counted them (strategy 2).

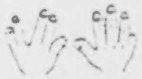

Fig 1

Children who used these two strategies had not yet created the strategy choice. It was created in the third strategy *'Don't divide up the first hand'*. Many of the children who used this strategy said something about 'five' in spite of the fact that the word five did not appear in the problem.

Susie, when she solved questions 1 and 3, had to count all the fingers up to the seventh or ninth in order to know what the finger number for these numbers looked like. Then she said: 'There, now we can start'. After more manœuvres with her fingers she finally answered 'Seven' to problem 3. When asked how she found that answer, she explained her procedure as follows (fig 2):

'First I sort of counted five ... Then I put up two ... Then I put up two more ... Then I put them down (the two last ones) and count them again (the seven first ones) like this (touching her lips with one finger at a time)

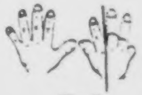

Fig 2

Susie illustrated one way in which the strategy 'choice' can be created through finger numbers under the restriction that the first hand must remain undivided. She does not *add* the fingers representing the last part to the two representing the first part. Instead of that she *'takes away'* the two fingers, representing the first part from the whole, and 2 + X= 9 has become 9 – 2 = X.

For that problem, children who knew their finger numbers put up nine fingers directly and folded the last two. Some did not even bother to manœuvre their fingers. They looked at their 10 fingers and answered 'seven' immediately, explaining that they knew, because 'this is nine' or 'this is seven', pointing to the ninth or the seventh finger. They used the fingers as if they had been a number line.

Through strategy 4 the idea *'Transform finger numbers'* was created. The idea governing that strategy was: if you move one finger between your hands (or between other finger groups) one finger group increases as much as the other finger group decreases. Some children used this strategy repeatedly in the guessing game. For example, they could put up the finger number for nine and start with the guess 'There are five in this box and four in that one', go on to guess 'Six in this one and three in that one' and so on, while they moved one finger at a time from one finger group to the other.

Strategy 5 – *'Think of finger numbers'* – was of a little more abstract kind, and no concrete fingers were used. Niclas, for example – after his answer 'Six' to problem four, and the explanation that 'he thinks with his hands' – explained in detail the way in which the finger-number 10 could be changed from 5 + 5 into 6 + 4 and again back to 5 + 5. He moved the thumb from one hand over to the other while he illustrated his thoughts with his fingers. Children who said that they 'thought with their hands' could also think according to the strategy 'Don't divide up the first hand'.

A lot of cubes, buttons etc were placed on the table in front of the children, but none of them used any material other than fingers, if they used any concrete material at all.

In strategy 6 – *'Abstract operations'* – no reference was made to fingers. This strategy can be illustrated by a child who in the guessing game, after having guessed 'Five here and four there', guessed 'Six here and three there ...'. He explained: 'I take one less than five to four and one more to the five ...' .This was the strategy most often used for problem 10, the problem formulated in order to test the strategy's existence. Other children could explain, after having answered 'four' in question 1: 'I thought, if it's three, so I'll take three from seven ...'. And a child who immediateiy answered 'Three' to question 2 explained that he knew, because three plus seven make 10.

If we analyse the Roman numerals in their ancient form, we observe that the two ideas 'Don't divide up the hand', and 'Transform numbers, related to an image of ordinal-cardinal finger-numbers, might easily have been used also earlier in history, when our figures once were pictures of our fingers: all numbers bigger than five can – independently of the structure of the problem – be divided up into two parts in such a way that we have immediate access to parts and whole without any counting (fig 3) .

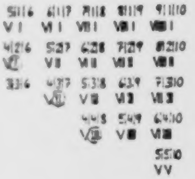

Fig 3

The reason is that the V-symbol is undivided (or can be divided into 4 + 1) and that the biggest part of the number can therefore be perceived – or thought of – in a simultaneous ordinal-cardinal way, exactly as the whole is perceived or thought of.

The old Chinese rods (fig 4) were also pictures of the hands, and, just as the Roman numerals, structured to one undivided five-symbol as soon as the numbers were bigger than five

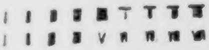

Fig 4

We always seem to have 'felt' that it is not possible to perceive or think of numbers larger than five unless the many small components are 'chunked together' into a few bigger ones.

*Structured and counted numbers*

Two strategies were related to this conception:

*1 Counting all from larger* *2. Counting on from larger*

As discussed earlier, the problems were formulated in order to test if the strategy 'choice' was used or not. The children who expressed the conception 'Structured and counted numbers' used it in order to choose forwards instead of backwards strategies when they solved problem 2 – the 'take away' problem – but never chose to count backwards in the missing addends. (A few did, but then *after* an 'Abstract operation' of the type: 'Four ... just took away three', for problem 1). The problem for them was that the largest part in the missing addends was unknown. Thus, it was impossible to use the comfortable strategy 'Count on from largest'. Yet, the children still found a way, which at least helped them to avoid the laborious creation of a concrete keeping track strategy. They just counted *all* from largest, for problem 3, for example: '1, 2, 3. 4. 5. 6. 7 ... 8, 9 ', before they answered 'seven'. An interesting finding in their strategy is that they listed the counting words related to the *unknown* part *up to* the two words related to the known part. Problem 2, the *'take away problem'*, was solved in the same way by 3 children. They counted *forwards*: '1, 2, 3, 4, 5, 6, 7 ... 8, 9, 10', before they answered 'Three left'. Baroody (1984) has illustrated this strategy in addition, and given it the name 'Counting all from largest'. Yet, since the larger part was known in problem 2, this problem also could be solved through 'Counting *on* from largest', and 11 children solved it that way.

The conception 'Structured and counted numbers' thus only seems to make it possible to choose counting all or counting on from largest, *not* counting *back*. The children seemed to feel unsure of how to count back, and preferred to use – or think of – finger numbers in their backwards operations. All children – except 4, who sometimes expressed the conception 'Structured and Counted numbers' – also used strategies related to 'Finger numbers'. The four other children used early strategies ending up in wrong answers – or double counting strategies – for the remaining problems.

The assumption has therefore been that the conception 'Counted and structured numbers' might be a precursor of the conceptions 'Finger numbers' and 'Counted numbers', but that it is mostly a strategy related to the conception 'Finger numbers' where the fingers' names just are listed .

**Conclusion**

Table 1 below illustrates that *number relations* within the problems – *not the structure* of the problems – seem to explain the choice of operation children made in this study.

**A teaching experiment testing the findings**

A teaching experiment was performed with two of the interviewed classes in order to test the didactical implications of the findings. It lasted for two years and illustrated that the strategy 'choice' could be developed by all children through the kind of concrete creation the school starters had used. This strategy gave the pupils access to all part-whole patterns of the 10 basic numbers as 'number images'. This in turn gave a readiness to add and subtract mentally, not only within the number range

1 – 10 but also within higher number ranges. All children in the two classes could add and subtract mentally within the number range 1–100 at the end of the second year.

Table 1.
Number of correct answers related to different conceptions and strategies which were used when the four problems were solved. The strategy most frequently used for each problem is written in italics.

| Conception<br>Strategy | Problem<br>1 | 2 | 3 | 4 |
|---|---|---|---|---|
| MEASURED NUMBERS BY 2, 3, 4 | *13* | 0 | 1 | 4 |
| COUNTED NUMBERS | | | | |
| Counting all/on | 1 | 0 | 0 | 0 |
| Counting on | 6 | 0 | 1 | 1 |
| FINGER NUMBERS | | | | |
| Count the last part | 0 | 0 | 1 | 0 |
| 'Don't divide up the hand' | 0 | *21* | *7* | 7 |
| 'Transform finger numbers' | 12 | 0 | 0 | 2 |
| 'Think of finger numbers' | 2 | 1 | 1 | 3 |
| Abstract operations | | | | |
| Transformations | 0 | 0 | 0 | *12* |
| Others | 4 | 3 | 5 | 0 |
| STRUCTURED AND COUNTED NUMBERS | | | | |
| Counting all | 1 | 3 | 3 | 0 |
| Counting on | 0 | 11 | 0 | 0 |
| DIRECT ANSWERS* | 9 | 12 | 6 | 5 |

*Direct answers without explanation

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# A KRUTETSKIIAN FRAMEWORK FOR THE INTERPRETATION OF COGNITIVE OBSTACLES: AN EXAMPLE FROM THE CALCULUS

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*In two research studies of college students' learning of calculus a number of cognitive obstacles were identified. We found that by interpreting these cognitive obstacles within a Krutetskiian framework — that is, by examining the roles of the cognitive processes of reversibility, generalizability, and flexibility — we were able to add to our understanding of the origins of a variety of these obstacles. It may be the case that using such a framework can be useful in analyzing cognitive obstacles to learning in a variety of other mathematical contexts, as well.*

Currently, there are a number of efforts across the country to transform the content and teaching of the calculus. One aspect of any successful reform — an aspect that has been too easily ignored in the past — is the nature of student learning. Indeed, understanding how and why students construct calculus-related concepts as they do is critical knowledge for those involved in pedagogical and curricular issues. The purpose of this paper is to propose a scheme for understanding students' difficulties in the learning of calculus.

Our investigation began with two projects in which we collected data on students' difficulties in applying concepts and procedures to solve a broad range of calculus problems.[1] The theoretical bases for our analysis of students' conceptions lie in two different but related areas of research. Krutetskii (1976) identified characteristics of the problem-solving processes used by students to solve mathematical problems. The three processes that we feel are most closely related to the understanding of the calculus are *flexibility, reversibility*, and *generalization*. The research on cognitive obstacles also provides a perspective for examining students' difficulties with a variety of calculus concepts.

## Krutetskii's Problem-Solving Processes

Krutetskii (1976) identified several abilities related to successful problem solving: flexibility in thinking, reversibility of thought, and an ability to generalize. Most of the current research in learning has focused on students' understanding of particular concepts — in the calculus venue, concepts such as limit, function, or derivative. The approach taken in designing our investigations is closer to that of Rachlin (1987) in his study of an experimental algebra curriculum. The problem-solving processes of generalization, reversibility, and flexibility guided the selection and creation of tasks that were used in individual student interviews and framed our interpretations of resulting data, ultimately focusing on how these processes emerge in the normal conduct of students' work in the calculus.

*Flexibility.* Krutetskii (1976) describes flexibility of mental processes as "an ability to switch rapidly from one operation to another, from one train of thought to another" (pp. 222-223). An instance of this is when students are able to solve the same problem in different ways. Students who are able to choose from several different problem-solving methods are usually able to solve a variety of problems posed in a variety of ways. Another aspect of flexibility is a student's ability to understand and use multiple representations of a concept (e.g., Kaput, 1987). In the calculus, the concept of function is central to the understanding of limit, derivative, and integral. Consequently, students who are comfortable with algebraic and numeric formulations and the graphical representations of functions are better prepared to solve many problems that arise in the study of calculus.

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[1] A more complete description of these studies can be found in Norman & Prichard (1991).

*Reversibility.* "The reversibility of a mental process here means a reconstruction of its direction in the sense of swithching form a direct to a reverse train of thought" (Krutetskii, 1976, p. 287). Reversibility is essential for a complete understanding of a broad spectrum of mathematical processes. For example, many students have a facility for factoring a polynomial to find its roots, but are unable to produce a polynomial with given roots. Likewise, reversibility provides a crucial link between the processes of differentiation and integration. Often integration is introduced as the solution to problems such as "Name a function, $F$, whose derivative is $f$." Yet, when learning rules for integration, many students do not reason reversibly to understand the underlying processes. They tend to view differentiation and integration as rather unrelated processes each of which has its own set of rules to be memorized (e.g., Eisenberg, 1991). As another example, many students have no difficulty in describing the properties of a given function and its derivative. However, when students are given the properties of a function and its derivative, many are unable to construct a graph of a function that satisfies these properties, much less the algebraic form of such a function.

*Generalization.* Krutetskii describes two aspects of this ability: "(1) a person's ability to see something general and know to him in what is particular and concrete (subsuming a particular case under a known general concept), and (2) the ability to see something general and still unknown to him in what is isolated and particular (to deduce the general from particular cases, to form a concept)" (1976, p. 237). The first ability is fairly well-developed among students, who must learn many rules for differentiating functions in a first-semester calculus class — e.g., when given a function to differentiate, students must recognize the function as a particular case to which one of the rules may be applied. Unfortunately, students are not given the opportunity to develop the second ability very often in their calculus classes. Instructors who feel the need to complete a syllabus, tend to tell students the rule or generalization rather than giving them the opportunity to discover it themselves. However, classroom experience has shown that, for example, students are able to generate the power rule for differentiating functions on their own when simply given the opportunity to investigate several cases.

A third aspect of generalization, not touched upon by Krutetskii, differentiates among two levels of general rules — rules related to a specific class of functions and general rules for all functions. For example, there is a rule for differentiating functions of the form $Ae^{Bx}$. There is also a rule for differentiating $f \cdot g$, for any differentiable functions $f$ and $g$. These two levels of generality are not usually acknowledged by instructors or texts, and certainly most students do not recognize the differences.

### Cognitive Obstacles

The notion of *cognitive obstacle* has appeared frequently in the recent literature on mathematics learning — particularly that related to algebra learning. Briefly, cognitive obstacles are idiosyncratic constructions or interpretations of concepts and processes which, while arising naturally in the process of cognitive development, tend to constrain or impede the development of a complete understanding of the concept or process. Without delving too deeply into the psychological basis for this, we note that a constructivist view of concept development provides a theoretical framework in which to view cognitive obstacles. Within this perspective the nature of an individual's conceptualization is influenced by cognitive structures and processes already extant. For example, to use Piagetian terminology, the accommodation of a new concept requires a reorganization of cognitive structures, an action that may be not be so easily or appropriately done. In fact, Herscovics (1989) speaks of "their [cognitive structures] becoming cognitive obstacles in the construction of new structures." (p. 62)

As mentioned, cognitive obstacles to a particular concept might simply be a natural stage in the development of that concept and for many students may be transitory. For some students though, the extent of the influence of the obstacles becomes problematic. Thus, understanding the origin of a particular obstacle, how it relates to an individual's unique mathematical experiences, and how it might be overcome are questions that are useful to examine.

## A Study of Problem Solving Abilities

To study the process of students' thinking rather than simply the products of their thinking activities, four clinical interviews were conducted with each of six students enrolled in a Calculus I class. A principal part of the interviews consisted of the students being videotaped or audiotaped as they solved a series of calculus problems. This approach provided the detailed trace of students' problem-solving behaviors that was needed for this study. To illustrate the data collected from these interviews, we consider the case of Paige, whose misunderstandings and difficulties are fairly typical of calculus students.

Ability to generalize. The following example is a typical task focusing on the recognition of a particular case of a known concept or rule.

If $f(x) = cx^n$, then $f'(x) = cnx^{n-1}$.

(1) $f(x) = -2x^5$, $f'(x) =$ ________;

(2) $f(x) = x^{-2}/3$, $f'(x) =$ ________;

(3) $f(x) = \frac{1}{2} \cdot x\sqrt{\frac{3}{2}}$, $f'(x) =$ ________.

Problem (3) above had been encountered by Paige while solving a related rates problem. Rather than recognizing this expression as a special case of $cx^n$, to which the simple power rule could be applied, she focused on the product of the two factors, $\frac{1}{2}x$ and $\sqrt{3/2}$, and applied the product rule. (In a similar problem, another student used the quotient rule to find the derivative of $\frac{\sqrt{3}x^2}{2}$.) While we would assume that the exponent rule is mastered by our students, nonetheless many students do not recognize that they can apply this rule to functions that involve irrational coefficients.

Flexibility in thinking. One aspect of flexibility of thought is the degree to which a successful solution process on a previous problem fixes a student's approach to a subsequent problem (Rachlin, 1987). A good problem solver knows when to "fix" and when not to. This process is closely related to the process of generalizing in that a problem solver may recognize that a new problem situation is similar to one recently encountered and that the same problem-solving approach would be helpful for the new situation. On the other hand, a good problem solver also recognizes when a new approach is most effective for a particular problem.

Paige was given only the graph of a quadratic function $[y = (x-1)^2/16]$ and asked to find the following:

a. $\lim\limits_{x \to 1} f(x)$ b. $\lim\limits_{x \to -3} f(x)$ c. $\lim\limits_{x \to 5} f(x)$ d. $\lim\limits_{x \to c} f(x)$

In questions a, b, and c, Paige located each point to correctly identify the limits. When she attempted d, she had no idea of how to describe what the limit should be. She had "fixed" on the method of locating points and was unable to use another strategy for solving the problem. It is also possible that her expectation of what her answer should be did not allow her to describe the limit in a way that seemed to her too vague — namely, $f(c)$. Paige's responses to these tasks and her inability to demonstrate flexibility in her thinking is related to her difficulty in working with the multiple representations of functions. Other students who understood and used the algebraic and graphical representations of functions were not challenged by these tasks. They were solved easily and quickly.

Reversibility of thought. The following two examples illustrate different aspects of this process.

I. Sketch a possible graph for a function $f$ such that:

a. $f$ is everywhere continuous

b. $f(-3) = 1, f(0) = -2$

c. $f'(x) < 0$ when $x < 0$

Like many of our calculus students, Paige was not very successful with problems of this type. This could be due in part to her lack of experience with problems of any type that require reversibility of thought. Paige hesitated before plotting the points $(-3, 1)$ and $(0, 2)$. Then she had no idea about what steps to take next. She could interpret the third condition in terms of the slope of tangent lines, but she did not know how to use that information to sketch a graph.

II. Find a function whose derivative is $3x^2$.

Paige had difficulty answering a question as seemingly simple as this one. Her understanding of the relationship between the derivative and integral was weak. When Paige began to study integrals and basic techniques of integration (power rule and the method of substitution) she approached differentiation and integration as two different sets of rules to memorize rather than as reversible processes. When the same task was presented in the form $\int 3x^2 =$ ________, she easily used the power rule for integrals to find $x^3 + c$.

Remarks. The three abilities — generalizing, flexibility of thought, and reversibility of thinking — provide a framework for studying and describing students' understanding of calculus and their performance on calculus problems. While these abilities are very important in successfully solving problems in calculus, they do not receive explicit attention by classroom teachers and curriculum developers. In particular, relatively few problems in a standard textbook show different methods for solving calculus problems. Paige's inability to use these processes hampered her progress in the study of calculus. Other students who demonstrated greater facility with these three processes tended to have more success in the course.

## An Analysis of Cognitive Obstacles

In the second investigation, we examined and catalogued the work of our calculus students. Data have been collected from several classes, but those from two classes (a first-semester calculus class and a differential equations class) have been completely and systematically analyzed. The cognitive processes of flexibility of thought, generalization, and reversibility provide a framework for analyzing and classifying specific cognitive obstacles identified during our investigation. In the first study, we identified a number of situations that provide insight into common cognitive obstacles from the perspective of this Krutetskiian framework. The results of the second study, as well as the work of other researchers, provide additional instances of cognitive obstacles that appear to be closely tied to these three processes.

We found a significant number[2] of errors appear to be tied to specific, sometimes well-known, cognitive obstacles. Most of the obstacles identified are simply analogues of those that have been identified in the extensive research on algebra learning [see Kieran (1989) and Herscovics (1989) for good reviews of the research]. In fact, difficulty with algebra itself seems to be a frequent impediment to students' success in solving calculus problems. We identified two general types of cognitive obstacles — (1) those associated with natural and mathematical language, and (2) those involving intuitions about mathematical representations and specific mathematical concepts. The cognitive mechanisms driving these obstacles can be tied to Krutetskii's processes of flexibility, generalization, and reversibility.

Linguistic/representational factors. The language of mathematics is a very peculiar and sometimes abstruse language. Mathematical concepts often defy description in natural language so that students who do not understand well mathematical syntax, conventions and, even, mathematical idioms, may find it difficult to fully apprehend the concepts they are trying to understand. Students sometimes impose a natural language grammar on mathematical situations (a tendency to generalize inappropriately) or make semantic judgements based on a misinterpretation of syntax. From our data we were able to identify cognitive obstacles related to (1) concatenation, (2) the name-process dilemma, (3) unitizing, (4) operation transference, (5) generalized distributivity, and (6) natural language. The following example illustrates one of these obstacles.

*Example: Operation transference* is a form of generalization. It refers to the tendency to transfer rules related to an operation in one context to a completely different context. Sometimes the transfer is legitimate; sometimes not. For example, $A + B = B + A$ whether $A$ and $B$ are real numbers or real $n \times m$ matrices. But, while we know that $AB = BA$ for real numbers, this is not necessarily so for matrices. These type of generalizations have been identified among algebra students (Matz, 1979), but is prevalent among calculus students, as well. Students enter calculus with the exponent rule $(xy)^n = x^n y^n$ well in hand. We have noted that some students have a tendency to apply this rule inappropriately with derivatives, writing, for instance

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[2] Approximately 10400 errors were analyzed of which about 22% were not simply arithmetic or procedural.

$$(fg)' = f'g' \text{ and } (fg)'' = f''g''.$$

A similar problem, which was noted frequently among several students in a differential equations class, is exemplified in the following problem solution

$$\begin{aligned} y'' + 2y' = 0 \quad &\longrightarrow \quad y'(y' + 2) = 0 \\ &\Longrightarrow \quad y = c, \quad y = -2x + c. \end{aligned}$$

Intuitive factors. While attempting to understand mathematical concepts, students construct their own idiosyncratic meanings for these concepts. In many cases, these constructs are very nearly what we, as mathematicians, would want. However, in other instances, a concept that a student is trying to understand may not be so easily conceptualized. In such cases, the student may rely on her or his intuition about the situation in order to attach meaning to the concept in question. While this is a natural mechanism for dealing with phenomena, a mechanism that can be applied very productively, often students' intuitions about mathematical phenomena tend to result in misconceptualizations.

There is a growing body of research on students' understanding of calculus concepts (see Graham & Ferrini-Mundy (1989), for a good review of recent research). Many of these studies have pointed to deceptive intuitions about concepts such as function, limits, continuity, derivative, and integral — intuitions that can act as cognitive obstacles to the development of appropriate understanding. The students we examined also exhibited similarly flawed intuitions, as the following examples illustrate.

*Derivatives.* While students are fairly successful at performing certain typical calculus routines, such as differentiating polynomials, their intuitive notions of the derivative of a function seen not nearly so strong. In particular, some of our Calculus I students had difficulty reconciling the definition of a derivative of a function at a point with the slope of the tangent line at the point. As has been noted by others, many students have difficulty in seeing that the tangent line is, in fact, the limiting line for a sequence of secant lines (see, e.g., Dubinsky, 1989). Given the graph of an unspecified, but differentiable, function nearly all of the Calculus I students exhibited difficulty in constructing a graph of its derivative and double derivative. Most of these students' intuitive notions of derivative centered around symbolic differentiation (i.e., most had no graphical intuition), consequently this limited intuition proved to be a cognitive obstacle to a comprehensive understanding of derivative. (See Figure 1.)

*Integration.* Geometric intuitions that we consider to be cognitive obstacles to the understanding of integration include those related to notions of area and change of area. These appeared most often as students tried to understand the limiting sum definition of the definite integral. For example, one intuition exhibited by some students was that the more approximating rectangles one had, the greater the area — regardless of the particular function or whether lower or upper limits were being examined. A second intuition that is quite natural but interferes with the computation of integrals is the notion (or lack thereof) of "negative" areas. In certain situations, we noted that students also seemed to associate in a peculiar way the value of an integral (or derivative) with a value of the function. For example, several students indicated that the slope of the curve in Figure 1 was greater at point B than point A because the function value at B was greater. Similarly, another student indicated that the area under the curve — and hence the integral $\int_0^x \phi(t)\,dt$ — in Figure 2 decreased as $x$ increased because the "graph is getting smaller ... closer to the axis." [This phenomenon has been noted by Eisenberg (1990), as well.]

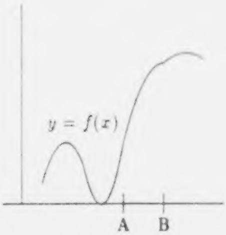

Figure 1. Which is greater — $f'(A)$ or $f'(B)$?

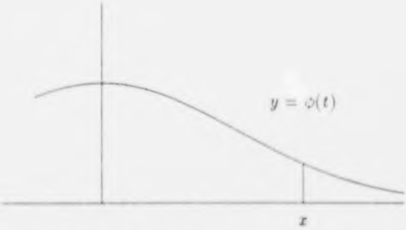

Figure 2. How is the area changing as $x$ moves to the right?

## Concluding Remarks

It should be made clear that the *particular* cognitive obstacles identified are very much tied to the current formulation of mathematical instruction in general and the calculus in particular. More broadly, as we mentioned early on, the factors influencing learning are extensive and varied. However, in no way should they be construed as representing a fixed and permanent set of influences. Instructional methods change; advances in technology open up new ways of representing mathematics; attitudes and beliefs about mathematics change; emphases are refocused. These sorts of influences will inevitably affect the nature of mathematics and mathematics learning.

We determined that cognitive obstacles to the learning of calculus arise in at least two different ways — one related to linguistic/representational aspects and the other to related to intuitions. Regardless of the evolution of mathematics and mathematics learning, it is difficult to imagine that these two aspects, whatever their future formulation, would not continue to be primary influences on the genesis of cognitive obstacles.

Consider, for a moment, the current conditions as an exemplar of linguistic/representational aspects and intuitions as sources for cognitive obstacles to the learning of the calculus. Given that so many of our algebra and calculus courses are immersed in symbolic manipulation, often at the expense of understanding, it is not surprising that linguistic/representational factors give rise to cognitive obstacles. Also, since learners basically *want* to understand — or at least make sense of — what they are being asked to learn, the intuitions that students bring to bear on the concepts of calculus often become obstacles to the appropriate construction of those concepts.

We found that a useful framework in which to embed considerations of cognitive obstacles lies in the cognitive processes of reversibility, flexibility, and generalization. We expect that this framework is potentially useful in other areas as well. Efforts to reform the mathematics curricula — especially in critical areas such as the calculus — must be based on an understanding of the nature of students' construction of mathematical knowledge and the variety of factors influencing that construction. This is essential knowledge for mathematics educators. Perhaps the Krutetskiian framework described here and the discussion of cognitive obstacles will prove useful to those mathematicians and educators who wish to play a role in the reformulation of mathematics teaching and — more importantly — learning.

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# LOGO MATHEMATICS AND BOXER MATHEMATICS: SOME PRELIMINARY COMPARISONS

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We describe the results of some preliminary research with *Boxer*, a new computational medium with some claims to being a successor to Logo. Issues concerning the pupil-computer interface are discussed, and we compare Logo- and Boxer-mathematical activities both generally and within a set of tasks centred around the notion of a parallelogram.

For the past two years we have been exploring the potential of *Boxer* for mathematical learning. Boxer is a new[1] computational environment with some claim to becoming a successor to Logo. Its intellectual and technical antecedents are rooted in the Logo tradition: it was designed by Andrea diSessa and Hal Abelson, originally at MIT and is now nearing completion under the direction of the former at the University of California, Berkeley. All interaction with Boxer is via a text editor and text can be entered, copied, deleted, cut and pasted in the normal way, but with one crucial difference. All text is contained in boxes, and boxes may contain other boxes. Boxes are not a screen effect — they are not, for example, 'windows'. The editor 'knows' about boxes in the way that a conventional editor 'knows' about alphanumeric characters. In addition, boxes come in three sizes: they can be closed and very small (all the user sees is the name of the box, if any, and a shaded rectangle). Or they can be open, in which case the box will be either exactly the right size for whatever is in it (including, of course, blank spaces or other boxes): or it will take up the whole screen (see figure 1). Among *types* of boxes are *doit* boxes, which contain programs and *data* boxes, which contain information of any sort — including graphics. Graphics can be produced by a Logo-style turtle (see figure 1) or pasted in by a scanner.

Any piece of text in Boxer can be accessed or executed and recombined with any other piece. Perhaps it is most helpful to think of Boxer as a system which combines text processing, data manipulation and dynamic graphics into a single framework glued together by programming. Like Logo, the programming language which lies at the core of Boxer is LISP. Unlike Logo or LISP, Boxer makes use of advanced graphics capabilities (consider how even 'friendly' Logo error messages still scroll up from the bottom of the screen — Logo was designed on teletype machines and it shows), and equally important, is a more general-purpose environment than merely a programming language. Boxer also differs from conventional languages like LISP or Logo in that it is built around a consistent *spatial metaphor* — that of the box — in which the meaning of text or boxes on the screen adheres in its position relative to other boxes. Thus programs are relatively devoid of the textual clutter which is usually necessary to elaborate, say, program structure: in Boxer, a subprocedure of a procedure is simply a box within a box (see figure 1). Thus Boxer is a viable candidate for overcoming some of the syntactic and semantic difficulties experienced by novices with traditional text-based systems (see, for example, Hoyles & Sutherland 1989) whilst maintaining the mathematical power derived from a formal language.

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[1] At least it is new in the sense that Logo was new in 1982: it has been under development for about ten years, and has only just recently been implemented on a machine which is in any way affordable. On the other hand, like Logo before it, it will be a little while before we might realistically think of schools in any general sense having access to it.

In this paper we report some research studies of children using Boxer in a mathematical setting — that is of Boxer mathematics. Assisted by comparisons with a decade of research findings concerning Logo mathematics, we investigate the children's appropriation of the medium and how it structures and is structured by their developing mathematical understandings. In so doing, we touch on some questions of HCI, and it is to these that we now briefly divert attention.

## AN ASIDE ON THE PUPIL-COMPUTER INTERFACE[2]

The literature on *visual programming* and *direct manipulation* (Hutchins et al. 1986; Shneiderman, 1987; Feurzeig 1991) has offered new approaches to the design of computational environments. Neither are unproblematic: Shneiderman, for example, suggests that graphical representations may sometimes mislead and confuse users. In domains such as mathematics, we suggest there is a further problem with direct manipulation from the point of view of the learner, that is, that the entities being manipulated are *not* depictions of physical objects but of conceptual entities such as functions. A possible reason for the disagreement concerning the value of graphical environments (including direct manipulation) is that there are two types of 'directness' (Hutchins, Hollan & Norman, 1985): *articulatory directness* refers to closeness of relationship between entities in the domain and their embodiment on the screen; *semantic directness* refers to closeness of relationship between a user's intentions and their expression in the interface language (facilitating the externalisation and subsequent manipulation of concepts). Optimising both types of directness is crucial: a learner will gain little from interacting with objects on the screen if either their interaction does not evoke the mobilisation of *mathematical* knowledge or the objects do not express his or her understanding of mathematics. As we see it, the challenge is therefore to build *evocative computational objects* (ECOs) — objects which *matter within the relevant knowledge domain* and which *matter to the learner*.

Direct manipulation and visual programming are often used as embellishments to the underlying 'real' language[3]. The graphical and direct manipulation aspects of Boxer, on the other hand, are *not* simply interface embellishments, they are part of the system, justifiably described as 'real'[4]; there is no need for add-on 'interface' (like menus or icons). At the same time, the conventional power of text-based programs (and simply of text itself) is preserved. Our belief, developed over a decade working with Logo, is that for mathematics education, the formality and precision involved in working with symbol systems is something to exploit rather than sidestep (diSessa, 1990; Abelson 1987, Hoyles & Noss, 1992a). This potential may now begin to be more adequately realised with the development of new computational environments such as Boxer, which maintain the structural integrity of traditional programming languages without many of the associated difficulties[5]. At the same time, we now have far more examples and a stronger theoretical base on which to build effective pedagogies within computer-based microworlds (see Hoyles and Noss, 1992b).

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[2] We would like to acknowledge the contribution to this section of Dr Mike Sharples, School of Cognitive Science, University of Sussex.

[3] Via help balloons, pull-down menus etc. A recent exception to this is *Function Machines*, developed by Wallace Feurzeig and his colleagues at Bolt, Beranek and Newman (Cambridge: US)

[4] diSessa describes the underlying model of Boxer as 'naive realism'.

[5] Such as the intrusion of textual clutter, the problem of helping the novice to conceptualise flow of data and control and so on.

In summary, the premise underlying the research reported here is that in order to actively interact with a complex domain such as mathematics, learners need to develop an appropriate notation and conceptual framework that will allow them to construct and reconstruct ECOs which create possibilities for mathematical expression and communication.

## THE STUDIES

The empirical core of this paper is in two parts. First, we present some snapshots of children's ongoing Boxer work in their classrooms over a period of a year[6], which seem to us to encapsulate some of the key issues arising from their activities. Second, we report results of some exploratory research based around a set of activities in Boxer concerning parallelograms — a Boxer version of a study which we have previously undertaken using Logo (see Noss & Hoyles, 1987; Hoyles & Noss, 1988).

### Some snapshots of Boxer mathematics

*Boxes invite opening and closing.* One of the least well-realised hopes of the Logo mathematics community, has been the propensity of children to gain access to and explore the programs which underlie Logo microworlds. It is true that with some pushing and shoving, children can be provoked into trying to read, understand and modify Logo programs but it is seldom a spontaneous or highly-motivated activity. We suggest that the situation is rather different in the Boxer setting.

Opening and closing a box in Boxer is simply a matter of clicking on it. In particular, programs (simply text that executes when invited to do so) can be opened (i.e. be perused) or closed (in which the detail of what is in them is suppressed). This ability of Boxer to control the visibility of program abstraction (i.e. by making all or any part of a program visible or invisible), seems to offer a powerful tool to the mathematics educator. We have collected many examples where this inherent *visibility* of boxes encouraged children to make sense of the underlying processes modelled by the tools which we (or other children or the teacher) had left lying around within the current Boxer world. Two simple examples which both arose from a noncomputer project on weather were a program to convert fahrenheit temperatures to centrigrade, and a block-graph drawing program to facilitate the display of the children's meteorological data.

*Opening new strategic apertures to mathematical expression.* In Hoyles and Noss 1992b, we discussed how computer environments can offer children access to approaches and solutions which are simply unavailable with paper and pencil — they open *strategic apertures*. We start with two examples from Boxer mathematics. First, values of variables can be assigned by directly inserting the value into a box with the variable's name rather than using an input to a procedure. This provides a powerful metaphor for understanding how variables are bound and one which is more 'concrete' and controllable than the Logo version: we found that children preferred to use this mode of interaction in order to 'keep track' of what was varying and how. A second example requires more explanation. One of the teachers with whom we collaborate announced that she wanted to work with *symmetry*, and wondered whether it would be possible to think of a Boxer world which would capture some of its essential features. After some thought, an idea took shape: define an errant sprite (called Celia) in which right and left were switched; so, whenever this sprite was invoked, it would draw the reflection of a well-behaved sprite (Richard). So far so good: clearly such an idea could be implemented in any programming language. We set about implementing it in Boxer, and only then realised just how *natural* such a microworld would be to define: basically, all we had to do was to define Celia's behaviour as an object (all sprites — indeed all boxes — are objects). Of course, Celia automatically inherits all of the properties which belong to his parents — in this case, definitions of, say, FD and BK. So what did we gain? First, we have here a nice instantiation of some of the fundamental ideas of object-oriented programming. Second, we have an immediately evident and natural way of representing the misbehaviour of our chosen sprite: the redefinition of RT and LT are written *where it is natural to do so*, and (almost) without any syntactic fuss over the trickiness of having two definitions of FD within one screen. Third, these two factors together combined to make it natural to generalize the idea — more importantly for the teachers, and ultimately the children to do so — to a *naughty sprites* microworld, in which FD is BK, RT is LT and so on (see Figure 2).

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[6] We collaborate with a Local Education Authority, which has provided two systems currently installed in a primary school, with children (aged 9-11 years).

*Generating novel modes of pedagogical intervention.* Over the last decade, we have developed an analysis of the variety of ways of intervening pedagogically in children's Logo work (see, for example, Hoyles & Noss, 1992b). Yet early in our Boxer work, we began naturally to develop a style of interaction that had *simply never occured to us* in Logo settings: we began leaving around the screen boxes in the form of messages for children, containing a range of information (programs, text, challenges, 'help' etc) — an example is given in figure 3. Why did this come about? In the first place, boxes can contain a mixture of contents, and therefore are easily personalized to the learner(s) in question; this is very difficult in Logo, where objects are essentially programs only[7]. Thus *boxes are like personal presents* — they invite the learner to open them, to scan them for information, or to use whatever programs they contain. Second, *boxes are not intrusive*. Programs and data within a box are evident (they simply exist on the screen), but because boxes are partially-closed universes, they do *not* intrude into any existing activities in which children are engaged. There is no danger, for instance, that some variable in a newly-created box will conflict with one already existing within a child's project. Third, *boxes can be ignored*. When a box is closed, it simply takes up a little screen space, nothing more. It does not clutter the 'workspace' and it does not appear, say, in the 'editor' whenever a program is being modified: children can choose whether or not to exploit a box's presence.

*From pragmatic to conceptual knowledge.* From our observations to date we suggest that the Boxer setting facilitates the growth of pragmatic or tacit mathematical knowledge into conceptual understanding. We have outlined elsewhere (Hoyles, 1986) our claim that a primary focus of interest within computational environments lies in the way that mathematical ideas and concepts can be used (i.e. instantiated in the form of programs and run) before they are understood, and that the former provides a means for the latter with the assistance of scaffolding provided by the medium. In fact, we have gathered several examples of this with Boxer: children's tacit knowledge of programming/mathematical ideas (such as scoping of variables) has become explicit via interaction with Boxer and we conjecture that this derives from the increased *malleability* and *visibility* of Boxer programs.

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[7] This is not quite true in LogoWriter, although it is more difficult to create and distribute interesting nuggets.

*Personal appropriation.* In our work with Logo, we have seen many instances of children personalizing their relationship with the machine, the most obvious example being through imaginative naming of procedures. In Boxer, we found many extensions of this phenomenon, with children developing single key-presses to load pieces of program, and writing programs which contained 'tricks' (such as boxes within boxes within boxes) to introduce an element of fun into their activities. What is important here is that such tricks were not optional extras, derived from 'features' of the environment known only to dedicated hackers: they were derived from the design principles on which Boxer is based, and thus offer a potential route of access to the computational power of the medium.

*Building procedures.* The Logo mathematics literature is replete with examples of children having difficulties with the notion of procedure and preferring to work in direct mode. Our observations of children working with Boxer seems to indicate a narrowing of this 'gap' between direct interaction and procedural abstraction. Within a turtle-geometric context, the fact that Boxer is an editor-toplevel system seems to encourage a deeper and more ready involvement with procedure than we have observed in Logo settings since programs can be tried out a piece at a time or *restrospectively* organised into boxes from fragments of direct-drive commands.

*Using ports.* Boxer offers access to *hypertext* through a simple extension of varieties of boxes to include 'ports': views of a box which can be made accessible to another box. Our observations suggest that, precisely because 'ports' are simply another view of a given box, they add considerable expressive power to the medium. The most obvious example of the utility of ports (we give a second in the following section), is in the context of debugging — error messages appear as boxes in which there is a port to the error itself. Errors can be debugged *in the port* (the question of 'where' the error actually is which plagues Logo debugging is thus effectively eliminated). A fruitful line of research would be to investigate if this makes interpreting error messages easier (some are still somewhat arcane in Boxer) or whether the global process of debugging programs is simplified.

**Paralellograms revisited**

There is an old maxim which declares that the first instinct of any educator when confronted with a new technology is to reinvent old educational ideas using the new technology. Educational software in the domain of mathematics (and elsewhere) is proof enough of this assertion. Our excuse, in the case of the study which we outline below, is that we were not inventing educational ideas in the sense of attempting to provide any pedagogical model for Boxer activities. Rather, we wanted to undertake a detailed investigation of children's understandings and activities involving a fragment of a conceptual field. To do so, we needed some baseline against which to interpret children's behaviours, and since our original study with Logo yielded rather detailed insights, we decided to repeat it in a Boxer context.

In our previous studies[8], we set out to provide a small number of children with a parallelogram microworld consisting of a structured set of tasks, some of which were to be attempted off the computer and some on the computer. Our hope was that these activies would enable us to gain insight into the development of the children's understanding of the essence of a parallelogram, and in particular how the computer served as a catalyst in this development — by affecting both the way the concept was represented and the range of methods of task solution available. Broadly, the relevant findings of the studies were as follows:

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[8] Reported to PME in Hoyles & Noss 1987, 1988 and elaborated in Noss & Hoyles 1987.

1. Using a Logo procedure generates a context for implicit action-based generalisations, although explicit awareness of internal relationships internal to a procedure tend not to arise spontaneously.
2. Discrimination of the components of a procedure develops from partial discriminations including:
   i. discrimination of the features of the figure without regard to its available symbolic representation(s);
   ii. discrimination within the symbolic representation; that is perceiving its structure and pattern without regard to the visual outcome.
3. The symbolic representation of a program can act as a form of scaffolding which allows the child to map out a solution in global form, and subsequently pay attention to local details.
4. Mismatches between pupils' fuzzy and intuitive ideas of a parallelogram and their formalised defintion were at least partially resolved.
5. The pupils clarified the set/subset relationships amongst parallelograms.

Additionally in the Logo context, two key questions emerged concerning the extent to which children could:

i. synthesise the symbolic descriptions in terms of programs (or fragments of programs) with the geometric image on paper or on the screen, and
ii. use the computer as scaffolding for the construction of generalisations.

*The Boxer version of the study*. The children, aged 11, had just left their primary school where they had undertaken over 100 hours of Boxer. We worked with three children: our intention was to compare the environments rather than the children, and our focus of interest was much more on what they *did* with the activities rather than on how they compared in their understandings.

We are unable to describe the activities in detail here. The main divergence between the Boxer and Logo versions (see figure 4 for some fragments of the Boxer activities) was that all the prepared tasks and the children's responses were *on-screen* — questions were given in boxes and answers constructed in boxes. Thus, whereas in the Logo setting, children's responses were partly a function of the different medium in which they were posed (i.e. pencil and paper, or Logo), in the Boxer setting the screen remained a focus of attention throughout changing the nature of the interaction between the children. Our videotape showed that a considerable amount of discussion was based around indexically referenced discourse (pointing at boxes etc).

*Some outline findings*. Not surprisingly, we found that children's prior intuitions about parallelograms were much the same in the two settings (e.g. a parallelogram is a lopsided rectangle: a rectangle is *not* a parallelogram). But in the Boxer version, we saw the children more able to use their visual representations as objects to think with and more likely to debug their intutions by connecting the symbolic representations with these graphical representations. This may be because graphical and textual objects are treated in a uniform way. We also observed a more experimental attitude with, for example, the numerical values of variables: it is tempting to attribute this to the direct manipulation aspects of Boxer's design as described earlier, but it may, of course, be a result of these particular children's Boxer experiences.

In the Boxer version, the children saw SHAPE — the central parallelogram they worked on and reconstructed — as an object, perhaps even *their* object. This was all the more so because of the way in which the Boxer system allowed us to use ports. We wanted the children to build up *their own* versions of procedures to use in subsequent activities. In our Logo version, we had no alternative but to say things like 'Take your version of parallelogram and do such-and-such'. In the Boxer version, we were able to simply make a port to a previously-made program where it was available for inspection and execution. Thus the children were more likely, it seems, to see the various graphical representations of a parallelogram (including squares, rectangles) as the *same* object transformed in different ways (flipped, rotated etc.), rather than *different* objects presented (by us) on pieces of paper. We suggest that there are interesting avenues to pursue here, along the general theme of the ways in which the objects and relationships on a computer screen structures pupils' perceptions and interactions.

The final point we want to make concerns the way in which the children used the computational environment as *scaffolding* for their emerging intuitions and understandings. We observed that the tendency of children in Logo settings to create pieces of program as objects on which to scaffold their emerging understandings was amplified in the Boxer setting by an exploration of multiple fragments of programs and data. These reified not only the current but the past states of pupils' thinking, making them available for inspection and reflection.

## CONCLUSION

The potential of Boxer is only just beginning to be researched — in general and in the context of mathematics. Nevertheless, our exploratory findings to date indicate some clear avenues to explore and suggest that the question of the pupil-computer interface is no longer one which mathematics educators can afford to ignore. We are provisionally persuaded by diSessa's argument that systems like Boxer — perhaps Boxer itself — might provide a new *medium* which, like reading and writing, may become integrated into our culture and repay considerable time invested in learning it; in doing so, it might provide a means for making accessible a *mathematical culture* which for so long has been available only to the selected few.

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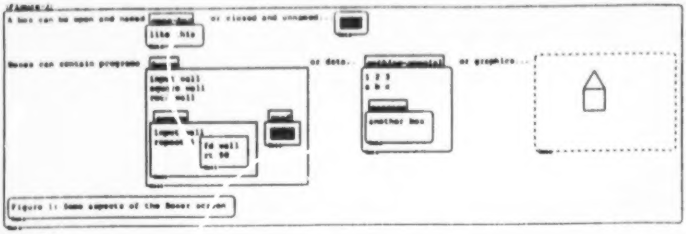
Figure-1
like this
1 2 3
a b c
another box

A box can be open and named ... or closed and unnamed.

Boxes can contain programs ... or data... or graphics...

```
input wall
square wall
rect wall
input wall
repeat 4
fd wall
rt 90
```

Figure 1: Some aspects of the Boxer screen

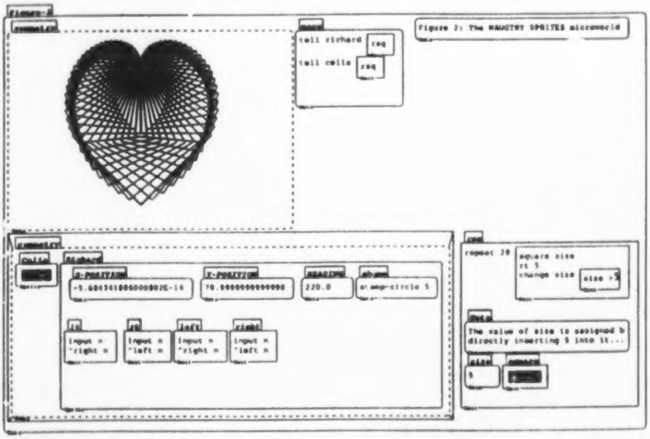
Figure-2
symmetry
Celia
Richard
X-POSITION
Y-POSITION
HEADING
shape
fd
bk
left
right
size

Figure 2: The NAUGHTY SPRITES microworld

```
tell richard req
tell celia req
```

```
-5.6843418860808E-14
70.000000000000
220.0
stamp-circle 5
```

```
input n
right n
```

```
input n
left n
```

```
input n
right n
```

```
input n
left n
```

```
repeat 20
square size
rt 5
change size size +5
```

The value of size is assigned b directly inserting 5 into it...

```
5
```

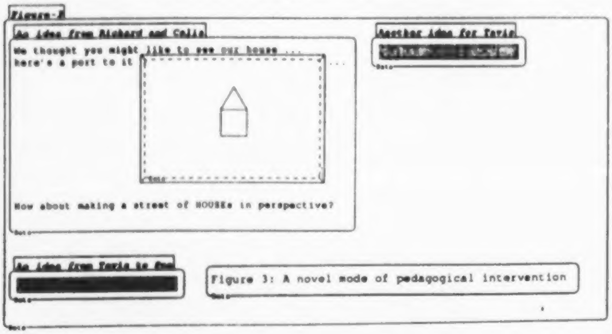
Figure-3
An idea from Richard and Celia
Another idea for Tevie
An idea from Tevie to ...

We thought you might like to see our house ...
here's a port to it ...

How about making a street of HOUSEs in perspective?

Figure 3: A novel mode of pedagogical intervention

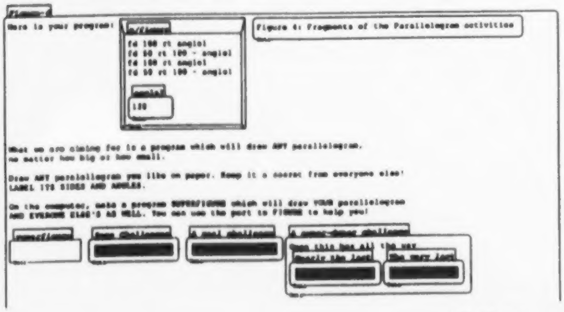
Figure-4
superfigure
Some challenges
A real challenge
A super-duper challenge
Nearly the last
The very last

Here is your program:

```
fd 100 rt angle1
fd 50 rt 180 - angle1
fd 100 rt angle1
fd 50 rt 180 - angle1
```

angle1: 130

Figure 4: Fragments of the Parallelogram activities

What we are aiming for is a program which will draw ANY parallelogram,
no matter how big or how small.

Draw ANY parallelogram you like on paper. Keep it a secret from everyone else!
LABEL ITS SIDES AND ANGLES.

On the computer, make a program SUPERFIGURE which will draw YOUR parallelogram
AND EVERYONE ELSE'S AS WELL. You can use the port to FIGURE to help you!

Open this box all the way

BEST COPY AVAILABLE

# REPRESENTATION OF AREA: A PICTORIAL PERSPECTIVE

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*A sample of 66 children in Years 1 to 5 were interviewed individually as they worked through a sequence of counting, drawing and measuring tasks involving covering rectangular figures. While most of the children had no difficulty covering a rectangular shape with tiles and finding the area by counting individual tiles, there were wide discrepancies in the children's ability to represent the results of such actions in a drawing, for tasks ranging from drawing the tiles in front of them, to using measurement to construct a tiling of unit squares in a given rectangle. It was evident from these results that young children do not automatically interpret arrays of squares in terms of their rows and columns, and that this could hinder their learning about area measurement using diagrams.*

## INTRODUCTION

A number of research studies have indicated that children often do not grasp the relationship between different forms of representation of mathematical ideas (Hart, 1987; Dufour-Janvier, Bednarz and Belanger, 1987). For example, children may be able to use the formula to find the area of a rectangle without knowing the number of unit squares that would be needed to cover it. As a result, concrete materials have been widely recommended to provide the basis on which to build abstract concepts. Hart and Sinkinson (1988), however, show that many children experience difficulty linking concrete activities such as covering rectangular shapes with the mathematical formalisation for area measurement that is the synthesis of the activities. They propose bridging procedures such as diagrammatic representation, to overcome this problem, but Mitchelmore (1984) suggests that the step from representing a problem using concrete materials to representing it pictorially may in itself be difficult.

The aim of this project was to investigate young children's difficulties in making the transition from concrete to pictorial representations of area measurement concepts for rectangular figures.

## METHODOLOGY

### The sample

The sample consisted of 66 children from three schools. Two of the schools were located in a medium to high socio-economic areas; in the first of these 28 children were randomly selected (10 from Year 1 and 6 from each of Years 2, 3 and 4), in the second, twenty children (7 from each of Years 2 and 3, and 6 from Year 4). The third was a research school for children with learning difficulties, primarily in reading. These children attend the school for one year to receive intensive individualised instruction and this group of 18 children (4 Year 2, 4 Year 3, 4 Year 4 children and 7 Year 5) represented a variety of schools and socio-economic backgrounds.

The children were interviewed individually as they worked through a sequence of tasks designed to tap the various skills involved in representing the covering of rectangular areas with squares (including counting, covering and drawing).

### Rationale for task selection

The tasks were devised on the assumption that they tapped skills involved in the procedure for finding the area of a rectangle pictorially, which is later summarised in the usual formula. These skills were considered to be:

a) To construct an array of the correct dimensions by first finding the number of unit squares along each side and then representing the array; and

b) To calculate the number of unit squares using multiplication.

It will be argued that the critical step is to see the array of unit squares as groups of rows or columns. Without this understanding, neither the significance of the lengths of the sides of the rectangle nor the multiplication principle can be grasped. Now finding the area of a rectangle using concrete materials does not require any understanding of the row-column structure, since the array is determined by the materials and not by the child's thinking; so using concrete materials alone is unlikely to lead to the area formula. By contrast, drawing tasks can both indicate children's understanding of the array structure and help them learn it.

### Description of the tasks

Several types of counting tasks were included to determine the type of strategy the child primarily used:

(1) Counting a 5 x 6 array of dots. Dickson (1989) felt this task was a good indicator of the counting strategy a child used.

(2) Counting the number of squares covered by (a) a 3 cm x 6 cm, and (b) a 4 cm x 6 cm rectangular shape superimposed on centimetre grid paper.

The following tasks only involved (freehand) drawing:

(3) Making a 3 x 4 rectangular array made of 4 cm square cardboard tiles and then drawing it.

(4) Drawing a rectangular array when no measurement skills are required, including completing a rectangular array of [one centimetre] squares and drawing the array when side marks are given.

The next set of tasks required measurement and drawing (or visualisation) skills:

(5) (a) using a concrete unit (a 2 cm$^2$ cardboard tile) to work out the number of tiles needed to cover a given rectangle,

(b) working out how many [2 cm$^2$] tiles would be needed to cover a rectangle shown, given one such square drawn in the top left-hand corner.

(6) Measuring a 10 cm line with a ruler, then using the ruler to work out how many one centimetre square tiles would be needed to cover a rectangle.

## RESULTS

To successfully complete all the tasks, the children had to link numerical, spatial and measurement skills. The responses in the three aspects of the tasks were categorised into levels and the children were assigned to a level on each aspect of the task. Since there were a number of drawing tasks, and some children improved on tasks as the interview progressed, a child's drawing level was taken to be the mode of all the drawing tasks s/he had attempted. The counting level was based on strategies used for tasks (1) and (2), the measurement level on tasks (5) and (6).

### The counting tasks (1) and (2)

These tasks gave a indication of a child's approach to counting an array. (In general children used the same types of strategies when counting the tiles they had drawn.) The approaches that were evident, listed in a supposedly developmental order, are shown below. The percentage of the sample at each level is given in brackets.

1. Counting by ones, using either systematic or unsystematic strategies (50%).
2. Counting by groups of either rows or columns (38%).
3. Calculation by array multiplication (12%).

Some children were included in level 2 on the basis of only one application of repeated addition (counting by groups of 5). The decision to include them in level 2 was made on the assumption that their inability to count by groups in other contexts was a consequence of their inability to count by the groups (other than 5) represented in the array structures, rather than to group by rows or columns.

### The drawing tasks (3) and (4)

While most of the children had no difficulty covering a rectangular shape with tiles and finding the area by counting individual tiles, there were wide discrepancies in the children's abilities to represent the results of their actions in a drawing.

None of the children had difficulty in covering a rectangular shape with cardboard tiles and counting the number of tiles they had used. The most surprising result, however, was that 30% of the children in the sample could not accurately copy the array that they had made and that was still in front of them; these the children drew an array with far more tiles. This problem was particularly apparent in the sample of children with learning problems, as 61% of these children could not do this task. Because of the difficulty of this task, an additional task was sometimes included, that of making an array from a pictorial model. None of the children to whom this task was given had any difficulty making the array.

Overall, tasks (3) and (4) indicated two critical aspects of children's skills in drawing an array; first, perception of equal rows (or columns) and second, use of parallel lines to draw the array quickly. The methods of drawing arrays that were evident in the drawings are listed below (in a supposedly developmental order). The percentage of the sample at each level is given in brackets.

1. Individual tiles only: no alignment of rows or columns or alignment in one direction only (14%).
2. Individual tiles only: reasonably well aligned (14%).
3. A partially structured array (i.e. some tiles drawn individually, some drawn using parallel lines (23%).
4. A structured array drawn as two (perpendicular) sets of parallel lines or visualised from side cues (50%).

About a quarter of the children in the sample seemed to be in a transition between drawing individual squares and drawing lines; their drawing strategies were not consistent across the tasks and appeared to be affected by different pictorial cues. A drawing problem common to the children who drew individual tiles was a loss in horizontal alignment because the squares tended to decrease in size from left to right, caused by a reverse-L method of drawing individual squares. This method resulted in lines that drifted upwards rather than remaining parallel to the top row of squares. The children were then tempted to add extra lines to try and equalise the sizes of the squares, and the array structure was lost. Many of these drawings were also inaccurate, so that the depicted units differed greatly in size.

**The measuring tasks (5) and (6)**

Both these tasks involved determining the number of squares along each side of a rectangle, when not explicitly shown. The measurement levels that seemed to be demonstrated in the tasks, listed in a supposedly developmental order, were as follows (the percentage of the sample at each level is given in brackets.)

1. Does not measure the line or the rectangle (32%).
2. Measures the line (either accurately or inaccurately) but not the rectangle (30%).
3. Measures one side, adjacent sides or internal space of the rectangle(s), either by repeating one unit or using the scale (38%).

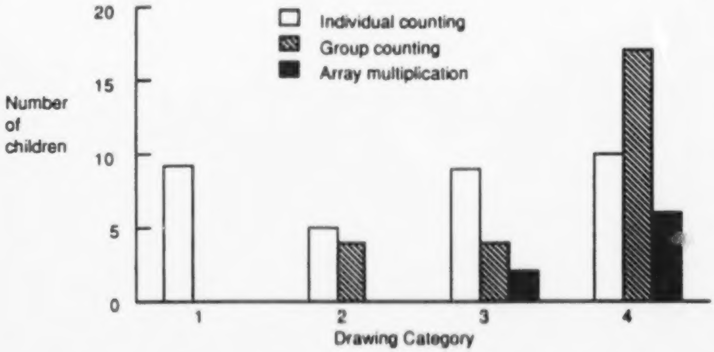
Working out how many tiles would fit on a rectangle, given one tile, was usually done by repeatedly tracing around the tile or by moving it and keeping count without drawing. Few children used the tile to mark the edges of the rectangle to determine the number of tiles along each edge.

Many children did not use a ruler correctly when measuring the line, with the most common error being to start at one and measure the line as eleven centimetres. It was noticeable that many of the children who primarily counted individually, did not use the scale on the ruler but repeatedly measured one unit. This was done by making a mark, sliding the ruler one centimetre along the line, and then repeating the procedure.

### Linking the numerical, spatial and measurement levels

The number of children at each of the possible combinations of levels for counting and drawing, drawing and measuring, and counting and measuring are shown in Figures 1 to 3.

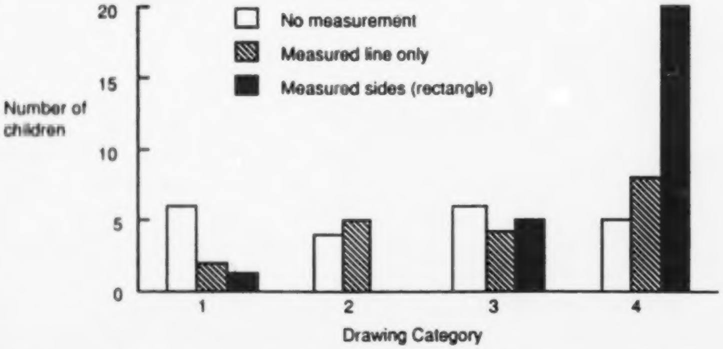
**Figure 1 Children's performance on the counting and drawing aspects of the tasks.**



The results in Figure 1 confirm that (in general), unless children can represent the structure of an array in terms of rows and columns, they are unlikely to be able to determine the number of elements in the array by multiplication or group counting. None of the children who primarily drew the array as individual tiles used array multiplication whereas the majority of the children who drew structured arrays determined the number of tiles by group counting or multiplication.

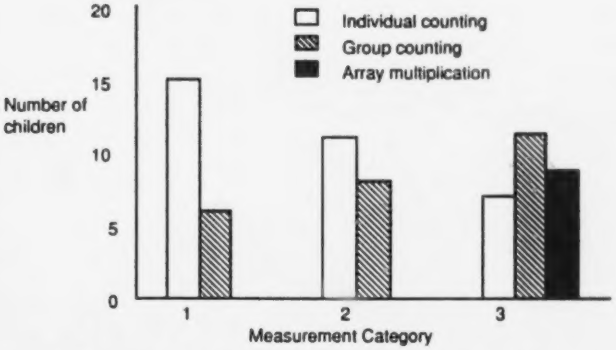
The measurement levels (see Figure 2) also seemed to be strongly related to drawing. Only the children who drew structured arrays at some stage in the interview measured the sides of the rectangle. Some children, however, who could measure the line and draw a structured array did not apply their measurement knowledge to determine the number of tiles that would be needed along the side of a rectangle. One child in Year 5 used a ruler in an attempt to measure the number of tiles which would cover a rectangle by measuring 1 cm across and then 1 cm down but was unsuccessful on the task because of his inability to draw an array.

**Figure 2** **Children's levels of performance on the measuring and drawing aspects of the tasks.**



It is evident in Figure 3 that there is a relation between counting and measuring on these tasks. Both operations seemed to be two-dimensional; 29% of the children measured in two dimensions and determined the number of tiles by grouping rows or columns or by multiplication; 39% primarily counted the tiles individually and did not measure either side of the rectangle to find how many tiles would fit.

**Figure 3** **Children's levels of performance on the counting and measuring aspects of the tasks.**



What is surprising is the number of children who could measure a line but who did not apply this skill to work out how many tiles were needed, either by using the scale or by repeating a unit. This result may reflect lack of experience in marking off units of a given length, or in view of the strong relationship between measuring and drawing, it may indicate that the children were not confident of the row-column structure of the arrays.

For the children without learning problems, both measurement and counting levels based on the tasks were strongly age related, but this was not the case for the other group of children; many of the latter drew arrays and measured the line correctly but still counted arrays individually, and in many cases inaccurately.

## DISCUSSION

The results of this study indicate the difficulty many children have in representing pictorially their concrete actions in constructing arrays. The first drawing task illustrates the aspects that make the drawing problem so much more difficult than the concrete task for children. To make an array from a picture seemed a trivial task compared with its inverse, presumably because drawing the array requires the child to analyse the array structure in terms of the number of rows and columns. The skill of representing an array of squares using two perpendicular sets of parallel lines is also more difficult than might be expected, indicating that the structure of a square tessellation is not obvious to children but must be learned. The measurement requirements of the pictorial task imposed additional difficulties compared with the concrete task; most of the children drew the boundary of the array first, then had to work out a method of subdividing each length into the requisite number of parts.

Although some drawing techniques must be developed, the most important skill in representing an array seems to be based on an understanding of a fundamental property of rectangular arrays in general: the elements of an array are collinear in two directions. It is this property which allows the partitioning of the array into rows and columns. One might think that the third drawing level, (the transition from individual tiles to a structured array) would be adequate for area measurement, but there were many students at this level, and still some at level 4 who could not apply their multiplication and linear measurement skills to determine the area of a rectangle.

The skill of determining the number of squares along each side of a rectangle when not explicitly given, was only successfully shown by children who multiplied dimensions or grouped by rows or columns and either drew structured or partially structured arrays. A few children who could draw structured arrays were still at the individual counting level and their progress was limited by poor numerical skills as they were apparently unable to count by groups. Other children counted inaccurately and in general these children did not use the row-column structure of arrays to systematise their counting. Some children, in general the older children with learning problems, successfully completed the tasks although they could not count the number of squares in an array using grouping by rows or columns.

It is evident from these results that young children do not automatically interpret rectangular arrays of squares in terms of rows and columns, and that this could hinder their learning about area measurement using diagrams. It would seem that linking counting by groups to the structure of rectangular arrays could be a powerful technique to develop both multiplication and area concepts. Activities involving drawing, measuring and counting arrays of squares or other objects are likely to develop children's perception of the fundamental collinearity property, and its relationship to measurement. Such activities would seem to be essential prerequisites to linking activities with concrete materials to the [evolution of the] formula for finding the area of a rectangle.

The results of this study may be of limited generality because of the sample is not representative of the school population. The more consistent age progression in the two regular schools would suggest that these skills develop together for the children who are successful at mathematics; however, this did not seem to be so for the children who are experiencing learning difficulties.

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# SPATIAL THINKING TAKES SHAPE THROUGH PRIMARY-SCHOOL EXPERIENCES

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*The study used a matched-group experimental design with 190 Year 2 and 4 students from three primary schools. Two groups were involved in a range of 2D tasks under different classroom organisations while the third group completed number puzzles. The performances of students on a range of 2D and 3D tasks were measured and results indicated that experience with two-dimensional mathematico-spatial activities has a positive effect on performance on the 2D tasks while gender differences were found with retention test scores on a three-dimensional task. Other findings included no differences between two types of classroom organisation except for Year 4 females working in cooperative groups for whom results were the opposite to the overall gender difference in three-dimensional thinking after practice. An analysis of video-recorded data indicated that the nature of students' learning varies with the type of student/student and student/teacher interactions and visual imagery was found to be an important factor in learning.*

## Problem Solving and Spatial Thinking Processes

Students are active participants in their own knowledge development and, for this reason, classroom learning experiences can facilitate the construction of mathematico-spatial concepts and skills. As Osborne and Wittrock (1983) have suggested, when students are confronted with a problem they attempt to make sense of the problem situation by locating and linking more familiar knowledge, concepts, skills, and images with the present problem situation. Visual imagery plays an important role in student's thinking at this stage (Goldin, 1987; Piaget & Inhelder, 1971). When students construct initial cognitive representations of a problem, they are then better placed to develop tentative problem-solving strategies.

Visuo-spatial concepts, images, and skills develop when one mentally manipulates visual images or uses other aspects of visual thinking that are basically analytical to puzzle over problem situations. Problems may require the use of both verbal-analytic and visual modes of thinking (Eliot, 1987; Suwarsono, 1982). Furthermore, visuo-analysis occurs, for example, when one attempts "embedded figures" tasks and other tasks that were formally considered as holistic rotation or alternate orientation tasks (see explanations by Carpenter and Just, 1986).

Concepts that embody operations such as "joining these two (congruent isosceles right) triangles forms a square" are regarded as spatial thinking processes in this paper along with visual imagery used to represent, recognise, or reproduce a shape or position. Imagery varies (Presmeg, 1986) from pictorial images to those that are dynamic, pattern-based, event-related, or conceptualised. Less emphasis was placed in this study on the recognition of a shape by using definitional concepts such as "a triangle has three sides".

## Classroom Learning Experiences and Spatial Thinking Processes

The teacher's task, then, is one of planning problem situations that are likely to encourage students to use thinking processes and sensory experiences that will assist the linking of perception with prior learning experiences, and will thereby facilitate quality problem representations and the development of appropriate problem-solving strategies. Hence the choice of tasks, their use, the ensuing discussion, and the students' expression of learning will need to encourage the use of visual imagery and the linking of images and concepts (Clements & Del Campo, 1989; Goldin, 1987). Modification and extension of a concept can be facilitated by analysis, physical comparisons, and language during the act of problem solving.

Learning develops not only through puzzling and reflecting, but also through interaction and communication, both verbal and non-verbal, with others. The quality of spatial discourse - that is how language, questioning, and labelling are used - also influences spatial problem-solving performance. It is open to question whether this discourse is best at the cooperative group level (Johnston & Johnston, 1990), in which case individual problem solving processes may be reduced and language limited, or at the teacher-pupil level (Clements & Del Campo, 1989), where a major danger is the possibility that the teacher's activities and thoughts and the classroom "routines" (Voigt, 1985) will be so dominant that the students do not engage actively in the problem-solving process.

## The Study

This paper reports a study designed to investigate whether a series of spatial problem-solving activities would improve students' performance on tasks requiring two-dimensional spatial thinking processes, and that this performance would be retained and transferred to a three-dimensional spatial task. The study also considered whether the type of student/student and student/teacher interaction resulting from different classroom organisations would affect learning.

## Method

### Experimental Design

The study employed an experimental matched-group design in which subjects from the same class and in the same cluster as determined by their ranking on total pretest scores, were randomly allocated to one of three Groups. For each Group, eleven half-hour learning activities were provided over six school weeks. Group 1 students worked individually on two-dimensional spatial problems and participated in whole class, teacher-led discussions. Group 2 students worked in small cooperative groups on the same spatial problems but Group 3 students, who formed the "control" group, worked in small cooperative groups on a series of number problems. Once students were randomly allocated to the Groups, those in Groups 2 and 3 were organised so that each cooperative group of three or four students had both male and female students and a student from the upper, middle, and lower range of scores on the pretest. Group 1 was as large as the other two Groups together and as Group 1 worked at a separate time the student/teacher ratio was equivalent for each student.

All students completed a spatial thinking pretest, and after the spatial (or "control" activities) completed a parallel posttest, and eleven weeks later (six weeks for one school because of annual holidays) completed a parallel retention test. An analysis of covariance with pretest scores as covariates was used to determine the effects of gender, Year at school (2 and 4), and Group (1, 2 and 3).

**Subjects**

There were 190 students involved from three school in Metropolitan South-West Sydney. The schools were situated in lower and lower-middle socioeconomic areas with a high proportion of students with English as a Second Language. Both a Year 4 and a Year 2 class were involved from each school plus an additional 18 students from one of the schools. This latter group of students completed the activities in groups of three and were video-recorded during the learning experiences and interviewed immediately regarding the way they were thinking during critical parts of their problem solving. All the classroom teachers were enthusiastic, quality teachers but classes varied in the amount of spatial activities that they had participated in before the study. During the study, classroom teachers did not give planned lessons on 2D Space and the same teacher (the author) provided all students with the learning experiences pertinent to the study.

**Test Instrument**

A test was developed by the author for the purpose of measuring spatial thinking processes at the recognition level of spatial concept development (van Hiele, 1986). There are six subtests:

*Subtest 1* requires respondents to recognise congruent shapes in rotated or reflected positions. Some of the items in this subtest are more easily solved by analytic than by holistic procedures.

*Subtest 2* is concerned with recognition of shapes made by tessellations of a smaller shape.

*Subtest 3* asks respondents to complete shape outlines by drawing the matches that could be added, and to mark the matches that could be taken away to make the configurations, and to recognise and trace over embedded shapes in outlines.

*Subtest 4* is about the construction of models made of parts from memory and/or under transformation.

*Subtest 5* contains items concerned with nets of three-dimensional shapes, and in particular with the mental transformation of two-dimensional shapes by folding.

*Subtest 6* is primarily concerned with the recognition of congruent angles in shapes in different orientations or contexts.

The items were analysed using Rasch techniques (Andrich, 1988). The analysis indicated that the items involving three-dimensional space (subtest 5) did not fit well with the other two-dimensional items suggesting another underlying trait was involved. There were 46 items used for the 2D tests and six for the 3D tests.

**Learning Experiences**

The spatial learning experiences involved a number of activities using commonly available materials. These are:

1. finding the similarities and differences of the shapes, making shapes from other shapes, making outlines using sticks, sketching shapes and configurations, and comparing angles - using both the seven-piece tangrams and pattern blocks;

2. making pentomino shapes from square breadclips, finding their symmetries, and tessellating the shapes;

3. making and rearranging designs with matchsticks and seeing shapes within the designs.

The activities were designed to encourage the students to solve problems through discussion and the invoking of visual imagery and pertinent spatial concepts. By using open-ended and multifaceted activities, the activities catered for the needs of students with a range of prior experiences and existing concepts. The activities were used to provide a basis for challenging students to reflect on and, whenever necessary, to modify existing concepts, images, and skills. In particular, it was expected that teacher/student and student/student interactions would encourage the use of language and communication that would facilitate the further development of concepts.

### Analysis of Video-recorded Activities

The participation of students in the activities has been analysed using categories based on Goldin's (1987) representation in problem solving. The categories used in the study are (a) visual imagery, (b) student/student and student/teacher interaction, (c) language representation, (d) conceptualisation of shapes, of operations on shapes, and of number, (e) heuristics - both metacognitive and strategy decisions, (f) affective variables, and (g) perceptual and manipulative behaviours. Particular note was made when one category influenced another category. The tangram activities will be used in this paper to illustrate the nature of students' learning.

## Results

An analysis of covariance of the scores on the posttests (post2D and post3D) and retention tests (ret2D and ret3D) using pre2D as covariate for the 2D tests and pre2D and pre3D as covariates for the 3D tests indicated no significant effect on the variance due to the three main variables or their interactions. However, as shown in Table 1 when the two spatial groups were combined, significant effects were shown. The improvement for the groups learning from spatial problems was significantly better on two-dimensional spatial thinking than the "control" group learning from number problems. This was attributable to improvements at Year 2 level (significant at 0.01). Furthermore, gender affected the three-dimensional results and post hoc analyses of the gender effects showed that males improved more than females except for the Year 4 girls who were learning as cooperative groups.

Table 1:

**Analysis of Covariance of Retention of Spatial Thinking Processes; Spatial versus Number Groups**

| | two-dimensions | | three-dimensions | |
|---|---|---|---|---|
| main effects | F | signif. of F | F | signif. of F |
| group | 5.072 | .026 * | 1.389 | .240 |
| gender | 0.037 | .848 | 4.146 | .043 * |
| year | 0.080 | .778 | 0.550 | .459 |
| 2-way interactions | | | | |
| group gender | 0.647 | .423 | 0.228 | .634 |
| group year | 1.576 | .211 | 0.719 | .398 |
| gender year | 2.248 | .136 | 0.408 | .524 |
| 3-way interactions | 0.011 | .916 | 0.588 | .444 |

* significant at the .05 level

## Video Analysis

*Visualisations and conceptualisations.* Students were fascinated to realise that the triangular shapes could be made in more than one way; that two triangles could make three different shapes; that many different squares, triangles and rectangles could be made; and that the similarities of shapes such as equality of lengths of sides resulted in different configurations. Some groups of Year 2 students when first given the pieces, set their investigations to lay pieces on top of each other and noted that the "corners" or "pointy parts" were the same.

When asked to remake a design, students remembered strategies and certain aspects rather than the pictorial image especially for more complex designs. Nonetheless, students used a variety of types of imagery although most seemed to involve an analysis of the design, often connected with a procedural concept, and patterns.

*Links Between Categories.* Each step in students' thinking was categorised and at least 20% of these steps for each activity indicated that visual imagery led to deliberate manipulation of pieces. A further 5% of steps indicated that an operational concept led to a manipulation while 7% indicated manipulation resulted from interaction between student/student or teacher/student.

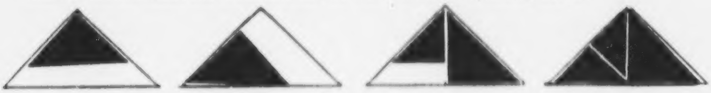
Nearly all interactions that led to heuristics about what to do or to conceptualisations were between the teacher and the student and not between student and student. One exception was in the cooperative groups, in which the students with lower pretest scores, did not get the opportunity to manipulate materials, some of these student would speak out and frequently summarise what was happening thus improving conceptualisations.

Two snippets from Year 4 illustrate some of the above points (fictitious names have been chosen to indicate gender) - the first snippet is from a small cooperative group.

> Natalie places a triangle on the square intending to make the square with the triangles but Tess takes it to compare the sides of several shapes. Tess then decides to make the parallelogram with the two triangles while Natalie watches patiently. Natalie finally gets a chance to make the square before Tess takes the pieces to match angles placing the right angle of the square on the right angle of the large triangle. At once all the friends perceive the pattern of triangles that remains and the large triangle is completed. There are some more tussles between Don and Tess to commandeer the pieces but Tess manages to make the large triangle with the parallelogram and two triangles. They could all see how it would work but it was all a matter of turning the triangle to fit it in. Watching, Natalie says "I know now, they all use triangles". ... Don wants to make the large triangle using the middle-size triangle but between snatches he keeps placing the triangle to highlight its similarity (see Figure 1). Finally he turns it to cover half the large triangle but he still has trouble placing the small triangles although he thinks it should work. It is only some time later when the teacher intervenes that he completes this design satisfactorily but with embarrassment.

The second example describes students working as individuals. Compare the different approaches in getting started and the typical interactions between students and the teacher and students.

Kathy matches two points and listens to the conversation between the teacher and her friend. She looks anxiously at the teacher and then at her friend. The teacher encourages her to say what she is doing. "I think these two points are the same, they are both going like this". She is unsure of herself, especially when she decides two right-angles are not the same. When the teacher suggests they are, she decides that she hadn't placed them carefully enough on top of each other. She now understands what she was suppose to be doing by the manipulation and comparison and she goes on to match other equal angles. Then she decides to make the middle-size triangle by turning and joining the two small triangles. She checks she has a shape by flipping them together so the triangle is in the usual position on its base. She becomes uneasy again, flips pieces distractedly, and glances at her friend but she does not talk. Eventually the teacher suggests she make another shape and she asks if she can put the shapes on top of the triangle to help. "Yes, of course". Happy now, she makes the large triangle with the square and two small triangles with a little effort, pleasing herself.



Dominance of Similarity Partial Coverage Complete Coverage

*Figure 1:* **Covering the Large Triangle**

Beside the affective variables and interaction patterns, these snippets illustrate how imagery directs manipulation and how manipulation assists in solving problems, in developing imagery, and in focussing learning.

## Discussion

### Effect of Learning Experiences

Studies of the effect of experience on spatial tasks have generally shown improvements in spatial abilities but not necessarily in mathematical problem-solving (see summaries by Clements, 1981; Eliot, 1987; and Lean, 1984). The present study involved younger children than most of the others and the tests of spatial thinking processes were based on mathematics classroom activities.

The study showed that experience with two-dimensional spatial problems affected two-dimensional spatial thinking processes when sufficient time had elapsed for these concepts and processes to be consolidated. However, these thinking processes did not transfer to the three-dimensional items. The difference in effect on three-dimensional and two-dimensional thinking may be an artefact of the testing as the two-dimensional test consisted of five subtests while the three-dimensional items formed only one subtest. However, support for these differences comes from a study by Rowe (1982) in which training, either in two-dimensional or in three-dimensional spatial thinking was effective at Year 7 only on two-dimensional tasks but not on three-dimensional tasks.

### Gender Differences

Although many studies report small differences favouring males, mainly with three-dimensional tasks or older students, overall studies show there were no differences on two-dimensional tasks (see the meta-analysis by Linn and Hyde, 1989) Wattanawaha and Clements (1982) found differences for Years 7 to 9 for items that involved three-dimensions or mental manipulation of images. In contrast, Rowe (1982) found no sex-related differences resulting from the training or testing.

## Conclusion

The study has shown that the concrete problem-solving approach taken to assist children in spatial learning has potential for primary teaching. The problems encouraged analytical processes as well as visual imagery. In fact, analysis of video-recorded activities indicated that the use of imagery by children was most significant in the solving of problems, in particular in directing student's manipulation of materials. Affect and heuristics were influenced by interaction between student/student and student/teacher. In reviewing the video-tapes, one is reminded of the excitement that children from all groups showed as they discovered new concepts, grappled to develop images, concepts, and skills, and tried to convey to me their findings.

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# THE INTERACTIVE CONSTITUTION OF AN INSTRUCTIONAL ACTIVITY: A CASE STUDY

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**ABSTRACT**: *The purpose of this study is to illustrate how a teacher's expectations for children to represent their solutions in written form by means of number sentences, changed the nature of the mathematical activity in that classroom. The findings suggest that the interactive nature of instructional activity renders the process of acculturating children into the formal language of mathematics problematic.*

## INTRODUCTION

With a growing consensus among mathematics educators that children should be actively involved in doing mathematics (NCTM, 1991; Richards, 1991), closer attention is being paid to the role of communication in the classroom.

> Communication plays an important role in helping children construct links between their informal, intuitive notions and the abstract language and symbolism of mathematics (NCTM, 1991; p.27)

Within this framework, the teacher has to facilitate children's dialogue while acculturating children to the abstract language and symbolism of mathematics. This goal of classroom communication rises an important question: how can the teacher keep the balance between guiding the children's process of symbolizing and encouraging children to learn mathematics meaningfully?.

This study illustrates how a teacher's expectations for her students to represent their solutions to word problem in written form, by means of number sentences, changed the nature of the mathematical activity in that classroom.

## THEORETICAL FOCUS

Understanding communication in the classroom requires the coordination of two levels of analysis: the psychological and the socio-interactional (Cobb, 1989). On the psychological level, the analysis focuses on the individual -the teacher's or a student's- personal interpretation of the actions of others and their personal construction of mathematical meaning. On the socio-interactional level, the focus is on the mutual construction of meaning and the interaction of the participants (Bauersfeld, 1988).

When the teacher presents his (her) students with an instructional task, (s)he has in mind certain goals and expectations for the students. (S)he communicates them to the students explicitly or through his (her) actions. On the other hand, given children's personal understanding of mathematics and their past experience, they will make their own interpretation of the teacher's actions and define their goals for the activity (Cobb, 1986; 1989). In a sense the teacher and students create the activity in the very process of interacting with one another.

## RESEARCH BACKGROUND AND RATIONALE

The case study emerged as part of a larger project intended to study children's development of place value concepts while engaged in money-related activities. This took place in a second grade classroom with a problem-centered mathematics tradition (Cobb, Wood, & Yackel, 1991). In this classroom, children work in small groups and then participate in a teacher-led whole class discussion in which they are expected to explain or justify their solutions.

One of the questions guiding this research was: Would children use the counting strategies and solution methods developed within the context of money in more general problem solving situations? To this end, a four-day period of word problems presenting a variety of situations, was added at the end of the money sequence activities. The problems presented the first two days included only money word problems. While the wording of these problems was similar to that of problems within the money sequence, the main difference was the format. Previously, the amount of money was presented with pictures of coins as in

> This is Jane's money (picture of 2 quarters and a penny). This is Tom's money (picture of 1 quarter, 2 nickels and one penny). How much more money does Jane have than Tom?

Now, the amount of money was stated. For example, "Barb has 56 cents. Paul has 28 cents. How much more money does Barb have than Paul?"

In the last two days, the problems included a variety of situations.

It became apparent that in the context of these new word problems, many children began writing number sentences. Some children wrote the number sentence before they even began analyzing the problem. They did so by looking for key words or other clues. Yet others, solved the problems first and then looked for key words. Although in most cases the answers were correct, the number sentences seemed unrelated to children's solutions and in many cases did not make sense. These events were unanticipated since the students rarely wrote number sentences before. The focus of this paper is to analyze in some detail the emergence of this phenomenon and its relationship to children's learning.

## DATA COLLECTION AND FINDINGS

The body of the data is comprised by a six-week observation period of daily lessons, videotapes of whole class discussions and the work of two pairs of children, clinical interviews, and informal interviews with the teacher and children. Field notes and copies of children's work were also collected.

## THE INTERACTIONAL CONSTITUTION OF AN INSTRUCTIONAL ACTIVITY

During the first day of word problems, as she interacted with pairs, the teacher began asking for number sentences, sometimes before they even had the chance to reflect on what they had just read. Later, during the whole class discussion there were several instances in which the classroom talk focused on the correctness of the number sentence. Those children who presented an incorrect number sentence were not asked to explain how they had thought of the problem. This was very unusual. So far, the reconstructed pattern of interaction had shown that the focus of the conversation during whole class discussion was children's solutions. It can be inferred that the teacher's intention was to help her students link the "underlying structure" of the word problems to the children's solutions (NCTM, 1989; pp. 26-27).

The second day, the teacher still asked children in small groups to write number sentences but, the whole class discussion focused on children's solution methods. She recorded children's counting strategies on the board using an arrow notation indicating from where to where the child had counted on or back.

During the third day, the teacher opened the lesson by specifically requesting children to write number sentences. Students had to be prepared to write a number sentence when called on to present their solutions. The whole class discussion focused on the correctness of the number sentences.

## CHILDREN'S PAST EXPERIENCE AND INTERPRETATIONS OF THE TEACHER'S EXPECTATIONS

All children in this classroom came from regular textbook-based instruction. At that time they had received specific instruction on key words or other clues to write number sentences.

Children attempted to interpret their obligation to write number sentences in different ways. Some of them began writing a number sentence but solved the problem independently. Others solved the problem first and then proceeded to look for key words in order to write a number sentence. Yet, one particular pair struggled to solve one problem because one of the children made the number sentence the center of his mathematical activity. During the whole class discussion, children offered their number sentences first. Most of them had the wrong number sentences although they had arrived at the right answers.

## JIM AND KATIE: THE CASE STUDY

Jim and Katie were chosen to exemplify the interplay between cognition and social interaction. During the clinical interviews that preceded the observations and the analysis of the classroom videotapes, Jim displayed more sophisticated counting strategies than Katie. For example, Jim would solve a problem like 55-28 by counting down by tens and ones from 55: 45,35,34...28. Katie, instead would count back by ones keeping track of her counting on her fingers. In several cases she would skip a number or two in the sequence.

In general, Jim and Katie alternated their responsibility to solve the problems. In many instances, they did not discuss with each other how or why they had arrived to a particular solution.

The first day of word problems, Jim and Katie worked on the first page with almost no reference to number sentences. Only in one instance, while he was stuck, Jim mentioned he might write a number sentence but he took Katie's suggestion that they were not required to do so. Katie was reading the first problem on the second page when the teacher came to them. Immediately, the teacher asked Katie to try writing a number sentence. She couldn't think of one so the teacher turned to Jim who translated the problem piece by piece. Jim wrote number sentences for the remaining problems.

During the second day of word problems, Jim did not show a definite pattern in his writing of number sentences. Sometimes he did not even think of one; other times he wrote a number sentence after he had solved the problem and it did not even make sense. It seems that Jim's main goal was to make sense of the problems and then solve them according to his interpretation.

The third day, Jim approached the solutions by first writing a number sentence. It also became apparent that Katie had no expertise on writing number sentences. After Jim wrote the first one, she said "let me ask you a question, how do you know when it's a plus?" Jim explained his use of key words.

The following problem illustrates Jim's interpretation of the mathematical activity after the teacher made clear her expectations for number sentences.

J: "April had some money. She spent 24 cents. Now she has 36 cents left. How much money did she begin with?"

He immediately proceeds to translate the key words into a number sentence.

J: "April had", blank, "she spent", minus. "Now she has", equals (writes down ___-24=36) 24; 25,26,.....,36. 12 (writes 12 in the "blank" slot.

Katie makes an attempt to solve the problem. She begins to count on from 36 and asks Jim's help to keep track of her tens. He does not cooperate with her and she stops at 48 in her second attempt. Katie says the answer is 12.

J: I just figured out you can't take 12 from 24 (meaning 24 from 12). I think it'll be a plus. It must be a plus (changes the number sentence to 12+24=36).

At this point Katie begins to argue that Jim's number sentence is incorrect.

K: It can't be a plus.

J: Why?

K: Because she spent some money. She spent, she spent, she spent! It can't be a plus!

Katie gives Jim examples such as "if you spend a quarter you cannot have 30 or 50 cents back."

Unlike his past approach, Jim does not go back to the problem. He finds his number sentence at fault and fixes it for the numbers to make sense. His justification to Katie's is the following.

J: I did it the right way and it don't equal.

Under the observer suggestion, Jim and Katie attempt to solve the problem with play money. Using only dimes and pennies, both make two piles, one of 24 cents and one of 36. Only Katie pursues the solution of the problem. The observer inquires into their solution.

I: What are you doing here?

K: We are trying to use money to solve this problem.

I: I see you have 24 and 36.

K: That's how much money she has left; 36. And that's how much she spent (pointing at her pile of 24 cents).

I: And what are you trying to find out?

K: We are, we are going to take this money. If we count all this money together we can figure out how much she had. 50,60 (counting the dimes first and then the pennies).

I: 60?

Now Jim seems to find Katie's solution in contradiction with her argument that "it can't be a plus" since she put both amounts together to reconstruct the original amount of money, that is, she "adds".

J: She has 60? It can't be a plus she said! She spent some remember?

Knowing that she is mathematically weaker than Jim, Katie is about to give up. Jim is using her own argument that it can't be a plus because April spent some money. The observer asks her what was wrong with her answer. Katie takes some time, reads the problem once more, underlines the question "how much money did she have to begin with?" and explains.

K: But look! (to Jim) How much money did she have to begin with? So there you go, 60. That's what I have.

Katie now writes the number sentence 60-24=36

## CONCLUSIONS AND IMPLICATIONS

This study illustrates how the teacher and students together create the instructional activity as they interpret each other's actions and adjust their meanings. By no means did the teacher's actions cause Jim to change the nature of his mathematical activity. Jim wanted to fulfill his obligation of providing a number sentence and based on his previous experience he looked for certain words to determine what kind of operation was "encoded" in those words. Previous observations of Jim's work show that he always analyzed the problem and solved it according to his interpretation. Furthermore, his justifications and explanations were made from his understanding of the problem. Now, his interpretation of the mathematical activity was to find and then solve a number sentence.

Jim's actions exemplify the change in the nature of the mathematical activity at the individual level. However, an analysis of the whole class discussion exemplifies this change at the collective level. That day instead of presenting their answers, children offered their number sentences first. For example, a student had solved a missing addend type problem by counting on from 57 to 70 obtaining the answer 13. He wrote the number sentence 57+70=13. The teacher turned to the class to discuss the correctness of the number sentence. Few students had been successful in writing number sentences. They could see that the number sentence did not make sense numerically but, they did not turn to the problem in question. The nature of the explanations and justifications had changed. In the history of this classroom, the focus of the mathematical talk had been children's solution methods. Consequently the arguments children had presented in the past had to do with their interpretation of the problem and their counting strategies.

In summary, the nature of the mathematical activity that the classroom participants constituted during the four-day period of word problems can be best understood by close consideration of the actions and mutual expectations of the teacher and the students. On the one hand, the teacher intended to help children bridge their mathematical thinking to the underlying structure of the word problems by means of written representations. She seemed to expect that those number sentences would do both, represent children's thinking and the underlying structure of the problem. On the other hand, children attempted to solve the problems meaningfully while fulfilling their obligation of writing a number sentence. But, in many cases, counting strategies can not easily be represented by a number sentence.

One interesting aspect of this classroom mathematics tradition is that it allows all its participants to reflect on their actions thus providing the students as well as the teacher with learning opportunities.

The teacher's reflection on her previous actions became apparent the fourth day of word problems. That day, she opened the lesson asking her students about the "rules" (i.e. general strategies) for solving problems. Children's first responses were to read the problem and to write a number sentence. At this point she said "If it helps you solve the problem, you may write a number sentence. But, don't you think you should think first? Take some thinking time?". Now, the majority of children, including Jim, did not write number sentences. Even more, one pair of children who used counting up or counting back strategies and had failed to write correct number sentences, chose the arrow notation that the teacher brought up the second day of word problems. This seems to indicate that, although it is desirable that children record their solutions, there is a need to help them develop a notation which more faithfully represent their counting strategies.

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# STUDENTS' VIEWS AND ATTITUDES TOWARDS MATHEMATICS TEACHING AND LEARNING: A CASE STUDY OF A CURRICULUM EXPERIENCE

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*This paper describes a qualitative study dealing with the views about teaching and learning mathematics held by the students of a school involved in a experience of new curricula, stressing active methodologies and group work. At 7th grade, the students clearly noticed important differences in the mathematics classes and welcomed them. At 10th grade, they had a general positive attitude towards mathematics but did not perceive major changes. However, they were very concerned with the implications of this experience for their academic progression.*

What views hold secondary school students regarding the teaching of mathematics? In what ways are these affected by a process of curriculum change mandated by the Ministry of Education? Are they sensitive to differences between old and new curriculum orientations? How does that influence their learning experiences?

These were some of the questions that we regarded as potentially interesting when we were asked by a Government agency to make a case study of a school participating in an experience of new mathematics curricula. Our work concerned a wide range of issues, including the nature of the innovation process and the perceptions of the participating teachers (see Ponte et al., 1991). In this paper we focus particularly on the views and attitudes held by the 7th and 10th grade students about teaching and learning mathematics.

## Students' Views About Mathematics and Mathematics Teaching

The conceptions, attitudes, and expectations of the students regarding mathematics and mathematics teaching have been considered to be very significant factor underlying their school experience and achievement (Borasi, 1990; Shoenfeld, 1985).

The general conceptions determine the way students approach mathematics tasks, in many cases leading them into nonproductive paths. Students have been found to hold a strong procedural and rule-oriented view of mathematics and to assume that mathematical questions should be quickly solvable in just a few steps, the goal just being to get "right answers". For them, the role of the student is to receive mathematical knowledge and to be able to demonstrate so; the role of the teacher is to transmit this knowledge and to ascertain that students acquired it (Frank, 1988).

Such conceptions may prevent the students of understanding that there are alternative strategies and approaches to many mathematical problems, different ways of defining concepts, and even different constructions due to different starting points. In consequence, they may miss significant aspects of mathematical experience, including formulating their own questions, conjecturing relationships, and testing them. They may approach the tasks in class with a very narrow frame of mind that keeps them from developing personal methods and build confidence in dealing with mathematical ideas.

Associated with these conceptions are students' expectations of what is a mathematics classroom. If the teacher tries some innovative activities an overt or covert reaction of the students may quickly develop further inhibiting the learning process.

How resilient are such conceptions and expectations, once formed in students' minds? Are they a simple consequence of the climate of the mathematics classroom or do they acquire some sort of independent existence?

Despite the interest that this topic has recently attracted, not much is known about the possibilities of influencing these general views of the students, and especially, what can be the effects of new curriculum approaches purposely designed to improve their views and attitudes regarding mathematics.

## Context of the Study

Led by the Portuguese Ministry of Education, a comprehensive global Reform of the state controlled educational system is under way since 1986. The Ministry lays out the plans concerning the different aspects of the Reform and publishes them as mandatory laws. Some of these are previously submitted for public discussion, but the final decisions are always made by the Ministry (in some cases contrarily to the majority of the expressed opinions). This process can well be viewed as a variant of the classical "top down" approach for educational innovation, since in its development there is no significant input from the major addressees (Howson, Keitel, & Kilpatrick, 1981).

As part of this process of educational reform, an experience of new mathematics curricula for 7th and 10th grades and for a new 10th grade discipline called *quantitative methods* was conducted, in selected schools, during the school year of 1990-91.

The new curricula are organized in three strands of objectives: knowledge, abilities and attitudes/values. According to those we interviewed at the Ministry, one of the major aims was definitively the improvement of the attitudes of the students towards mathematics. The new curricula suggests a more intuitive approach to the mathematical concepts, with emphasis in graphical representations and real world situations. Other new features include the introduction of probability and statistics from an earlier grade level and a greater attention to geometry. In terms of teaching methodologies, the use of calculators is recommended from grade 5 on and some attention is given to active methodologies and group work[1].

The new discipline of quantitative methods was ment for students of humanistic areas who did not have formerly mathematics in their plan of studies. The content is (a) logic and complex and real numbers, (b) statistics, combinatorics, and probability, and (c) functions. Its main purpose was, according to the offical documents, to give students tools and ideas seen as "required for an integration in social and economical activity", and also as "necessary for the development of a dynamic thinking structure".

Since the subject matter of the experimental curricula underwent significant changes regarding the present content, the students involved in this experience had no textbooks. Instead, they used as their main study source support materials provided by the Ministry of Education, eventually complemented by other materials made by the teachers themselves.

The experimenting teachers had 3 hours of reduction in their weekly working load (which is 22 hours for beginning teachers). They were supported by a small group of accompanying teachers, especially created by the Ministry to assist this experience. The parents and the students were informed in the beginning of the school year that the school would be using a new curriculum. It was possible to transfer a student to a different school, not in the experience, but only a very small number opted for doing so.

The research team was composed by the authors of the present paper. None of us had so far a close relation either with this experience or with this school. However, we were generally critical of what we considered to be the absence of clear orientations in the new curriculum proposals.

## Methodology

The study had three main phases. The first phase, preparation, included the formulation of research questions, the planning of field work (with the elaboration of interview and observation guides and of criteria for selection of informants), the outline of the final report, and a first contact with the field. The second phase included the conduction of field work and the third was devoted to writing the research report. The methodological approach and the field activities were strongly influenced by an interpretive conception of educational research, as described in Goetz and LeCompte (1984), Merriam (1988), and Patton (1987).

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[1] Topics in the new 7th grade curriculum were: numbers and variables, proportionality, similarity, rational numbers, statistics, solids, triangles, quadrilatrals and equations. The 10th grade proposal included: numbers, algebraic expressions, analytical geometry, trigonometry, functions and derivatives, statistics, and probability.

A detailed plan of activities was designed from the earlier beginning. It specified the actions to be carried out, the responsibilities of each of the members of the research team, and the approximate completion dates. In that respect, many of the suggestions regarding the design of a case study provided by Yin (1989) were taken into consideration.

Data was collected for this study through interviews (which were audio-taped and later transcribed), observations and documental analysis. A number of people was interviewed, including teachers, administrators, Ministry officials and nineteen students, who were interviewed in groups. Each group had at least two boys and two girls (of whom two were high and two were low achieving students). The selection, based in these criteria, was made jointly by the teachers and the researchers. Observations were also made of classes (three at 7th and two at 10th grade) and of other school activities. Documental analysis was made of the text of the new curricula, the materials produced by the Ministry and by the teachers, the reports of the accompanying teachers, and other school documents.

The field work was conducted in groups of two researchers (with one group focusing on 7th and another on 10th grade) and amounted to 130 hours. The research team had the cooperation of three assistants who transcribed about 50 hours of audio-taped material.

Data was analysed cross-checking all the information available regarding the study questions and subquestions and collectively discussing the emergent main issues. The final report (in which this paper is based) underwent a substancial process of sucessive drafts and revisions.

## Overview of the Curriculum Experience

This experience had many components, including the selection of the participating schools, the elaboration of the new curriculum documents and their transmission to the schools, the preparation by teachers of the classroom implementation, regular classroom and *interdisciplinary activities*, the provision of ongoing support for the participation of the schools in the experience, and the feedback to the curriculum authors to make the appropriate revisions. In this paper, as we are especially concerned with the students reactions, attention is focused on the classroom and interdisciplinary activities. Despite our initial skepticism, the closer contact with the actual school practice made clear to us that some innovative activities were taking place in the schools.

Classroom work. In the absence of textbooks, students made wide use of their own notebooks. The 7th grade classes were mostly structured around worksheet based activities (containing exercises, conceptual questions, and problems). They were carried out in pairs, but sometimes also in groups of four students, followed by discussions on the blackboard involving the general participation of the class. Students used the calculator naturally, when it was relevant for the task in hand.

The 10th grade class tended to be structured around the discussion conducted by the teacher. The main conclusions were written on the blackboard and then readly copied down by the students to their notebooks. It was quite noticeable that one of the teachers highly valued extra classroom activities, which very often were called to the discussion.

In the quantitative methods class there was little discussion. It was essentially based in worksheet activities, most of which were computational exercises requiring the use of the calculator, with the students working in groups of four in a very relaxed atmosphere. The teacher circulated in the class attending to requests from groups or individual students.

Overall, we witnessed a great variety of classroom activities, depending on the grade level and also on the teachers' styles. Even so, some common differences can be noted regarding traditional mathematics teaching: much more use of technology than usual, more attention to relations of mathematics and the real world, and in some cases more group work and more exploratory activities.

Interdisciplinary activities. Besides the change in curriculum contents for all disciplines, this experience also required the schools to organize interdisciplinary activities, to be supported by regular classroom work, up to a certain number of hours. The teachers at first did not realize all the implications of such requirement and hoped that it would be droped. When it become clear that the Ministry insisted on the realization of these activities, they strongly complained and searched ways of avoiding them.

As educators, we feel that interdisciplinary activities are an excellent pedagogical idea. Activities of this nature are common in many schools, sometimes supported by some release time provided to the teachers, in other cases just based on teachers' volunteer time. But we must stress the striking difference between the reluctance with which teachers regard this sort of activities when they are mandatory and the willingness to organize them when they stand on their initiative.

## Students' Views about Mathematics and Mathematics Classes

Of special interest in this paper is the issue of how were the students involved in this experience perceiving mathematics teaching and learning, as compared with previous years.

7th grade students. In general terms, they were satisfied with their mathematics classes and with the new curriculum. In the beginning, they were concerned because there was no textbook, but now they felt all right without it. Echoing a common complain of their own teachers, they also blamed the supporting materials (made by the Ministry), saying that these were not available when they should.

The students felt that classes were different. In their view, there was much group work, reports, investigations; there was more discussion, less work on the blackboard, more work on the notebook. They felt that the new curriculum implied more work and more thinking.

For some of them, mathematics classes were sharply split in two groups: theoretical classes (mostly done through writing on the blackboard) and practical classes (exercises on the notebook), the second being far more common. The mathematics class was seen as practical and active:

> "In mathematics we are always doing something."
>
> "Mathematics is exercises. It is a more practical subject."

Since there was no textbook, they said that more attention was required in class. They mostly felt that the use of calculators was all right (with some reservations from two girls). Regarding the computer, which they did not use in class, their opinions were sharply divided: some thought that it would help learning and wellcomed it into the classroom, others did not.

These students felt that some changes had occurred in their assessment. Besides the test other things now also were taken into account: absences, attention, behavior, and notebook. Some of them felt that the tests had themselves changed:

> "A new test with a few questions corresponds to an old one with many questions."

10th grade students. They felt particularly insecure in the beginning of the school year: how would it be with no textbook? Some of them even considered the possibility of changing to another school. Now they were still insecure about the implications of this experience in the process of application to an University (that will take place after they completed the 12th grade).

But these students felt that all the disciplines were going all right. Regarding their previous experiences, mathematics was the subject that had less changes:

> "Mathematics is a practical subject... [You go through] new contents and then solve exercises... Mathematics is really to make you think."

They did not like very much the supporting materials made by the Ministry since they felt that the exercises were too easy; they preferred the teaching materials made by their own teacher. Calculators were used in the topic of statistics but now were left at home because they were no longer necessary (the required computations being quite simple). Some of them were in favor of group work, but major group projects only took place twice during the year, and concerned work done outside the classroom. Others wondered if doing investigation-like activities was appropriate for mathematics or better suited for other subjects, such as Portuguese.

These students were much concerned with their grades. They felt that the tests were the dominant factor, but indicated that the weight of class participation and other works had also increased.

They particularly disliked the interdisciplinary activities. They felt that they were not well conceived and implemented, and regarded them as a distraction to the more important learning activities. They were so serious about this that even made a document to give the interviewers stating their position on this matter!

Quantitative methods students. These are students of the humanistic strand, who in the past usually finish their study of mathematics at 9th grade.

They felt that this new discipline was nothing but mathematics with a different name, which they did not like since they did not like mathematics. These students considered that the subject matter was quite boring (in their words: "it is always the same thing") and missed the security of the regular textbook. They did not like what they called the "horrible" supporting materials and did not feel any particular enthusiasm about the use of calculators.

These students also did not like group work. However, they felt that the evaluation had changed somehow: the analysis of the notebook and the results of group work were now also taken into account.

## Conclusion

This experience implied quite significant changes in the life of the students. For the first time in their school experience they did not have textbooks. They were presented with a different sort of mathematical activities which were perceived as requiring more thinking. Also, although that was not necessarily new for all of them, they were required to do much group work, made a significant use of the calculator, and engaged in interdisciplinary activities.

The students reacted in a very positive way towards the absence of the textbook. They took care in organizing their notebooks and found them a useful study base. The challenge of different and more demanding mathematical activities was well taken by them. At 7th grade they were positive about having problems requiring more effort and thinking. And at 10th grade the students took themselves the initiative of asking the teacher more difficult problems. They were quite positive about the active nature of mathematics classes and their reactions to group work were generally favorable.

The most notable exception to this general pattern were the students of quantitative methods, who must be regarded as a very special case. They were very unhappy about having to study mathematics at this point of their school life and were negative towards everything that had to do with this experience. There is a remarkable parallel between the lack of enthusiasm of the students and that of their teacher. It is difficult to know which one come first, but at this point they just seemed to be reinforcing each other.

The most striking difference in the attitudes of the 7th and 10th grade students concerns the interdisciplinary and project-like activities. The students in the 7th grade were favourable and those in the 10th grade were very negative. Younger students still have interest in what the school may offer them, while the older ones appear to be led mostly by personal extra school interests or long term concerns with University admission. They saw the process of Reform as mostly bringing them serious handicaps.

All the students were clearly aware that they were participating in a curriculum experience. For 7th graders, it seemed to have an impact in their vies and attitudes towards mathematics. For 10th grade students, the experience seemed not to affect their view of mathematics or mathematics learning.

This experience suggests in a remarkable way that there is a relationship between the views and attitudes of the teachers and those of the students. When teachers are negative towards something, so are the students. When they are positive, they seem to influence the views and attitudes of the younger students. At the 7th grade these can afford to be optimistic and willing to accept some of the values and challenges coming from the teachers. On the contrary, the 10th grade students transmitted clearly an image of independence. Therefore, the changes that bear with their expectations and affect their normal routines need to be introduced with especial care and subtlety.

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# KASIMIR:
# A SIMULATION OF LEARNING ITERATIVE STRUCTURES

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*The elaborate use of control structures is an important aspect of programming proficiency. In particular, novices and experts differ substantially in their ways of representing repetitions in programming code. Even simple commands have to be recognized as parts of a problem and must be represented cognitively. This process will be described in the following contribution. We present a computer simulation that describes the single steps during problem solving.*

A large part of learning how to program is dedicated to the acquisition of control structures. We investigated this particular kind of knowledge acquisition and identified three different representations of control structures. They may be described as sequential, iterative, and recursive approaches (cf. HAUSSMANN & REISS, 1989; HAUSSMANN & REISS, 1990).

- The *sequential* way of writing programming code for graphics tasks is similar to the way one would produce a drawing using paper and pencil.
- *Iterative* solutions take into account the structural aspects of the problem such as repetitions of specific shapes. Iterative solutions can be differentiated into subclasses depending on how students recognize nested substructures (iterations within iterations).
- *Recursive* solutions use the opportunities the programming language offers to define procedures calling themselves and in that way constructing a drawing most elegantly with respect to programming code but not with respect to the efficiency of the program execution. Tail recursion is included in this broad definition of a recursive solution.

The corresponding problem solving strategies were modeled in a computer simulation using a production system approach. We identified aspects involved in transforming a sequential solution to an iterative solution. The program gives evidence of a change in memory organisation during the learning process. While the geometric shapes are first stored as declarative knowledge in the form of nodes during the learning process the proceduralization of knowledge occurs. The concept is transformed into an operation (ANDERSON, 1983). According to this theory it is no longer stored in memory as a specific instance but as a general concept to be used in different tasks. The load on working memory can be reduced and is used for other components of the task like self-monitoring.

Declarative knowledge is necessary to elicit some form of behavior but it is only performed if certain productions coincide. There are productions which are frames for specific declarative information. In case of a sequential solution declarative knowledge about a graphics task (i.e. drawing a polygon) is knowledge about the nodes and the relations between the nodes. It is integrated into a general how-to-draw-a-picture frame. Acquisition of knowledge in Anderson's theory (ANDERSON, 1983) begins with using productions independent of skills on declarative knowledge. Then skill-specific productions are compiled.

In programming, declarative knowledge is used in the context of general productions which are not specific for the actual problem. Learning by analogy (HOLYOOAK & THAGARD, 1989; THAGARD, HOLYOAK, NELSON & GOCHFIELD, 1990) uses productions from a similar domain. Rules of the actual domain are integrated. A main difficulty for the problem solver in the initial stage is a possible overload of working memory. This may lead to a repeated computing of critical intermediate results, as mentioned by ANDERSON (1983, p. 231). It may as well result in forgetting intermediate steps and in a faulty solution of the problem. Knowledge compilation has two aspects, composition and proceduralization. Composition collapses two productions which are used in a direct sequence to a single production. Proceduralization builds productions which include declarative information. Proceduralization results in domain specific procedures. Composition assumes all relevant goal structures to be available in working memory.

ANDERSON, KLINE, & BEASLEY (1977, 1980) explicate three principal learning mechanisms in terms of their theory. These three mechanisms are generalization, discrimination, and strengthening. Generalization broadens the range of applicability of a procedure, discrimination causes rules to become narrower, strengthening implies that better rules are strengthened and poorer rules are weakened. An example for generalization is deleting a condition in a procedure. If there are two productions which have identical implications but differ in some conditions, only the common conditions are stored in the generalized production. Another form of generalization is applied if two productions differ in a certain aspect which might be represented by a variable. Using the variable results in a new generalized production. This kind of generalization should be applied if the use of variables is guided by newly detected relationships between the formerly included constants. The hierarchy of solutions will be related to the mechanisms of generalization, discrimination, and strengthening.

There was a report about an investigation on the acquisition of iterative control structures by high school students between 13 and 15 years of age earlier (HAUSSMANN & REISS, 1989c). The students were enrolled in a LOGO course for eight months and took part in interviews at the beginning of the course, after just a few hours of programming instructions, and at the end of the course. In both interviews the students were asked to solve identical problems. In particular, we intended to compare the results of the first and of the second interview as an informal evaluation of the course. Most of the tasks included the use of LOGO graphics commands. Every problem could be solved elegantly by using either iterative or recursive control structures. The groups of tasks represented the four areas of programming prerequisites as identified by PEA & KURLAND (1983): planning a program, writing programming code, understanding a program, and debugging a program. Our presentation of the results will concentrate on the second programming proficiency, the ability to write a program. The problem presented here asked the students to write a LOGO program that resulted in a given drawing; consisting of elementary geometric shapes.

The analysis of the students transcripts confirmed that the three different strategies mentioned above could be distinguished: They may be described as sequential, iterative, and recursive approaches (cf. HAUSSMANN & REISS, 1989b; HAUSSMANN & REISS, 1990).

The cognitive representation of the knowledge which is necessary for a sequential solution can be differentiated into three aspects according to the ACT* theory of ANDERSON (1983). He distinguishes declarative knowledge stored in long term memory, productions stored in production memory, and knowledge which is available in working memory during the problem solving process.

Applying this framework to our domain of novice programming three aspects which are important for the solution of the problem have to be represented in the declarative knowledge: (1) The problem solver must have a representation for the nodes involved in the problem and has to be able to differentiate between the various nodes. (2) The connections between the nodes should be combined in a network showing all possible relations and the directions leading from a specific node to its neighbor nodes. (3) There has to be a representation for the direction a problem solver will choose while trying to find a path through the network. The productions necessary in the process of finding a path through the network represent two main functions:

(1) Get a starting node. Whenever there are nodes which are not yet connected to each other one of these nodes has to be selected from which a path through the network or a part of such a path may be initiated.

(2) Find a suitable neighbor node for a selected starting node. Connect these two nodes. Try to use the same direction as in the last step but change the direction if necessary.

The flow of control of the involved actions is presented in figure 1.

The sequential solution of the problem is first refined when the problem solver recognizes that an iterative structure can be recognized in a drawing. After just a few hours of LOGO instruction the problem representation changes significantly. The problem is no longer regarded primarily in the context of everyday experiences and practical knowledge but in a way more appropriate to the programming language. There is an obvious shift of focus. Using not only elementary LOGO commands but also primitive objects like triangles or squares repeatedly during lessons results in an elaborated search for such shapes by the students. Moreover, using the REPEAT command in various situations enables the students to apply it in different tasks. A first refinement showing these characteristics is called the simple iterative solution.

A simple iterative solution takes into account, that a specific part of the figure may be drawn repeatedly. In the first task this part is a single triangle which will be drawn on the screen four times

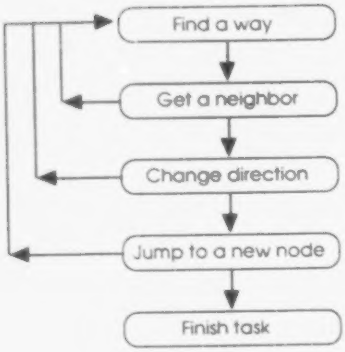

*Figure 1*: Control of actions in a simple programming task - sequential approach

in order to obtain the desired figure. It is important *to note* that triangles are not considered to be units with their own iterative structure. They are represented by three nodes which are connected through three lines. The representation in the second task is quite different. While working on this task the problem solver identifies a square as having an iterative structure. A square may be drawn on the screen by repeating the steps of drawing a line and performing a ninety degree turn four times. The repetition of the square is not considered as iterative element due to the increasing sizes. We assume that squares were used so often during the LOGO lessons that their iterative representation has become part of the declarative knowledge (cf. the role of episodic memory in WENDER, WEBER & WALOSZEK, 1988; SCHMALHOFER & KÜHN, 1990). On the other hand triangles were not discussed intensively during instruction so that their iterative aspects have to be identified with the help of procedural memory.

Accordingly we differentiate between two kinds of simple iterative solutions with respect to the specific task. In the first case the object endowed with an iterative structure is a row of equal objects, in the second case this object is a single square. In the following we will mostly refer to the triangle task and report on some aspects of the production system for a simple iterative solution and its Prolog implementation.

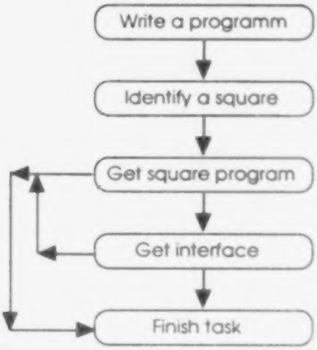

*Figure 2.* Control of actions in a simple programming task - simple iterative approach

A third group of solutions may be classified as complex iterative solutions. Complex iterative solutions take into regard that a nested iterative structure can be recognized in the problem. A problem solver using a complex iterative strategy takes into account that the drawing includes four triangles which may be drawn with the help of a REPEAT command but s/he uses the iterative structure of a single triangle as well. S/he recognizes the triangles, takes one instance of a triangle as well. S/he recognizes the triangles, takes one instance of a triangle, and determines a direction using only two points. The knowledge about triangles leads immediately to a representation using the REPEAT command.

Now the problem solver concentrates on the interface between the two triangles. Using this strategy s/he knows that this interface has to be defined only once and may then be included in REPEAT loop. Using a complex iterative strategy enhances the probability for a correct solution. The problem solver uses her/his declarative knowledge on triangles and interfaces.

The flow of control of the involved actions is presented in figure 3.

A comparison of figures 1, 2, and 3 showing the flow of control for the different problem solving strategies immediately reveals that these threes modes have different prerequisites in the supervision of the process. Whereas the sequential solution requires numerous backtracking steps, their number is obviously improved using a simple iterative approach. Performing a complex iterative solution does not require any backtracking during flow of control and puts only a minor load on working memory.

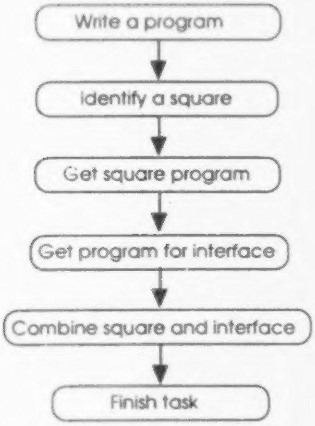

*Figure 3.* Control of actions in a simple programming task - complex iterative approach

The students' problem representations are quite different in all three situations. Accordingly, understanding the task of writing a program or translating this assignment to a specific cognitive model varies strongly among students.

Sequential approach:

„There are nodes which are in some way connected by lines. The point of writing a program is to produce a drawing for this situation. Every line one would draw with a pencil must be translated into a FORWARD command with suitable input." Most probably the sequential solution will not result in a correct program but can be used only in direct mode.

Simple iterative approach:

„There are certain nodes connected by edges. There are parts of the drawing which look like triangles. A program for such a triangle is needed, its execution has to be repeated four times." LOGOs direct mode is left, edit mode is entered, and a program is designed without seeing the immediate results in a graphics window.

Complex iterative approach:

„There are four triangles beside each other. Each triangle may be represented by repeating three times a FORWARD command with a suitable input and a RIGHT command. One needs an interface between the nodes for which a specific LOGO program is written. The triangle program and interface program are combined to a complete program that solves the task". The complex iterative approach is sometimes a prerequisite for a recursive approach. Usually students who have no difficulty in defining a complex iterative program are able to switch between this representation and a recursive representation without any problems.

The sequential and the iterative solution have in common a representation of the nodes, a representation of the network which has to be created by the problem solver, and a representation of the order of possible directions. Moreover, in both cases there is the possibility of translating directions into suitable LOGO commands, that is into LEFT or RIGHT commands. On the other hand, the most important difference between the sequential approach and the simple iterative approach is the representation of triangles in the iterative solution. Triangles are represented as a set of three nodes in any order.

The declarative representation of a triangle or a square by its nodes leads to a strategy which enables the problem solver to get a representation of the polygons in an ordered set of nodes. Moreover, the different polygons are ordered one after the other. The first kind of ordering results in a representation of a polygon using nodes and directions and thus in programming steps for a single polygon. These steps are missing the command for the turtle to change directions after arriving at the initial node. This causes errors in the definition of intermediate steps.

The complex iterative solution has three important features. The first one is the reason why the problem solver must not get a flexible representation of a polygon by walking from one node to the next. The representation is obtained by looking only for the first step which is generalized. The direction a problem solver gets is the direction which may be used repeatedly. Accordingly a polygon is defined by using the REPEAT command. Problem solvers using the complex iterative approach usually give definitions for the intermediate steps between the drawing of the polygons. These intermediate steps are included in the brackets and thus repeated four times. Problem solvers using simpler approaches will mostly not succeed in writig a correct solution.

Learning can be modeled by production systems. Moreover modeling helps to specify the formal structure of the learning process. Whereas earlier investigations (KURLAND & PEA, 1985; HAUSSMANN, 1986; HAUSSMANN & REISS, 1989a) distinguished between iterative and recursive strategies we now have a refinement. We can differentiate between simple and complex iterative strategies and now have a formalized way to distinguish them from a sequential strategy on one hand and from recursive strategy on the other. Furthermore the efficiency of the different approaches is explained in terms of memory load. The sequential strategy is only applicable to simple problems because the memory load becomes a problem in the process towards finding a solution. In contrast, the complex iterative approach minimizes memory load and thus allows capacity for the supervision of the process.

Some questions still have to be answered: The four strategies can be regarded as steps in a learning process. But how does the transition of one step to the next take place? Why do some students reach the next step and others do not? The results of our investigation are limited to a very restricted domain (i.e. learning to program in LOGOs graphical environment). It still has to be shown that the findings are valid outside the graphical environment (e.g. in using lists), for other programming languages (MÖBUS & SCHRÖDER,1988), investigating iterative thinking in the Tower-of-Hanoi problem (SIMON, 1975; HAUSSMANN & REISS, 1989b; HAUSSMANN & REISS, 1990). It seems to be worthwhile continuing to look for solutions within the theoretical framework described in this contribution.

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• Kristina Reiss is identical with Kristina Haussmann cited in many references of this publication.

# SELF-CONCEPT PROFILES AND TEACHERS OF MATHEMATICS: IMPLICATIONS FOR TEACHERS AS ROLE MODELS

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*This study examines the mathematics and problem solving self-concept of teachers and the possible modelling effects that depressed teacher self-concepts may have on students. In particular it explores for differences in self-concept among male and female teachers and the possible influence of such differences on the self-concept formation of girls and boys in primary and secondary classrooms. Sixteen teachers, eight from primary schools and eight from secondary schools participated in this study. An equal number of males and females were chosen, four of each gender displaying either very high or very low self-concept in mathematics and problem-solving. The teachers were interviewed to gain insights into how they formulated individual perceptions of themselves as mathematicians and their perceived influence as role models on their students in mathematical activities. The results suggest that primary teachers with depressed self-concept profiles may not provide adequate role models as consumers of mathematics for their students and that teacher training Institutions need to consider implementing specific strategies which address the amelioration of low self-concepts in mathematics, especially in the primary teacher training programs.*

## INTRODUCTION

The last two decades have spawned many research studies examining issues related to gender differences in the learning of mathematics. While there is still some debate about the nature of these differences and where and how they manifest themselves, some general conclusions are possible. The large body of research on performance variables is typically equivocal and not particularly conclusive. On affect however, there is considerable evidence that males are more positive about personal aptitudes in mathematics when compared to females despite the fact that similar performance differences are not substantiated by research evidence.

Brandon, Newton and Hammond (1987) reviewed major findings in studies located in the United States and concluded that boys' achievement levels surpass that of girls' levels at some point but that this point varies considerably from study to study. Although biological reasons have been advanced (e.g., Benbow & Stanley, 1983), sociocultural reasons are more widely accepted, particularly sex role expectations and gender identity (e.g., Meece, Parsons, Kaczala, Goff & Futterman, 1982).

Generally, some of this emergent male superiority is attributed to greater exposure to formal mathematics training (e.g., Etherington & Wolfe, 1986) but nevertheless evidence suggests that males are not superior in all aspects of mathematics (Barnes, 1983). A recent review by Willis (1989) suggests that while some differences in mathematical achievement do exist, albeit small, these change from culture to culture. A very recent American study (Friedman, 1989) which employed Meta-Analysis to investigate recent studies on sex differences in mathematical tasks concluded that such differences are now very small though on the side of male advantage and that such differences in performance are decreasing over the years.

Eccles (1982) claims that the greatest differences between boys and girls can be found in their attitudes to and self-confidence in mathematics rather than in actual achievement. Marsh, Smith and Barnes (1985), in a recent Australian study with fifth-grade students, found that although the girls outperformed the boys on a standardised maths test they nevertheless had lower mathematics self-concepts than the boys! Other studies (e.g., Thomas & Costello, 1988) also provide evidence of boys perceived superior competence and girls undervaluing their achievements in mathematics.

The link between the affective domain and performance has not to date been adequately established. The direct link between self-concept and achievement may be tenuous but the evidence that teacher attitude affects student performance is stronger. Midgley, Feldlaufer and Eccles (1989) clearly demonstrated that teacher efficacy has a strong relationship with students' self-perceptions about abilities in mathematics. In effect teachers' attitudes in mathematics may have an influence on how students perceive their own abilities to deal with mathematics. This association was found to be strongest for low achieving students who are taught by a teacher with low mathematics efficacy.

While there is no evidence that lower self-concepts of teachers directly impinge on the achievement of students, raising self-concept among novice teachers would seem a desirable goal. After all if they are going to impart knowledge it would seem advantageous to regard one's knowledge highly. Is it possible that we may be producing teachers who lack confidence in their own abilities as problem solvers and perhaps as mathematicians? Other authors in Australia pose similar questions, notably Sullivan (1987) and Watson (1987), as a result of research on attitudes conducted with novice teacher groups but generally in primary training.

In this study, sixteen primary and secondary teachers with extreme self-concept profiles in mathematics and problem solving were identified and attitudes and teaching styles among those with very high and very low self-concepts were examined. Observed differences in attitudes and behaviours suggest that teacher trainers should focus on the amelioration of low mathematics self-concept profiles, particulary among female primary student teachers. There is some evidence that teachers, particularly primary teachers with low mathematics self-concept, may have some deleterious effects on the potential of students to learn, appreciate and react positively to mathematical concepts. The major purpose of this research was to probe for how teacher self-concept evolves, the impact that such self-concept may have on the teachers ability to teach successfully and influence students undertaking studies in mathematics and in particular whether patterns are manifestly evident in differing attitudes to male and female students of mathematics.

## METHOD

### The Sample

Self-concept profiles were determined for 171 teachers - 85 primary teachers and 86 secondary teachers involved in the initial phase of the study. Sixteen teachers with extreme self-concept mathematics and problem solving profiles were chosen for the second phase of the study. Two male and two female teachers from each of the following categories, Low self-concept primary; Low self-concept secondary; High self-concept primary; and High self-concept secondary were invited to participate in the study. A total of 20 invitations were issued in order to attain the sample of 16. While there is no suggestion that this sample is representative, it does cover fairly extreme ranges of this population on the self-concept variable. Each individual who participated was generally at least one standard deviation above or below the mean of the group on these two measures.

### Instrumentation

Although a variety of measures have been used to determine attitudes one of the more salient measures of self-concept for adults is the Self-Description Questionnaire III (SDQ III) developed by Marsh and O'Niell (1983). The SDQ III is a multidimensional self-concept instrument with 13 subscales for which individuals respond to statements, approximately half of which are negatively worded, on a response scale which varies between "1 = definitely false" and "8 = definitely true". The subscales relating to general, academic, mathematics, problem solving and verbal ability were included along with a random selection of ten other items for a total of sixty items. This reduced the time required by teachers to respond to about ten or fifteen minutes. For the SDQ III the range of scores varied from a maximum of 80 to a minimum of 10.

The interviews focussed on four major issues related to self-concept formation, teaching strategies and subsequent fostering of attitudes in specific teaching environments. The unstructured format allowed for fairly open-ended response and the potential for the interviewers to probe further into areas which suggested particularly fertile implications for how teachers teach and, subsequently, students learn mathematics. The general areas of investigation included: formation of mathematical self-concept; the perceived role of the teacher and personal beliefs about the nature of mathematics and mathematics learning; gender differences in attitudes and approaches to learning; and, the quality and nature of teacher pre-service and in-service education and their possible consequences on professional development and in particular, self-concept development in mathematics and problem solving.

### Collecting the Data

Each school was visited and, in consultation with the principal and the Maths staff, completion of the SDQ's by staff was arranged. Staff were given the questionnaire and briefed on how to complete it. The questionnaires were collected on a subsequent visit. This approach though somewhat cumbersome guaranteed a large percentage of response. This was particularly important for the second phase of the research for which identification of participating teachers was required.

Case study methodology was chosen for the second phase of the study because the researcher was interested in subjective definitions of teachers' experiences in formulating their personal self-concepts and their perceptions about how these impinged on their own teaching capacity and the possible effects on their students. Every attempt was made to ensure that the interviewers followed the same protocols and a schedule of questions was provided. One factor which potentially could have biased the interpretation of responses may have been the fact that interviewers were aware of the self-concept profiles of the participants. However, while there is no direct evidence that this was the case and the researchers are confident that the reported outcomes are not biased as a result, some caution in the interpretation of the data, given the small number of the sample, and the fact that three interviewers were used, may be wise.

## RESULTS

Once all the data had been gathered from the participants and properly transcribed, the researcher looked for specific themes and trends in attitudes as expressed by the eight distinct self-concept groups. Major contrasts in attitudes were found between self-concept profiles but few identifiable differences were evident based on gender or school type. The expected differences based on gender did not materialise, rather there was a contrasting finding which pointed to an acute awareness of factors related to gender issues and an attempt to address these issues. Despite this, some common stereotypical attitudes did emerge.

The results here are presented as a synthesis of the perceptions of those with low self-concept compared to those with high self-concept in mathematics. Gender differences in perception are alluded to, in the instances where they occur, under the umbrella of the self-concept variable which contrasts characteristics, attitudes and perceptions of teachers to a much greater degree. Differences are also evident between primary and secondary teachers but again these seem to exist as sub-themes within the self-concept variable.

Detailed analysis of the transcripts revealed no real patterns of differentiation among teachers based on gender. None of the subjects reported any perceived bias based on gender in their own education either general or specific to their training in mathematics. The majority simply reported that if this were the case they had not been aware of it. Two female teachers, one in primary and the other in secondary stated quite firmly that they did not experience such bias. Interestingly, they represented both self-concept orientations but the primary teacher further commented "It was assumed I was like my mother, not good at maths and good at the social studies area. Typically, my brother was good at maths, and Dad was an engineer"!!

Views on gender achievement and attitudes were marked by a fairly consistent declaration that boys and girls were considered of equal ability by all of these teachers though some alluded to differences in attitudes and learning styles. Low self-concept teachers were more likely to allude to noticing such differences but the two primary female low self-concept teachers clearly expressed fairly stereotypical views about girls' attitudes to mathematics. Many of the comments qualified any noted differences by statements that boys and girls were equal in performance. Major differences noted included levels of persistence (boys more persistent), need for reinforcement (girls need more) and behaviour (girls quieter, work harder). In general, most teachers commented that in group work they prefer mixed groupings presumably on the rationale that this represents a balanced and socially aware position that purports equality of opportunity for boys and girls. Significantly both female primary teachers expressed views about how girls learn mathematics which would at best be defined as stereotyped and quite possibly as influential and perpetuating fairly anachronistic attitudes towards girls studying mathematics.

In general, subjects who reported low self-concepts in mathematics and problem solving tended to report negative and unrewarding experiences in learning mathematics. Some reported traumatic school experiences (i.e. failures, teacher over-reaction, poor test performance, parental disapproval etc.) which soured their taste for any activity that was mathematical in nature. In some cases traumas resulted in loss of confidence and the liking for mathematics that had initially existed. Such experiences tended to occur in the upper primary years.

Expressions of dislike were more strongly presented by the primary teachers. The secondary teachers tended to be more neutral in their reflections on the formation of their self-concepts and in two cases a recovery at the tertiary level was quite evident. This was ascribed to improved performance associated with a master teacher who ameliorated an increase in confidence through patience and what was perceived as personal attention to them as individuals. Both expressed their belief in the importance of teachers as role models.

Predictably, the high self-concept teachers expressed general feelings of confidence, a liking for their subject and reported positive recollections in learning mathematics. There was greater indication of parental support in all cases but more importantly, five of this group recollected a teacher who had a very positive influence on their performance and attitudes. Such teachers were always described in similar ways. They were laid back, obviously knew their subject but more importantly enjoyed their subject and seemed to establish a personal rapport with individual students.

Almost without exception the low self-concept group described themselves as traditional in their approach, chalk-talk was mentioned on several occasions. Six of the eight (3 male, 3 female) declared their preference for the use of texts particularly at secondary level. While a number recognised that there were many options open to them in terms of possible teaching strategies to use most declared a lack of time, disinterest, lack of motivation or inspiration to do much about this. Some expressed the need to make mathematics relevant and more concrete (2/8) but the need for making students aware of the real life applications of mathematical principles tended to be more generally recognised by the high self-concept cohort (5/8). In general, the attitudes of the low self-concept subjects tended towards the negative and expressed low expectations of their students. The consensus among this group was that family and societal influences were very strong in determining attitudes and therefore self-concept in mathematics. Most expressed the view that once this attitude was formed it was difficult for the teacher to affect change.

Quite clearly the high self-concept cohort expressed a much more positive view of themselves as teachers of mathematics and of their own students. Unlike their low self-concept colleagues, this group viewed their influence over their students as important although they still conceded that societal and family influences would be difficult to overcome. They also expressed a need to relate mathematics to real world activities and to offer a variety of concrete based activities, interestingly at the secondary level (2/4) as well. They tended to allude to language which accentuated positive results and saw maths as "do-able" by most students providing it was properly presented as an enjoyable activity. Their approach to teaching tended to be less traditional, more group and individually oriented, far less dependent on tests and always monitored in order to determine how students progress.

In discussing their pre-service training and on-going professional training through in-service, three major themes emerged. Many commented on the contrast between content and methodology courses, some preference for in-school professional development and, though not as frequently as the first two, the need for a problem-solving based approach to pre-service training. A high proportion of the participants (11/16) strongly suggested that their in-service training was not entirely successful in introducing them to the skills required to become proficient in teaching mathematics. The high self-concept teachers tended to be more positive about the value of in-service but also tended to prefer in-school sessions and co-operative efforts which didn't require time outside the normal school hours.

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## SUMMARY AND DISCUSSION

Analysis of the data clearly reveals some important contrasts in the manner in which low self-concept and high self-concept mathematics teachers approach teaching their subject. There are also some gender differences between male and female teachers' attitudes and approaches to teaching mathematics but these tend to be minimal and much more likely to be found at the primary level. Male and female teachers in secondary schools have similar training profiles and therefore are more likely to be homogenous in their approach.

The contrast in attitude and enthusiasm for the subject between low self-concept and high self-concept mathematics teachers does flag some important differences. The high self-concept cohort seemed more motivated, more inventive and creative about how to conduct maths lessons. Unlike the low self-concept teachers they not only verbalised the need to make maths relevant and understandable through concrete activities, they generally tended to provide the appropriate working environment. High self-concept teachers also tended to remember some teacher in their student experience who was inspirational and provided a positive role model for them. Early experiences with mathematics, here defined as school experiences at the primary level, and parental attitudes towards the subject undoubtedly had a strong influence on the self-concept profiles of this set of teachers. Traumatic experiences, usually in the upper primary grades, could potentially have a life-long impact on self-concept.

In contrast to the high self-concept group, the low self-concept group were more likely to be negative and complain about lack of time and resources to implement what they clearly understood to be the more successful ways of teaching mathematics. Their admitted lack of love and appreciation for the subject and limited creativity constrained their likely success as teachers in this area. They perceived a lack of proper pre-service training, that is, a much greater emphasis on methodology rather than content, and inappropriate or infrequent in-school inservicing and professional development opportunities as key factors which limited their potential as teachers. Lack of resources and time were also often alluded to as disrupting influences which made the teaching of mathematics a more difficult proposition.

Based on the interview data, the belief that self-concept must influence the way in which teachers interact with students is also supported by Midgley's *et al.* (1989) findings. It was evident in their study that teacher efficacy, a measure of their attitudes and beliefs about their role as teachers, had a powerful effect on their student's affective states, in particular those of low achieving students. In essence, teachers with negative attitudes and low expectations of their students tended to inculcate similar attitudes in their own students.

It is difficult not to conclude from the evidence presented that teacher self-concept in mathematics does lead to qualitative differences in the teaching of mathematics. From the teachers' perspectives concerning their schooling, early experiences were clearly influential on the formation of their own self-concepts, but it was not possible to make an inferential leap about what influences they themselves have on their own students. To find out much more in-depth interviewing of students over a long period of time, at least a term, would be required. Given the importance that low self-concept teachers put on traumatic events undermining their own attitudes to mathematics and the importance that the high self-concept teachers put on the influence of master teachers, one wonders whether they are perpetuating their own past experiences and projecting those on to their own students?

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# THE ELABORATION OF IMAGES IN THE PROCESS OF MATHEMATICS MEANING MAKING

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## ABSTRACT

*There has been increasing interest recently in the role of imagery in students' thinking as they attempt to make sense of mathematics tasks. In this study, five key informants provided information about their use of imagery in mathematics problem solving. Based on this research, we believe that meaningful mathematics is image-based and that rule oriented activity may not encourage the use of imagery in students' mathematical thinking.*

## Introduction

The role of imagery is increasingly being recognized as important to the activity of sense making in mathematics (Dreyfus, 1990, Brown & Wheatley, 1989, Cobb, Yackel & Wood, 1992). Lakoff (1987) and Johnson (1987) have proposed a theory of meaning based on a concept of 'image schemata', which Johnson describes as: "structures that organize our mental representations at a level more general and abstract than that at which we form particular mental images" (p.23). Lakoff develops a theory of Idealized Cognitive Models which give shape and pattern to human reasoning. In attempting to translate Johnson and Lakoff's work into the mathematical setting Dorfler (1991) suggests, "The cognitive manipulation of mathematical concepts is highly facilitated by the mental construction and availability of adequate image schemata" (p.20).

## The setting for this research

This research attempts to identify the ways in which the learner's activity of mathematical meaning making relies on the use of imagery. For three semesters the researchers have been present daily in a primary grade classroom during the mathematics lesson. During their second and third grade years the students experienced mathematics in a Problem Centered Learning Environment (Wheatley, 1991) where they were encouraged to make sense of mathematical tasks and where algorithmic procedures were not taught. During the first semester of their Grade 4 year four "key informants" from this group were identified for this research - Kristen, a mathematically gifted student, Tracy, considered by her teacher to be "weak" in mathematics, Elaine and Neal, considered to be of "normal Grade 4 competency". Louis, a six-year old student whose mathematical development is well beyond his chronological age was included as a fifth key informant. Thus far two hours of video recorded interviews with each student have been conducted.

During an interview the student is presented with a non-routine mathematical problem to solve. S/he is encouraged to think about the problem, talk about how s/he is "seeing" the problem and, when necessary, use pencil and paper to aid in finding a solution. At the end of this time the student is shown the videotape of her/himself attempting to solve the problem and is encouraged to reflect on her/his activity and "recall" any images s/he had while thinking about the problem.

## Analysis and discussion

Each of these students shows a rich but different use of imagery as they attempt to solve problems, and to explain their mathematical activity. It is evident that image construction is important as they attempt to make sense of a mathematical task. Three students were given the following task (Presmeg, 1989):

> A dog gave chase to a fox, which was 30 meters away. The dog's leap is equal to two meters and the fox's leap is one meter. While the fox makes three leaps, the dog makes two. How far does the dog run to catch the fox?

Each attempted to solve this problem by the use of some form of imagery, but each in different ways. Elaine began by saying she had a picture in her mind of a dog at one end of a path and a fox at the other, and described each animal in detail. She then drew that picture on paper using two parallel lines across the page and a small rectangle at each end with the words "dog" or "fox" written above each. Kristen also described a picture of the dog and the fox at opposite ends of a path, then drew a line across her page and put a circle at one end for the dog and a triangle at the other end for the fox. There the similarity ends. Elaine began reading the problem again and then marked off 30 segments along her path. She explained that her "brain was not big enough to hold the 30 meters" so she drew it on her paper to help her think about the problem. Thus she began to interact between the image she had in her mind and the pencil and paper diagram, something she continued to do throughout her problem solving. Eventually she decided (after several diversions using other images, including picturing the possibility that the fox would be "over the hills" and gone before the dog ever caught him) that every four squares indicated a gain of one meter for the dog. She then extended her path sufficiently around her page so that she could count off 30 blocks of four squares, at which point she explained that the dog had caught the fox. While this became a lengthy counting activity, fraught with errors (she lost count and needed to repeat her actions several times) in the process of developing this scheme she grew increasingly sensitive to the interrelationships between the various aspects of the problem and, through her increasingly elaborated imagery, brought together the mathematical relationships she had constructed in making sense of the problem statement.

Kristen took a different path. When she drew her line to indicate what she was thinking, she marked off 30 strokes in order to position the fox 30 meters away from the dog. She then said: "This is way too slow". This comment suggests that Kristen was thinking critically about her image and its practicality for her. After examining the problem again she began to organize her image differently and, as she tried to coordinate the various relationships of the problem, wrote the following scheme on her paper:

| | | | | |
|---|---|---|---|---|
| dog | 2 | 4 | 8 | 3 |
| fox | 1 | 3 | 6 | 4 |

She was searching for some pattern of relationship in these numbers. Eventually she said that the dog would gain one meter every time. At this point she said her answer was 60 meters because "double 30 was 60". The interviewer tried to encourage Kristen to reflect on her answer; however Kristen indicated she had found a procedure that she thought would work (referring to her doubling of 30) and so far as she could tell, her answer was 60.

Then something interesting happened. Kristen said she was "unsure" about her answer. When asked if she could find a way of becoming "sure" she said that she would need a "checkbook" (an answer key or someone to check the correctness of her answer). She was no longer thinking about the problem in terms of sense making but in terms of the procedure she had used to reach an answer. This difference in the pattern of interactions with the problem is explained subsequently as she begins to discuss her "unsureness". She explains that if she had been given a problem that had an addition sign she would know what to do, and unless she made a counting error, would be "sure" of her answer. Kristen was encouraged to use the imagery she had developed in thinking about the problem as a way of making sense of her answer. However, she appeared to have switched from the meaning making mode to the procedural mode in her mathematical activity. This raises the question: Is imagery involved in the procedural mode of the doing of mathematics? And here an incident that occurred with Louis offers some insight.

I. What's 150 + 25?
L. 175.
I. What did you use to figure that out?
L. Well you use 20 and first you said 150 and a hundred, and five plus two equals seven and you said 25 so I said 75. My brain is sleeping. I already knew the answer so I just had to say it out loud.

A short time later the following problem was given to him:

I. If you had 15 cookies and you needed to share them fairly among three people how many would each get?
L. You mean like my teacher, you and me?
I. O.K. so that's the 3 of us. How many would we each get?
L. Hm, that's a hard'n. That's one I don't know and I have to scream to wake braino up.

Louis immediately used imagery to think about the problem. He put the problem in some meaningful setting in order to make sense of it and he was aware of his need to think about it. He is using a sleep/wake metaphor to describe his experiences with tasks that suggests that in settings where an answer is routine he is not conscious of any imagery as he answers a question, while for questions which require sense making on his part he needs to build images in his mind to help him think about the problem.

Neal experienced the dog/fox problem somewhat differently from the other two students interviewed using this task. He said that he did not actually see the dog and the fox but focused on the numbers in the problem. He explained:

> I was just running through my head, the dog makes 2 leaps, the fox makes 3 leaps. But you know after all, the fox does make 2 leaps; but every leap it makes it is 2 meters. 2 times 2 is 4 so it is coming ahead and it has 4 meters, but the fox it jumps 3 leaps but the, the fo. . ., dog still catches up, because it's 3 leaps. Still, you know, every leap is only one meter so the dog's still catching up.

He was 'running' through the parts of the problem until they came together and suggested a pattern for him. He said, as did Kristen, that the dog would gain one meter each segment so the answer was 60. He did not use pencil and paper for the task. In the second part of the interview he explained his thinking in this way:

> I. You said at the beginning there that you were running those through your head. How were you running them through your head?
> N. Just thinking about the fox's three leaps and the dog's two leaps and finally I ran it through my head enough and thought about it enough and saw that the dog would still jump ahead by one.
> I. And was it the words running through your head or was it a movement? What was it running through your head?
> N. The words. The thoughts.

When he used pencil and paper in attempting to explain his answer he drew a picture of a very short line with stick figures at either end and with his hands indicated a hopping movement as he explained how he worked out the relationship between the various parts of the problem. The drawing was a very primitive illustration of the problem and it became obvious that it was not part of his imagery in thinking about the problem.

Somehow in relating the numbers in some fluid way in his mind he constructed some coordination between the various mathematical relationships of the problem. However, as with Kristen, the quality of the images he developed as he arrived at a solution did not adequately include all the dynamics of the situation.

Tracy's use of imagery in her first and second interview were in sharp contrast to each other. In the first interview she had to find a way to put 15 tigers into four cages so that no two cages had the same number of tigers. At the end of fifteen minutes she had one solution. Her demeanor as she struggled with this task indicated that it was very difficult for her. During much of the working time and while she was watching herself subsequently her mouth was covered by her hand/s and her shoulders were stooped; her body stance reflected her hesitancy with the task. In the second session she was to find 16 coins (quarters, dimes and nickels) which add to $1.85. Her body stance during this session was much more erect and expressive of openness; she was at ease with the problem. At the end of the interview she said that she enjoyed doing this problem more than the tigers and cages problem because "I had so much to think about on this" (referring to the tigers and cages problem). This difference can be explained in terms of the images Tracy was able to form and 'unpack' from previous problem solving in dealing with the second task, images which were not available to her on the first occasion. The money setting allowed her to think in terms of familiar images, particularly in dealing with 25. She had an image of a quarter as 25 and of a nickel and two dimes as 25, as well as an image of four of these to make $1. She confidently explained she could use a quarter and then three groups of a nickel and two dimes to make $1. The quarter has thus become a metonomy (Lakoff, 1987) for thinking more generally with groups of numbers.

For Tracy, this image was still in need of development. When she attempted to arrange the coins to make the 85 cents she did not initially think of groups of 25 within 85. She drew a quarter, then a nickel, while she counted on with her fingers; "25, 26, 27, 28, 29, 30", then added a dime and counted on with her fingers, and so on until she reached 85. This part of the task was more difficult for her. However, as she explored a second and then a third solution, she 'unpacked' the quarter and its related "twenty-fiveness" several times, so that by the time she was making her third arrangement she began to think of 85 as having 'quarter' parts, saying "I am going to use 3 quarters equals 75, so I did not have to use so much change". As with Elaine, in thinking about the dog/fox problem, Tracy elaborates her image as she interacts with it in her mind and as she draws what she is thinking about on paper. In the language of Lakoff (1987), Idealized Cognitive Models, are being established and elaborated in this setting. Few such images were present in her activity with the tigers and cages task.

## Conclusion

Each of these students used imagery as they attempted to give meaning to mathematical tasks. As Lakoff (1987) suggests, interconnections are formed and reformed as the student attempts to make sense of a task. On several occasions a dialectic between the images in the mind and pencil and paper activity led to the student constructing and coordinating problem relationships. Based on this research, we believe that meaningful mathematics is image-based. Our research suggests that images may not be used when students perform prescribed computational methods.

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# TEACHER CHANGE: A CONSTRUCTIVIST APPROACH TO PROFESSIONAL DEVELOPMENT

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*This paper identifies conceptual links between constructivist ideas and effective professional development practices and argues that if constructivist principles are to underpin mathematical learning environments for children, they should also provide the basis for professional development programs. The paper reports on the way in which one model, which is consistent with this thrust, has provided the opportunity for teachers to construct new understandings about teaching and learning, the roles they assume and the nature of change.*

## Introduction

For some time now, the issue of teacher change has been of concern to mathematics educators, for it is widely accepted that mathematics teaching practices have been too narrow in focus and have relied too heavily on a teacher-directed, transference of knowledge approach. Teachers are being urged to take account of constructivist ideas when planning and implementing their programs, but the same need has not been emphasised in regard to professional development. If it is desirable that constructivist principles should underpin mathematical learning environments for children, then it seems appropriate to suggest that they also be adopted as a basis for the professional development of teachers.

Scant attention has been paid to the philosophies underlying professional development in the past. Approaches have been traditionally characterised by what Fried (1980) refers to as the 'delivery of services' mentality, where there is centralised decision making about problems, needs and remediation strategies. Teachers themselves have had little input into this, in spite of their well developed understanding of how schools operate and how children learn.

Whilst it is beyond the scope of this paper to deal adequately with the complex nature of links between social constructivist ideas and the vast literature on professional development, I wish to draw attention to some of the key ideas of both as they relate to teachers as learners. This paper also reports on research into the impact of one approach to professional development which is consistent with constructivist principles and which represents a radical shift away from the once dominant transmission approaches to professional development.

## Constructivist principles for teaching and learning

Constructivist thought generally assumes that people constantly participate in the social construction of 'reality', where forms of understanding depend on the fluctuating circumstances of social processes. The idea that knowledge cannot be taught but must be constructed by the learner is central to the various constructivist perspectives which have been articulated across a number of disciplines. Control over the learning process therefore resides with the learner. Candy (1990, p.8)) maintains that

> constructivism in education is concerned with two things: how people construe (or interpret) events and ideas and how they construct (build or assemble) structures of meaning. The constant dialectal interplay between construing and constructing is at the heart of a constructivist approach to education, whether it be listening to a lecture, undertaking a laboratory session, attending a workshop, reading a text, or any other learning activity.

It follows from these remarks that educative processes need to provide the maximum opportunity for learners to 'reconstrue' ideas and events so that they are not locked into particular interpretations and constructions. Socially valued knowledge is continually created and re-created by members of a community through an interactive negotiation process.

One influential proponent of constructivism is von Glasersfeld who focuses on these fundamental principles in his articulation of a radical constructivist epistemology which has built on and extended the work of Piaget. He contends that knowledge acquisition is an adaptive process whereby the cognising subject organises knowledge so that it fits experience rather than matches an external reality outside the mind of the knower. Furthermore, he asserts that knowledge cannot be acquired passively: learning can only take place when existing cognitive structures meet with perturbations. Von Glasersfeld (1989, pp.136) emphasises that "the most frequent source of perturbations for the developing cognitive subject is the interaction with others". The implication for educators is that group learning is an important practice because it enables learners to construct a viable model of the world through trial and error. A socially constructed 'reality' can be built up through the shared meanings which arise out of social discourse.

Von Glasersfeld (1983) concedes that problems arise when epistemological considerations are applied to education. Nevertheless, a consideration of constructivist principles has given rise to what he sees as five consequences for teaching practice, which are paraphrased below.

- Teaching for understanding is distinguished from training which results in repetitive behaviour.
- The thought processes of a learner are given more attention than overt behaviour. The validity of the state of thought of a particular student is emphasised, therefore the deficit model of a learner is unacceptable and indeed is rendered irrelevant.
- Transmission of knowledge approaches are seen as being undesirable and ineffective: rather linguistic communication in the form of negotiation is used to guide the students as they construct their own knowledge.
- Errors are seen as being a natural and valuable part of the learning process and provide information about a learner's current understandings.
- Teaching interviews are directed as much towards modifying student's cognitive structures as inferring them.

If the pedagogical principles outlined above are juxtaposed with the professional development practices which are known to be effective, it is evident that there are some clear links.

## Effective professional development practices

During the past 20 years, there has been an accumulation of research literature on in-service education, so much more is known about the problems associated with change in schools. The work of Doyle and Ponder (1977), Joyce and Showers (1980), Huberman and Miles (1984), Pinks (1989), Fullan (1990) and others has clearly indicated that there are a number of factors influencing teacher change which had not previously been taken into account by those pressing for reform. These factors relate to the nature of teachers as people, schools as organisations and change processes themselves.

Meta-analyses of research on staff development (see Showers, Joyce and Bennett 1987, Wade, 1985) demonstrate that there is a high degree of consensus about what constitutes effective professional development practices. Sparks and Loucks-Horsley (1990) have summarised them as follows:

> (a) programs conducted in school settings and linked to school-wide efforts; (b) teachers participating as helpers to each other and as planners, with administrators, of in-service activities: (c) emphasis on self instruction, with differentiated training opportunities: (d) teachers in active roles,choosing goals and activities for themselves: (e) emphasis on demonstration, supervised trials and feedback: training that is concrete and on-going over time: and (f) on-going assistance and support available on request. (p.13.1)

These practices are quite consistent with constructivist ideas. In particular, the notions of learner direction and control, trial and error, learning in context and the recognition of individual understandings and construction of shared meanings are key features in each case.

The issue of professional development for teachers of mathematics is now being treated as problematical and worthy of attention. In developing in-service programs, more consideration is being given to the pivotal role of teachers as well as to the institutional and organisational nature of the process of change. Discussing the need for an entirely new approach to professional development, Robinson (1989, pp.274) has stated,

> The attempt to implement a specific reform or the application of a specific change model to a school situation generally only leads to disillusionment and frustration, because it focuses on end products, on what things should be like and not on what the reality is. Change processes are fitful, irrational, unpredictable. ....Professional development is, or should be, an educative process and in education the ends or objectives are always problematic.

He also suggested that since there is no prescription for "exemplary practice which fits all teachers in all situations, any model of change that relies on determining ends, then devising means will be inadequate, including all those within the management paradigm".

Some of the most recent professional development initiatives have arisen out of the Basic Learning in Primary Schools (BLIPS) program which was a project funded by the Australian government in 1984. Its aim was to increase the literacy and numeracy skills of children in infant grades by raising the confidence, competence and interest levels of teachers and by ensuring that teachers were central to the change process. There are now a variety of ways in which mathematics teachers can be involved in professional development. The most popular programs are the school-based models where teachers are encouraged to increase their understanding of the processes of teaching and learning by researching their own practice in collaboration with their colleagues. One such model is known as Key Group.

### The Key Group model

Key Group is a school-based professional development model which was developed by Alexopoulus (1985), Barry (1985) and Robinson (1986) in response to a perceived need for a model which placed teachers in control of the directions and momentum of change and which took account of the latest research findings on factors most likely to effect lasting change. It is based on the premise that imposed change is seldom effective.

The model involves the formation of key groups: each group consists of three teachers from one school, an outside consultant and, in many cases, a parent. At the commencement of the project, a number of Key Groups attend a three day live-in conference where they participate in seminars and workshops which focus on current ideas about mathematics teaching, action research principles, and research on change processes. They are also involved in discussion groups where they are encouraged to reflect on their teaching practices, identify their strengths and weaknesses, decide on an area which they would like to change and subsequently plan a course of action for achieving such change. This action is referred to as the group 'focus'. An important aspect of the conference is that teachers begin to understand what it means to be in control of the change process. When they return to their schools, they continue to work collaboratively towards implementing their action plan. Continued support in a number of forms is provided by the Ministry to enable them to do this.

This approach to professional development is consistent with constructivist principles in the following ways. It does not rest on the prescription of desired practices or the transmission of knowledge: rather, the workshops and discussion groups provide teachers with the opportunity to construct and refine understandings about themselves as professionals and about teaching and learning processes. They are encouraged to become reflective practitioners and assume the role of 'critical friend' for their colleagues. Risk-taking is a valued feature of the model, for it is recognised that trial and error leads to further learning. The beliefs and ideas of participants in the program are considered to be valid and are used as starting points for the development of new understandings.

Key Group offers a workable alternative to the many managerial style approaches which have dominated teacher training and professional development programs in the past. It is one example of a professional development model which places teachers squarely at the centre of the action, for it allows them to construct the change process in the way they see fit. It is not just content and issues based however: it provides teachers with the opportunity to learn new ways of thinking and working. By removing some of the constraints to change, it allows teachers to concentrate on implementing their change.

Initial evaluations of the model have found that teachers reacted very favourably to this form of professional development and there were a number of highly successful projects developed during the period of teachers involvement. This research has been undertaken to determine long-term effects of the model, that is, the extent to which teachers have maintained and continued to build new practices and knowledge about practice.

## Methodology

Individual interviews were conducted with five classroom teachers from the infant grades of one primary school, five years after they participated in a Key Group program or worked closely with participants. Numerous mathematical activities devised by these teachers were made available as was a video featuring an interview of two of the teachers by a television presenter. The interviews sought to collect data about the extent of teachers growth as professionals, their development of new understandings about teaching and learning and the way in which they constructed a process for change in their school.

## Results and discussion.

The overall impression reflected in research results was that Key Group has had a significant effect on the professional lives of participants and on the programs in the infant grades. It has led to new insights into pedagogical processes and forged new ways of working. Prior to Key Group, mathematics classes in infant grades were characterised by teacher-directed lessons. Each topic was dealt with as though it were a separate entity: few links were made with other mathematical topics or other curriculum areas. As one teacher commented:

•*We were doing really interesting things in language and general studies which related to overall themes. Maths was different and so boring in comparison because we seemed to be always doing the same sorts of things. My maths was a completely separate box. I always seemed to struggle and think what am I doing for maths this week.*

Teachers had reached the point where they realised they would benefit from professional development in the area of mathematics teaching and so became involved in Key Group.

### Construction of the change process.

During the time of their involvement in the Key Group conference, teachers came to realise that they were in control of the focus, directions and momentum for change.

•*I went along to Key Group thinking I would have new ideas handed to me on a plate, but it wasn't like that. I learnt that it was not meant to provide all the answers, but what it did was set up a way of thinking which helped us to find our own answers. We learnt how to question everything we did and why we did it.*

There was also an increased awareness of the notion of self-direction.

•*We were looking for some direction in our maths program because it wasn't going anywhere but rather than telling us what to do, Key Group encouraged us to set our own goals and make our own decisions. It also allowed us to address an area of our teaching which we were concerned about.*

During discussion sessions teachers began to understand the nature of collaborative processes and the feeling of support that can be derived from this way of working.

•*Although I have always lacked confidence in maths and in teaching it, I found that I was able to participate in the decision-making processes. Talk with other teachers was so valuable. It made me aware of the kinds of changes I needed to make and it stimulated my thinking.*

Another teacher was reassured that expected change was not forced on her.

•*Key Group gives the confidence to change as and when we are comfortable. There is no pressure but much encouragement to work at improving our teaching in our own way.*

When deciding on their focus for change, the teachers started out with the idea of making their program more activity-based and continually searched for ways to do this.

•*The brainstorming sessions were just so valuable. We were intending to make up as many activities as we could to fit our maths topics. But then, Sue said "What I'd really like to do is fit all of our maths into our theme work, so that our whole curriculum is integrated". That really started us thinking. I suggested that if we line up our maths program with our themes, we would probably find some overlap. Once we did that, tit was amazing how many connections we could make.*

The task then became one of identifying the mathematics which they wanted to include in their program, then constructing a number of appropriate activities around the designated themes. This process significantly broadened their view of the nature of mathematics.

After deciding what they wanted to do, the next stage of the process was to plan how to achieve it. This was a problem because it was apparent to teachers that the amount of work which needed to be done was daunting. It was evident here that they began to see that problem solving is an important part of the change process. It was assumed that they would have to do all the work themselves until one teacher mused: *Perhaps we could get the grade 6 children to help make the activity sets.* This comment opened up new possibilities in terms of how the change could be achieved. *What about parents?* The idea of enlisting help grew from there and they decided to approach other teachers, children and parents to work with them. Here there is the realisation that change is the concern of the wider school community, not just classroom teachers.

With the knowledge that Key Group would release them from their classrooms for a few days, they returned to their school to begin constructing the activities. They encouraged other teachers in their school to work with them. Some teachers responded positively.

•*Their enthusiasm was contagious and their choice of focus was the sort of thing I had been looking for. I have learnt that working together is so valuable and generates so many more ideas. Their ideas influenced me and my ideas influenced them, so we were always coming up with better and better ideas.*

The collaborative process which began at the Key Group conference maintained momentum back in the context of their school. The opportunity for colleagial reflection and discussion, which was provided by the program, allowed the formation of a strategic action plan based on a newly clarified and shared philosophy.

### Development of understandings about teaching and learning.

As they shared and developed their understandings about teaching actions, their self esteem and confidence grew. By working collaboratively and continually exchanging ideas, new insights were gained into the process of learning. This confirmed their beliefs about the value of child-centred, activity-based learning and the crucial role of language.

•*We realised that we were actually practising what we preached. Just as the children were working co-operatively and learning from each other as they talked, so were we. Because we had experienced this way of working, we were absolutely convinced that children would learn better this way.*

Being placed in the position of learner heightened teacher's awareness about how learning occurs and enabled them to theorise about their teaching to some extent.

•*We found at the Key Group conference that if we wanted help in sorting out our ideas, there was always someone to ask. The consultants there didn't force their ideas on us, but they made us think, usually by asking questions. In fact one of the workshops dealt with questioning techniques and the power they have in stimulating thinking. We then began thinking about how we learnt and translating that to children's learning.*

*•Some of the presenters were talking about constructivist approaches to learning and I found that what they were saying was similar to my own beliefs about teaching. When I want children to learn something new, I know that just telling them won't work. They have to be able to link the new ideas with what they already know and to be involved in some sort of activity. It occured to me then that this program does the same thing. It doesn't tell us what to do but it provides starting points.*

Reflective teachers are able to theorise their teaching which in turn informs their practice, setting up a dialectal process of reflection and action.

Mathematics classrooms subsequently became quite different environments. They were more learner-centred and allowed for much more interaction and discussion about mathematics than had previously been the case. Activities were organised in such a way as to enable children to work at their own pace and level and to have some choice about which activities to complete.

Construction of new roles. The change in classroom learning environments meant that teacher's roles were significantly different. In fact, from the point of view of professional development, one of the most notable outcomes of Key Group was the way in which teachers constructed new roles for themselves, not only in the classroom, but in the wider social context.

In the classroom, teachers acted as facilitators to the learning process. They were less directive and allowed children the opportunity to explore and question ideas as they negotiated shared meanings through discussion. They also made more use of questioning techniques to extend children's thinking. In other words they often provided the perturbations which were necessary for children to build new cognitive structures.

The development and use of new materials and strategies aroused the interest of other teachers in their own school and in other schools in the region. Three of the Key Group teachers became facilitators in the learning process of their colleagues by creating a collaborative environment. The opportunity to share and discuss ideas enabled other teachers to be exposed to and reflect on new ways of working. They also undertook to give demonstration lessons so that colleagues could observe how children used the new materials.

*•The interest was incredible. We were virtually in servicing and had teachers in and out of our classrooms all the time, not just from our school but from other schools as well. The thing that amazed me was that the kids were seldom distracted from their activities.*

One of these three teachers subsequently obtained a position as a regional consultant. In this capacity, she became one of the organisers and presenters of the Key Group conference. The other two have become well known throughout the region and have since published a collection of the activities they constructed.

*•If someone had told us a few years ago that we would be authors of a teacher resource book we would have laughed. But it's happened and I really put it down to Key Group.*

The success of the program in their school and beyond has strengthened their perceptions of themselves as curriculum developers and decision makers.

Another teacher constructed a different role for herself. She had always valued the importance of being in control, so the most valuable aspec: of Key Group for her was the fact that she became aware of the political processes which affect so much of what happens in schools. She became familiar with a number of administrative procedures to which she had not previously had access. Her sense of empowerment was heightened as these remarks indicate:

*_It was always the senior male staff who had to deal with administrative matters. Female staff were usually left uninformed,but now we have a better idea of what is going on and can make decisions which suit us._

## Concluding comments

As a model for professional development, this study indicates that Key Group is effective because it provides the opportunity for teachers to actively engage in a continuing dialectal process of reflection and action aimed at creating circumstances which facilitate the construction of new understandings about teaching and learning. Participants in this study have an increased awareness of their worth as knowledgeable, capable professionals. The model incorporates practices which research has shown to be effective and allows teachers to choose directions which suit the context of their teaching. At the same time, it offers some guidance as a starting point from which ideas can be generated, created and recreated. This process is ultimately empowering because it creates a professional development ethos in teachers which can be carried on throughout their teaching lives.

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# CONNECTEDNESS IN TEACHING ALGEBRA: A NOVICE-EXPERT CONTRAST

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*This study examines differences between two novice teachers and an expert teacher in presenting mathematical materials in a connected manner. Three types of data related to lessons on equivalent algebraic expressions were collected: lesson plans, lesson observations, and post-lesson interviews. Although connectedness is an important characteristic of mathematics teaching and learning, only the expert teacher used both vertical and horizontal connections to guide her teaching. Differences in the teachers' view and use of connectedness are discussed and illustrated.*

## Introduction

During the last decade, the study of expertise in teaching has received a great deal of attention (Berliner, 1986; Borko & Livingston, 1989). In the realm of mathematics, the study of expert teachers has focused, so far, merely on expertise in the teaching of arithmetic (Lampert, 1988; Leinhardt, 1989). These studies have identified several dimensions in which competency differences between experts and novices are likely to occur, including planning actions (agendas), managing action systems, and building explanations of mathematical materials. In respect to planning actions, sound differences in the levels of connectedness of the agendas were found between experts and novices: "The expert teachers always started their planning statements by telling what they had done the day before, whereas none of the novices did so... expert teachers saw lessons as connected and tied together... their agendas were richer in detail, in connectedness..." (Leinhardt, 1989, p. 64).

Connectedness is an important characteristic of mathematical knowledge (e.g. Even, 1990; Hiebert & Lefevre, 1986). One cannot understand a mathematical concept in isolation. Connections to other concepts, procedures and pieces of information deepen and broaden one's knowledge and make the learning of mathematics meaningful. It is thus important that teachers represent mathematics as a network of interconnected concepts and procedures.

The present study is part of a project aimed at identifying dimensions of expertise in the teaching of algebra. Much like other studies on expertise in domain-specific knowledge, we used the expert-novice research paradigm as a means of characterizing teachers' dimensions of expertise. Our first explorations suggest that expert teachers largely differ from novices in the role that connectedness plays in both their planning and teaching of lessons in algebra, as well as in their reflections on their own lessons. The present paper centers upon this issue.

## Method

### Participants

A comprehensive long-term, in-school, in-service project for improving mathematics and science teaching was carried out in six disadvantaged junior and senior-high schools. Teachers worked with master teachers and participated in a workshop course. As most of the teachers taking part in the project were novices, it was decided to investigate and examine patterns of thinking and actions of both novice and expert teachers. Finding out how they differ might contribute toward developing ways of helping the novice teachers become better teachers.

Two novice teachers from one school and one expert teacher from another school were selected for this study. The novice teachers were both in their second year of teaching. The expert teacher had more than 15 years of experience and acquired the reputation of being an "excellent teacher".

### Data Collection

Three types of data related to the first three consecutive lessons on equivalent algebraic expressions (open-phrases) were collected: 1) Lesson plans--each teacher was asked to submit a lesson plan, before each lesson, either in writing or verbally. 2) Observations--all lessons were observed by one of the researchers. Field notes were taken during observations and were supplemented by audio-taped records. 3) Post-lesson interviews--semi-structured interviews after each lesson aimed to examine teachers' reflections, and clarify incidents or episodes that had not been obvious in plans or observations. All interviews were audio-recorded and transcribed.

## Results

Major differences between novice and expert teachers were observed in several aspects related to subject matter knowledge (e.g., quality of knowledge) and pedagogical content knowledge (e.g., knowledge of students' preconceptions and misconceptions, reactions to student responses, and quality of explanations). As stated in the introduction, in this article we deal in detail with only one aspect of teaching expertise--connectedness. Table 1 summarizes novice versus expert characterization of teaching related to three components: planning the lesson, teaching, and post-lesson reflections.

Table 1: Differences in connectedness between novice and expert teachers.

| | Expert | Novices |
|---|---|---|
| Planning | Horizontal connections: Connections with previous and future material mentioned. | No horizontal connections mentioned. |
| | Vertical connections: A spiral process within and among the lessons. | Segmented planning within and among the lessons. |
| Teaching | Horizontal connections observed at least 6-7 times per lesson. | Almost no horizontal connections observed. |
| | Emphasis on vertical connections observed. | Weak vertical connections observed. |
| | Exploiting opportunities to make connections | Neglecting opportunities to make connections. |
| Reflections | Connectedness mentioned as a major goal of instruction. | Novice 1 argued that there is neither time nor need for connectedness. |
| | | Novice 2 had not thought about connectedness but once it was mentioned, conceived it as important but difficult to perform. |

Table 1 clearly demonstrates that the expert teacher views both vertical and horizontal connections as important factors that guide her thinking about teaching. This was not the case with the novice teachers. The following sections further discuss and illustrate these differences.

### The Expert

*Planning and Teaching*. Connected, organized structures of activity were evident throughout the expert's planning of the lessons as well as in her actual teaching. Every stage of a lesson was connected with the following one. For example, when planning an activity with two algebraic expressions: $7t+1$ and $8t-(t-1)$, the expert teacher wrote in her lesson plan: "Keep the two algebraic expressions in the corner of the blackboard -- *we will deal with them later*". Another example, taken from an in-class observation, demonstrates vertical connections between two activities; the students were asked to determine whether the two algebraic expressions, $2x$ and $|x|+x$, are equivalent. After concluding that they are not, the teacher moved to the next stage of the lesson and asked: "Are the inverse expressions of non-equivalent algebraic expressions equivalent?" She thus connected the two activities by suggesting to check the inverses of the two former algebraic expressions: $\frac{1}{2x}$ and $\frac{1}{|x|+x}$.

Connections were found not only in the expert's development of the lesson towards the main objective (vertical connections) but also across various topics (horizontal connections). For example, in a lesson plan about equivalent algebraic expressions she wrote:

> "Let's look at the algebraic expression $-2a$.
> a) What is the domain?
> When are we going to get a negative number? positive? zero?
> b) What is the relationship between the algebraic expressions $-2a$ and $2a$?
> That means the algebraic expressions are..."

Domains, substitutions and opposite algebraic expressions were not the topic of the lesson--they had been dealt with in previous lessons. Nevertheless, the expert teacher planned to make connections between these topics and the lesson's main objective and then executed her plan. These horizontal connections were made fluently and efficiently throughout the lessons.

*Reflection*. When reflecting on her teaching during the interviews, the expert teacher said that she considered the making of connections between different stages of the lessons and different topics very important and deliberately planned to make them in her teaching. She said, for example: "Every lesson is connected to a previous lesson either through the topic, if possible, or through an activity they [the students] did before." In another interview the expert teacher said: "It is my objective to show that there are connections between the topics."

## The Novice Teachers

Planning and Teaching. While planning the lessons and actually teaching them, the novice teachers did not seem to acknowledge the importance of connectedness. The different stages of the lessons were unconnected and there were almost no connections with previously learnt materials. The following example from a novice teacher's lesson plan illustrates this. The students have just learned the definition of equivalent algebraic expressions; i.e., that the substitution of any number in the two expressions gives the same result. The teacher has also shown them how to check 2-3 numbers and (wrongly) conclude that if the results are the same then the algebraic expressions are equivalent. For the following lesson, the teacher planned to teach "another method" for checking equivalency, based on reducing algebraic fractions and the distributive law. In his plan, the teacher made no connection with the "method" learnt in the previous lesson, nor did he explain why the new method was appropriate or needed. Moreover, he started with the definition of equivalency and immediately moved, without any connection, to the "new method". Following is the first part of his lesson plan:

> "Let's remind ourselves what equivalent algebraic expressions are.
> Repeat the definition.
> And now let's check if the algebraic expressions are equivalent
> $$m+6, \quad \frac{2(m+6)}{2}$$
> Are they equivalent?
> Hint: We can open the parentheses and try to reach the smallest expression possible. At the end we reach the expression m+6.
> That means that we got from one algebraic expression to the other.
> Let's take another example: the algebraic expressions 4a, a+3a. Are they equivalent?
> Yes--because we can add (collect terms) $4a=3a+a$.
> We used the distributive property.
> 'a' has the coefficient 1.
> 3a has the coefficient 3.
> We take 'a' out and get $(1+3) \cdot a$.
> We see that we used the distributive law. What did we use in the first example?--reducing.

Therefore, we can show that algebraic expressions are equivalent by using rules and conventions of number operations."

The novice teachers seemed to be "goal-oriented", trying to reach the objective of the lesson without paying attention to the processes that lead to this objective. The following excerpt from a lesson on equivalent algebraic expressions illustrates this. The teacher wrote on the board the following two algebraic expressions: $4a+3$ and $\frac{3a+6+5a}{2}$.

Teacher: Substitute $a=\frac{1}{2}$.

Student$_1$: You get the same result.

T: Are the algebraic expressions equivalent?

$S_2$: No, because we substituted only one number.

$S_1$: Yes.

$S_3$: It is impossible to know. We need all the numbers.

$S_4$: One example is not enough.

T: We can conclude -- it is difficult to substitute numbers in a complicated algebraic expression and therefore we should find a simpler equivalent algebraic expression.

While the substitution of $a=\frac{1}{2}$ in the two given algebraic expressions can lead naturally to the conclusion that "we should find a simpler equivalent algebraic expression", this was by no means the kind of conclusion appropriate to this discussion.

Usually, the novice teachers did not use opportunities they encountered during the lessons to make horizontal connections between the main topic of the lesson and previously learnt materials in order to strengthen students' understanding. For example, in a lesson whose objective was simplifying algebraic expressions, the teacher wrote the algebraic expression $7a-2+3a$ on the board and asked: "What is the equivalent algebraic expression?". The students did not answer. Then, the following exchange took place:

T: Who can explain what are equivalent algebraic expressions?

$S_1$: [Gives a correct definition].

$S_2$: You should substitute all the numbers.

$S_3$: How come?

T: How can we show that the algebraic expressions are not equivalent?

$S_1$: One example is enough.

$S_2$: If they are equivalent we need two examples.

Even though it was clear that $S_2$ (and others) did not understand how one checks the equivalency of algebraic expressions, the teacher did not use this opportunity to improve students' understanding. Instead, he moved on to another algebraic expression.

Reflection. Neither of the novice teachers emphasized the making of connections in planning and teaching. But the two teachers differed in the way they reflected on this issue. One said in his interview that the main objective of the lesson was the most important and that therefore he should not touch upon other topics. Otherwise, he claimed, he would never have enough time to cover the topic of the lesson. He also said, as a response to an interviewer's question, that he was not aware of any "jumps" between different stages of the lesson, even though many "jumps" were observed in his teaching.

In contrast, the other novice teacher said in his interview that he did not plan to make connections with other topics because he "had not thought about it". But he mentioned: "Now I think that it could be important and interesting".

## Conclusion

A major role of the teacher is to help the learner achieve understanding of the subject matter. One way to do this is by creating classrooms in which the making of connections is emphasized. In such classrooms, as is envisioned by the National Council of Teachers of Mathematics (1989), "ideas flow naturally from one lesson to another, rather than each lesson being restricted to a narrow objective. Lessons frequently extend over several days so that connections can be explored, discussed, and generalized. Once introduced, a topic is used throughout the mathematics program. Teachers seize opportunities that arise from classroom situations to relate different areas and uses of mathematics" (p. 32).

As we saw in this study, the expert teacher considered the issue of connectedness to be very important. She thoughtfully connected the different stages of her lessons as well as made connections between the lessons' main topics and previously learnt materials. This latter kind of horizontal connections did not cause "jumps" in the lessons. The students rather seemed to feel comfortable with this and had no difficulties switching back to the main topic of the lesson.

The novice teachers, on the other hand, did not emphasize connectedness in their lesson plans and teaching. They also tended to stick to their lesson plans regardless of what was happening in the lesson, and drew conclusions that suited their plans but bore little connections with what really went on in the classrooms. After explicitly being asked about the role of connectedness in the learning process, one novice teacher still did not consider it to be important nor was he aware of the unconnected aspect of his lessons. The other novice teacher became aware of the importance of connectedness during the interviews but felt that he lacked the appropriate pedagogical content knowledge of making this part of his own teaching.

This study suggests that there are substantial differences between the novice and expert participant teachers in their view and use of connectedness. In the light of these differences, it is suggested to examine means by which novice teachers might be guided to view connectedness as an important characteristic of teaching. Furthermore, we need to find adequate ways of helping teachers emphasize both vertical and horizontal connections.

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# ANN'S STRATEGIES TO ADD FRACTIONS

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*This article presents an interpretation of two nonconsecutive teaching episodes with a third grader, Ann, who participated in a six-month constructivist teaching experiment. During the first of these two teaching episodes, Ann generated fractional quantifications for the pieces of a segmented square pattern and then used this knowledge to generate her own strategy to add fractions in the subsequent teaching episode.*

An interpretation of the twelfth and sixteenth teaching episodes out of the nineteen teaching episodes in which Ann, a third grader, participated during six months (Sáenz-Ludlow, 1990) is presented in this paper. During these two teaching episodes, the teacher used the same segmented square pattern from which Ann generated fractional quantifications of its parts and constructed her own mental strategies to add fractions. All tasks posed by the teacher as well as Ann's solutions where expressed verbally; for this reason, no symbolic fractional notation is used to report the results.

## Theoretical Rationale

Analysis of the second National Assessment of Educational Progress (NAEP) (Carpenter, Corbitt, Kepner, Lindquist, and Reys, 1980) indicates that most 13- and 17-year-olds were able to add fractions that have a common denominator, but only one third of the 13-year-olds and two thirds of the 17-year-olds were able to add unit fractions or non-unit fractions with different denominators. Their procedure to add two fractions with different denominators was to add numerators and denominators to produce a new fraction, without noticing that the result was a fraction smaller than one of the two given. Moreover, they were unaware that such procedure was not analogous to the algorithm they used to add fractions with the same denominator. Furthermore, less than one fourth of the 13-year-olds and only one third of the 17-year-olds were able to estimate the answer to $\frac{12}{13} + \frac{7}{8}$. One interpretation of these findings is that students need a conceptual understanding of fractions before they are introduced to standard symbolic computational algorithms. The algorithms we use today to perform computations with fractions have been generated and refined in the course of the development of mathematics (Courant and Robbins, 1969; Dantzig, 1954; Smith, Vol. 2, 1953; Waismann, 1959). The algorithms, to be meaningful to the students, must represent the coordination of numerical quantitative reasoning and symbolic procedures. Piaget (1967) observed that symbolic numerical operations are the result of mental actions that first take place when working in experiential contexts. That is, the acquisition of numerical quantitative reasoning seems to be due to the individual's construction of numerical meanings in the midst of problematic situations posed in concrete and familiar contexts.

## Methodology

Ann participated in a six-month constructivist teaching experiment which purpose was to promote and follow her constructions of fractional numbers. The initial conjecture of the study was that a) children would generate fractional number units by using their natural number units and knowledge about natural numbers; and b) children would use natural language as a first source to symbolize their mental operations.

The constructivist teaching-experiment methodology (Cobb and Steffe, 1983; Steffe, 1983) takes into consideration relevant aspects of Piaget's and Vygotsky's theories of cognitive development. Teaching episodes are the backbone of this method of inquiry and their main goal is to facilitate the child's formation of concepts by means of tasks that challenge and further his or her prior conceptual structures.

In this study, teaching episodes lasted 60 minutes and they were videotaped for retrospective analysis and for the preparation of subsequent teaching episodes. The objective of the retrospective analysis was to infer the student's constructive paths to generate fractional numbers and operations with them.

## Finding Relations in a Pattern

In the twelfth interview (February 2, 1990), Ann was given a square pattern as in Figure 1 and it was referred to her as a decorated cookie (letters are used in this figure for descriptive purposes).

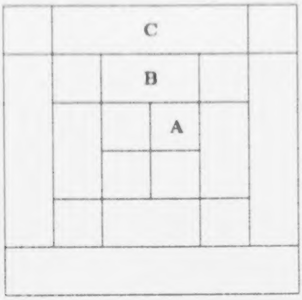

Figure 1

The square was segmented into different pieces that had different colors according to the size of the piece. The goal of the tasks, in this teaching episode, was to provide Ann with an opportunity to generate fractional quantifications for the parts of the cookie. With this objective in mind, the teacher had the following dialogue with Ann that, for the purpose of the analysis, is broken into parts. (In the dialogues *T* stands for teacher and *A* for Ann.)

> *T: This part [showing* **A***] is what part of the whole cookie?*
>
> *A: [Ann uses her right index finger as a means of segmenting the larger pieces into pieces as large as* **A***] One thirty-sixth.*

Ann's answers indicated that her fractional quantifications were linked to the multiplicity of the parts in the whole. She found a fractional quantification of **A** by partitioning the whole cookie into parts as large as **A** and counted them to recast the whole as a multiple of **A**; then she used this multiplicity to quantify **A** with respect to the whole.

> *T: This part [showing* **B***] is what part of the whole cookie?*
>
> *A: [Fast] One eighteenth.*
>
> *T: How did you figure out your answer so fast?*
>
> *A: See there is thirty-six of these [Ann shows a part* **A***] fifteen and fifteen equals thirty; you half out six, that is three; you add three to fifteen and that is eighteen.*
>
> *T: So, one eighteenth is the same as how many thirty-sixths?*
>
> *A: [Ann looks at the figure] Two thirty-sixths.*

To find a fractional quantification of **B**, she quickly visualized **B** as a unit of **A**'s and then generated eighteen units of two out of thirty-six units of one (**A** was taken as one) to conceptualize one eighteenth as two thirty-sixths.

> *T: This part [showing* **C***] is what part of the whole cookie?*
>
> *A: [Fast] One ninth.*
>
> *T: Why?*
>
> *A: Because half of eighteen is nine.*
>
> *T: One ninth is how many eighteenths?*
>
> *A: Two eighteenths.*
>
> *T: One ninth is how many thirty-sixths?*
>
> *A: [Ann looks at the figure] Four thirty-sixths.*

To quantify **C** as one ninth of the whole cookie, it seems that Ann could visualize **C** as twice as large as **B** and could anticipate that the number of units that were twice as large as **B** was only half of the number of units of size **B**. Therefore, she gave as an explanation "because half of eighteen is nine." She generated a sophisticated inverse reasoning by using her concept of a half. The teacher's inference, at this point, was that Ann had constructed a concept of one half to a high degree of abstraction. To test the strength of such conceptualization, the teacher posed to her the following tasks.

> *T: Can you show me half of the cookie?*
>
> *A: [Ann uses her right forearm to partition vertically the cookie in two parts and shows with her left hand the left part of the cookie] That is eighteen thirty-sixths.*

*T: [The teacher uses the diagonal from the upper left corner to the lower right corner of the square to make a new partition and, in doing so, two triangles are determined. The teacher shows Ann the upper left triangle] Is this part half of the cookie?*

*A: [Ann joins the smaller pieces and partitions larger pieces in that triangle to make pieces as large as* **A** *and counts them] There are still eighteen thirty-sixths.*

*T: Can you see more halves?*

*A: [Ann divides the cookie in four equal squares and shows two non-consecutive squares] Here is another half.*

*T: Here, what is more important, the shape or . . .*

*A: [By her own initiative Ann completes the teacher's thought and pointing to square* **A***] . . . the number of squares.*

Her answers indicated that a half, for her, did not have a specific shape and that the shape of a half could be discontinuous or continuous as long as it had eighteen units of size **A**. It could be said that, within this concrete situation, Ann had constructed a concept of a half to a high level of empirical abstraction. The quantitative relations of a half with the other parts of the cookie were indicated in the remaining part of the above dialogue.

*T: How many ninths would make half of the cookie?*

*A: [Ann seems to be counting while silently pointing at pieces of the cookie with her right index finger] Four ninths and two thirty-sixths or four ninths and one eighteenth.*

Ann seemed to use a particular half of the cookie and then reconstituted it using ninths and thirty-sixths or ninths and eighteenths. Since the different partitions of the cookie were only implied in its segmentation, Ann seemed to have visualized that half by using certain parts of the cookie that she had already identified quantitatively. An indication that she had constructed quantitative relations between the parts was her willingness to give a second answer without being asked. Because of her answers, a natural question to ask was the number of ninths in the whole cookie, expecting that maybe she would double the prior answer. However, her answer was based on natural number units as the following two questions indicated.

*T: How many ninths would make the whole figure?*

*A: [Ann thinks for a while, looking at the ceiling] Nine ninths.*

*T: How did you figure that out?*

*A: [Fast] Counting by fours.*

Her willingness to give an explanation very fast indicated her use of a mental representation to find her answer. Her looking at the ceiling instead of observing the cookie and her answer "counting by fours" indicated that she mentally represented the cookie as composed of thirty-six parts **A** and at the same time, she could see thirty-six segmented into nine units of four. The teacher's inference at this point was that Ann could be able to deal with any number of ninths greater than nine, so she asked Ann the following question.

> *T: If I give you forty-five ninths of this cookie, how many cookies would you be able to make?*
>
> *A: [After some seconds] All I need to know is forty-five divided by nine. 9-18-27-36-45 [She uses her right hand to keep track of her counting by nines] You will be able to make five cookies.*

Ann's awareness of the unity of the whole, represented as a composite unit of nine ninths, prompted her to divide forty-five by nine using counting and her fingers to keep track of her counting by nines.

In summary, it could be said that Ann's ways of operating with fractions in the course of this teaching episode was based on her ability to construct composite natural-number units and her self-directed coordination of fractional-number units and natural-number units.

### Adding Fractions

During the three consecutive teaching episodes (February 8, 14, 20, 1990) Ann generated fractional quantifications for parts of known amounts of money, sets of blocks, and different segmented patterns. The following dialogues are excerpts of the sixteenth teaching episode (February 27, 1990). The tasks were posed using the segmented square pattern in Figure 1. These excerpts indicated that Ann used the generated fractional quantifications as well as equivalent fractional quantifications for each of the parts of the pattern in order to add fractions with same and different denominator.

> *T: Suppose I give you one thirty-sixth and one eighteenth of the cookie. What part of the cookie would I have given to you?*
>
> *A: [She observes the cookie] Three thirty-sixths.*
>
> *T: Why?*
>
> *A: Because one eighteenth is two thirty-sixths, and one thirty-sixth is one thirty-sixth.*

Ann's awareness of the fractional quantifications among the parts of the cookie prompted her to use the smallest of the two given unit fractions in order to find her answer. It could be said that her independent choice of the smallest unit fraction in order to add the given fractions was the product of her awareness of the size of each unit fraction due to her knowledge of the relations among them.

To further the development of her arising algorithm, the teacher implicitly asked her to add two fractions that were multiples of the unit fractions that she had added before.

> *T: Suppose I give you two thirty-sixths and three eighteenths, what part of the cookie would you have?*
>
> *A: [Looking at the square] Eight thirty-sixths.*
>
> *T: Why?*
>
> *A: Because three eighteenths is six thirty-sixths, and two thirty-sixths is two thirty-sixths, and two and six is eight.*

Ann's explanations of her process to get the answer "eight thirty-sixths" indicated her initiative to find the equivalent fractions "three eighteenths is six thirty-sixths and two thirty-sixths is two thirty-sixths" in terms of the smallest unit fraction one thirty-sixth. Because she was dealing with the same unit fraction, she continued her explanation "...and two and six is eight."

Ann's solution to the above task seemed to be dependent on her looking at the square pattern (decorated cookie) as well as the known fractional relations. Thus, Ann's strategy to add fractions was dependent on a particular contextual situation. The teacher decided to pose a task similar to the prior one but yet more involved.

> *T: What part of the cookie would you have if I give you three ninths and two eighteenths of it?*
>
> *A: [Looking at the square pattern] Twelve thirty-sixths is three ninths and two eighteenths is four thirty-sixths, then sixteen thirty-sixths.*

Ann's need to look at the square pattern and the verbal explanation of her solution indicated that she selected the smaller unit fraction implied in the square pattern (one thirty-sixth) to generate equivalent fractions to the given ones, in order to have a common unit fraction. To inquire about the consistency of her awareness for the need of having a common unit fraction in order to add fractions, the teacher gave Ann the following task.

> *T: This time suppose that I give you one sixth and one ninth of the cookie, what part of the cookie would you have?*
>
> *A: [Looking at the ceiling] Ten thirty-sixths because one sixth is six thirty-sixths and one ninth is four thirty-sixths and six plus four is ten.*
>
> *T: How did you find out that one sixth is the same as six thirty-sixths?*
>
> *A: [Fast] Because 6-12-18-24-30-36 [Ann shows one full hand and one finger of the other hand].*
>
> *T: How did you find out that one ninth is the same as four thirty-sixths?*
>
> *A: [Fast] Because 9-18-27-36 [Ann shows four fingers of one hand] 1-2-3-4 [Ann counts those fingers].*

Ann's independent choice of finding equivalent fractions for one sixth and one ninth in terms of thirty-sixths indicated her consistent awareness of the need of having a common unit fraction in order to add two fractions as well as her ability to form composite units. Her looking at the ceiling and her willingness to explain quickly her way of finding the equivalent fractions in order to accomplish her task, indicated her interpretation of whole number units as fractional units. Such coordination presupposed Ann's ability to see thirty-six as a unit of units in two ways a unit of nine units of four, a unit of six units of six, and a unit of thirty-six units of one prior to any counting activity. Ann was also given a task that involved fractions that were multiples of the fractions used in the above dialogue.

*T: If I give you two sixths of the cookie and then three ninths of the cookie, what part of the cookie would you have?*

*A: [Ann looks at the ceiling and takes some seconds] Twenty-four thirty-sixths.*

*T: Why?*

*A: Well, six plus six is twelve, twelve plus four is sixteen, sixteen plus four is twenty, twenty plus four is twenty-four; that is twenty-four thirty-sixths.*

Ann's conceptualization of a fraction as a multiple of a unit fraction was indicated by her explanations. "Six plus six is twelve" meant, for her, two sixths are twelve thirty-sixths because one sixth is the same as six thirty-sixths; "twelve plus four is sixteen, sixteen plus four is twenty, twenty plus four is twenty-four" indicated that she added to the prior result (twelve thirty-sixths) four thirty-sixths for each of the three ninths because she knew that one ninth was the same as four thirty-sixths.

In summary, it could be said that Ann's way of adding fractions was consistent with her awareness of using units of the same kind when having an additive situation and her ability to generate fractional units of the same kind. The unit fraction chosen in each of the above tasks (one thirty-sixth) was implied by the pattern but yet it was her own initiative to choose the convenient unit fraction to accomplish the completion of each task. To provide Ann with an opportunity to distance herself from a concrete segmented whole with implied partitions that she could depend upon, she was given a task that involved fractions of a well-determined, however unknown, whole.

*T: Suppose that I have* **a certain amount of money** *and I would like to give to you one fourth and one eighth of that money. What part of that money would I have given to you?*

*A: You gave me one twelfth ..., yes?*

*T: [The teacher repeats the question]*

*A: This is the worst question. Give me a hint?*

*T: Which is bigger, one fourth or one eighth?*

*A: One fourth. [Pause] A fourth and half of a fourth.*

*T: Excellent, but how many eighths would I have given to you?*

*A: [Fast] If one fourth is bigger than one eighth, two eighths equal one fourth; one fourth is half of a half of the money. So you gave me three eighths.*

The given whole "a certain amount of money" provided Ann with a cognitive conflict that led her to be uncertain about her answer of "one twelfth" and to ask for a hint. She operated with the hint by incorporating it in her way of thinking rather than taking it as a question to be answered. Then she found an answer in the form of a sophisticated mixed number "a fourth and half of a fourth." This mixed number made the teacher infer that one fourth was for her "a unit" of which she could find a half. When the teacher asked Ann to find her answer in eighths, she generated a self-directed chain of reasoning to find her answer of "three eighths."

Ann's successful struggle to find her answer indicated that she was ready to operate at a higher level of abstraction with well-defined yet unknown wholes.

## Conclusion

It seems that the decorated cookie (segmented square), with the different implied partitions, was useful in providing Ann with concrete experiences that prompted her to mentally represent such partitions to generate equivalent fractional quantifications of a part. The equivalent fractions that Ann generated were the result of interpreting integral composite units in terms of unit fractions and multiples of unit fractions. Ann's idiosyncratic way of generating equivalent fractions to add fractions seems to indicate that once children have a conceptual understanding of fractions, they can generate meaningful procedures to add them. Her strategy, analogous to the standard algorithm to add natural numbers in which only units of the same kind are added (ones with ones, tens with tens, etc.), consisted in adding fractions after representing them as multiples of the same unit fraction.

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# PROSPECTIVE ELEMENTARY TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE ABOUT EQUIVALENT FRACTIONS.

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This study is part of a project whose objective is to examine the prospective elementary teachers' pedagogical content knowledge about fractions. Two main aspects have been studied. One, the bond of the idea of fraction equivalency to the concrete referents and, the other, how prospective teachers (PTs) utilize those referents to generate explanations about the procedure carried out to obtain equivalent fractions.

The participants were twenty-six prospective elementary teachers. The instrument used was the clinical semi-structured interview. First of all, a descriptive analysis of the followed processes was carried out. Secondly, a conceptual analysis of the obtained categories was performed, basing them on the sucession of cognitive processes proposed by Hiebert.

The difficulties experienced by the prospective teachers originated when the type of given tasks entered into conflict with their procedure of generating fractions and also when attempted to explain with concrete referentes the procedure followed.

If we consider that the pedagogical content knowledge generates upon the understanding of the mathematical content, the type of understanding displayed in some of these cases may not be the most suitable when having to use instructional representations to model the mathematical processes in elementary teaching.

The understanding of the idea of equivalent fractions is important because of its role in different aspects of the development of the concept of fraction and its applications. Based on equivalence, fractions are compared and operations are carried out. At a higher level, it allows the conceptualization of rational numbers as elements of a quotient field (Llinares and Sánchez, 1988).

There are numerous investigations related to the understanding of the idea of equivalent fractions in children. These investigations suggest that there could be considerable underlying confusion in the equivalence notion, and that greater difficulties may appear in simplification tasks, whose resolution the children base themselves on, in many cases, in nonsignificant rules, without there being a real understanding of the steps followed in the procedure (or algorithm).

On the other hand, the researchs about the prospective teachers' (or teachers') knowledge have explored how teachers think in relation to their mathematical knowledge and their understanding of specific curricular topics (analysis of teacher knowledge from the teaching perspective)(Ball, 1990, Even, 1990; Llinares, 1991a,b; Graeber et al., 1989; Post et al., 1988; Llinares and Sánchez, 1991, Sanchez and Llinares, 1991). These studies have shown the influence exerted by the knowledge of subject matter on their pedagogical orientations and decisions (Mc Diarmid et al., 1989).

The teachers' understanding of subject matter influences presentation and formulation as well as the instructional representations which are used when the teacher tries to make it understandable to the students. Shulman (1986) has denominated this type of knowledge, teachers' pedagogical content knowledge.

This study is part of a research project, to a further extent, whose objective is to examine the knowledge of fractions that have prospective elementary teachers (PTs), in relation to the usage of different instructional representation systems (Llinares, 1991a, b, Llinares and Sanchez, 1991;

Sanchez and Llinares, 1991). Two aspects have been focused on: one, the bond of equivalence notion to the concrete referents and, the other, how PTs utilize these referents to generate explanations about the processes followed to obtain equivalent fractions.

## METHOD

### Participants

The participants were twenty-six elementary student teachers, enrolled as members of the program for Initial Elementary Teachers Education at the University of Seville. Throughout the three years of the program, the PTs may choose among different specialties, which differentiate in subject matter accordingly. Twenty of these students were second-year students, while the other six were in their third year. During their university studies, they all studied elementary mathematics in the first course, concentrating fundamentally on algebraic contents. The mathematical contents considered in this project pertained to their elementary level (ages 6-11), and thus, common to all of them. These students were ramdomly chosen from a volunteer group. Of the twenty-six students, nine were men and seventeen were women, ranging from nineteen to twenty-three years of age with no outstanding characteristics.

### Instrument

The instrument used was the semi-structured clinical interview, the same was complemented with the notes taken by the investigators and the participants. During the interview, the interviewers were allowed to ask their own additional questions about the aspects they deemed interesting. There were playing chips on the table, which the interviewee could freely use and paper and pencil to draw with. Among other questions relating to different aspects of fractions, the students were asked:

a) Find a fraction equivalent to one given (e.g. find equivalent fractions to 9/12, 5/3, 4/6, etc.), utilizing the playing chips and/or rectangles (or circles) to describe the processes followed. The fractions were given verbally, and the participants could write them out.
b) Find the fraction equivalent to one given which would have a numerator smaller or larger (e.g. 9/12=6/?, 9/12=1[illegible]?, 8/12=2/?, 4/6=8/?, 5/4=?/12, etc.), utilizing the playing chips and/or rectangles (or circles) to describe the processes followed. The item fractions varied depending on the dynamics of the interview itelf.

During the same, an attempt was made to maximize the depth of the understanding of the process on behalf of the PT and on the type of explanation given to that which they were doing. Although at first, it was thought that the tasks proposed were the same for all the participants, they changed in some cases, in relation to the answers obtained. The open character of the interview must not be forgotten nor that the questions formed were part of a wider interview about different aspects of fractions. While analyzing the tasks, the type of item presentation, the procedure used to solve it, and the explanation given by the PT were considered. The interviews were fully recorded and transcribed for later analysis.

### Procedure

To begin with, after carefully reading the answers obtained, a first descriptive analysis was made of the processes followed in the resolution of the items presented, based on the different referents. The procedures followed by the PTs were grouped by similarities. Afterwards, a conceptual analysis of the categories was performed, based on the cognitive processes proposed by Hiebert (1988) to reach the understanding of written mathematical symbols. Out of the five processes identified, the first two will be discussed in detail:
I) Connection of individual symbols with concrete referents and
II) How these referents are used to generate explanations about the processes followed.

In the next section, some characteristics of explanations and arguments used by the PTs, in their attempt to model with concrete referents (playing chips and drawings) the algorythimic procedures developed with symbols, are shown.

## RESULTS

Of the twenty-six PTs interviewed, twenty-one based algorithmic procedures on memory. On some occasions, the mentioned rules allowed for the adequate resolution of the item at the symbolic level. Some PTs had difficulties in generating explanations on the connection of referents with the followed procedures with the symbols. Five of the twenty-six were unable to solve correctly the items presented.

A) Unappropiate rules to solve the asignment are remembered.

### A1) Fraction as division and equivalent fractions.

Within this group, one of the participants, in seeking equivalent fractions used as strategy "obtaining the same quotient by dividing," even though he also remembered the algorithmic procedure of "multiplying across" to prove if the fractions were equivalents. Like so, in the item "find out an equivalent fraction of 9/12" he expressed:

> B.12.11:...Well, I think this could be done in two ways....
> B.12.12:...Let's see if I recall correctly...equivalent fractions...that dividing the numerator by the denominator would give the same, right?
> B.12.13:...multiply across....

This procedure seemed linked to proving processed of whether two fractions are or not equivalent, more than the search for new fractions.

> B.12.14: Can I divide between these two? (pointing to the nine and the twelve).
> B.13.5:...Whatever I want?... I divide nine by twelve....

But this result does not provide the desired information:

> B.13.5:...zero and seven...I have to find...for example...wait ...four sevenths...I don't know...can I check if they're equivalent? (PT writes 4/7)

At that moment, because another fraction was needed, a step was taken towards an erroneous additive strategy. This additive strategy was transfered to his explanations with playing chips and rectangles.

> B.13.8:...I've counted from nine to twelve and from four to seven and it's the same.

without even verifying by means of cross multiplication. In this case, since the pedagogical content knowledge generates on the understanding of the mathematical content by the prospective teacher, a limited understanding of the procedure to obtain equivalent fractions can not generate good pedagogical content knowledge.

### A2) Nongeneralized algorithmic procedure

Six of the participants remembered as the preferred algorithmic procedure to "divide (or multiply) numerator or denominator by the same number" (the rule "multiply top and bottom - or divide top and bottom - by the same number to obtain an equivalent fraction"). In answering "the equivalent fraction to 9/12", one of the students gave as an answer:

> MM.9.16:...eighteen twenty-four....
> MM.9.17:...I've multiplied the numerator and the denominator by two.

This allowed for the construction of equivalent fractions to those given, but failed in the item 9/12=? For the new item, three PTs used an erroneous additive strategy with symbols. One of them explained

> MM.9.1: I've subtracted three...from the twelve, three (then writes on the paper 9/12=6/9)

To explain the process followed, a drawing is chosen as a representation:

> MM.9.7:...I'm dividing twelve in four parts...and for them to be nine ... it would be this (coloring in nine parts of the twelve in which the rectangle has been divided.)

At the interviewer's petition to explain how he would now go to six ninths, he comments:

> MM.9.8:...we would have to divide the square into nine...but you gave me six ...
> MM.9.9:...we would have to divide again by six, but ...I don't know how I could explain it to....

The other two PTs proceeded in analogous ways. It must be recalled that these three PTs followed a memorized rule, but did not use it to prove the results obtained by applying the erroneous additive strategy. The memorized rule was attached to the working out of specific items. The modification of the items leads to the need of generalizing the rule usage. This generalizing procedure of the usage of the rule was not practiced by the PTs. Also, in this case, as the previous protocol points out, they could not generate any explanation using referents of the process followed with symbols.

Another PT changed to the additive strategy in attemptig to explain with playing chips the process followed. Likewise, for the item 6/18=3/?:

> A.7.2: 3/9... I divide by two on top and on the bottom...

When the interviewer asked for an explanation of the process followed with playing chips:

> A.7.3:...but I don't know how to do it with playing chips... What should I do?, Should I take away the same number of playing chips from the two and I get the equivalent?

Of the two remaining PTs, one expressed that he could no do it with a fraction with a numerator that was not multiple (or divider) of the numerator given fraction. Using rectangles, the difficulty was justified based on the fact that it was imposible to divide the unity into another number of parts which were not the ones obtained by breaking each one of them in half.

The last, although recalling the algorithmic procedure, since it was not appropiate for the item proposed, derived into senseless calculus, without being able to find any connections with conrete referents.

### B) Concrete referent connection with equivalent fraction processes obtained at symbol level.

#### B1) Origin of rule meaning. Symbol manipulation or concrete manipulation?

Six PTs remembered an algorithmic procedure that allowed them to solve properly the proposed item, but they explicitly recognized the difficulty of explaining it with playing chips or any type of drawing. If the interviewer's persistence to explain the process was great, each represented separately two fractions. So was the explanation of one PT of the item 8/12=2/?

> I.11.6: Honestly, I know how to do it...but with playing chips it's harder ...
> I.11.8: You divide by factors and you go dividing, you'd go taking from one side and another until you have the two...

and, when asked to explain the process with playing chips:

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> I.12.1: I know how to represent it, but as a separate fraction, in other words, right now I can take four unities and from four take two, if the whole would be four I'd take two...then I have represented two fourths, but I don't know how to connect it with...
> I.12.3:...it would be divided by four...no, I'm sorry, they're two thirds...

The same difficulty shows up when faced with the continuos representation:

> I.12.5: The same thing would happen to me...I could represent eight twelfths, right?...and I would take eight of them...but, in moving on to two thirds then...the same thing would happen to me, I'd draw another cake and...there I'd point out my two thirds, but nothing more....

The difficulty in representing at the concrete level the different steps in the procedures developed at the symbolic level leads to what Hiebert (1988) terms as a transfer in the source of significance (of the referents to the symbols) and a change in the results evaluation. The validation of the results comes only from carrying out correctly the prescribed steps of the algorithm.

In two other cases, there was also a recall of procedures based on multiplicative strategies, that allowed the solving of the items at a symbolic level, but the interviews did not provide information on the referents modeled.

### B2) The influence of the type of referent on the connection process.

Another three PTs generated correct explanations to find equivalent fractions to one given with playing chips, but were unable to make the transfer of the explanation to the rectangles (or circles). So, one PT in having to find the equivalent fraction to 8/12 comments, explaining the process with playing chips:

> M.9.3:...I would take eight from twelve...
> M.9.4:...and it would be this (pointing to playing chips on the table)
> ...now I would take half of the playing chips... and I'd take four from six...that would be half of twelve and half of eight...half of the total and half of what I have taken...

This same reasoning, based on duplicating, is used for 16/24, but when asked to use rectangles the following is stated:

> M.10.2:...I can't do it without using the numbers....
> I. In other words, you draw the rectangle...
> M.10.3:...and I make sixteen from the numerator...and I'd continue drawing squares until the twenty-four...(laughs)...I can't think of anything ...

It is obvious at this point the difficulty in going from a discrete representation system, where unity in representing each fraction varies, to one where you take as a unity the geometric figure.

### B3) Influence of the type of item

The type of multiplicative relation between numerators in the proposed item introduced some modification in the modeling process. With the item 9/12=6/?, a PT gives the following answer:

> F.9.3: I'm thinking...Now I'm not thinking of rectangles, I'm thinking of finding a fraction that...well, what I do is...I multiply the fractions two parts by the same number...

This PT was able to generate a correct explanation with rectangles, going from 9/12 to 3/4 and from there to 6/8.

F.9.9:...I'm going to the rectangle...I'm drawing that it's nine twelfths ... I divide the unit into twelve parts (he makes eleven lines on one of the greater sides of the rectangle)...three, six, nine...(he marks every three separations with a verticle line and in the ninth space a dark line) ...this would be... three fourths....

It is at this point that the PT goes to a symbolic level to establish the connection between the given numerator:

F.10.3:...I've drawn the rectangle, divided it into eleven parts and then I realized that nine twelfths is the same as three fourths...to make it six I multiplied three times two...and the four times two....

Once 6/8 is obtained from 3/4, a rectangle (unit) is drawn the same size as the previous one and divided by four, three parts are colored in while over marking the eights, showing that the colored part in 6/8 is the same as in 9/12.

F.10.4:...I've said six eights...then I paint the drawing and divide it into eight parts...(he paints a rectangle and marks eight separations to one side)...two, four, six and eight...means that I have the same as nine twelfths...

When the item 9/12=15/? is proposed, difficulity is found in generalizing the previous explanations:

F.11.2:...I'm thinking to see how...the fifteen are...fifteen parts of...if the numerator is fifteen, but the demoninator...how do I find it? I mean they could be fifteen parts of whatever....
F.11.4:...I don't know how to do it....

In this case there's an inability to extend the previous reazonings.

Only two students out of the twenty-one we had initially considered, modeled the process correctly. One of the PTs provides information on the explanation of the process with playing chips or rectangles but only on items seeking equivalent fractions to one given. The other PT modeled with playing chips, based on a part-to-part relationship for 9/12=6/?, which allowed the construction of a multiplicative strategy based on the ratio concept.

M.16.3:...in total, twelve, but I will put nine (playing chips) black and three red...the black part corresponds to the nine twelfths, OK?
M.16.5:...the numerator six...which means...since six is the two thirds of nine...and at the same time I have to take the two thirds of twelve ... which is twenty four divided by three, eight. In other words, I take away one...
M.16.6:...Because, for the total I need eight, and since I already had six of the black ones...to have a total of eight I have to take one red.

However, he has difficulties expressing it with rectangles.

C) Application of an incorrect rule at symbol level

**C1) Erroneous overgeneralization of a rule**

One PT recalled a rule for finding equivalent fractions, but memorized without any meaning and in an incorrect way. So, to the item "finding some equivalent fraction to 8/9":

C.18.5: Well, adding, subtracting, multiplying or dividing top and bottom with the same number...
C.18.9: Well, eight by two, sixteen, nine by two, eighteen.

At the interviewer's petition for another fraction:

> C.18.10: Another...eight minus three, five, nine minus three, six.

When asked to explain the procedure with playing chips, only the original fraction was shown, and then took away (or added) playing chips to obtain the others.

**C2) Additive rule at symbol level.**

Lastly, four PTs out of the twenty-six directly applied additive strategies at the symbol level, and later limited themselves to justifying the results obtained.

Like so, in this case, when given the item "finding any equivalent fraction to nine twelfths", a PT proposed:

> L.8.8: ...seven eights....
> L.8.9: Because what I have done is to subtract from the numerator and the denominator the same amount, which in this case has been two units....

If what was wanted was an equivalent fraction to 9/12 but with 6 in the numerator:

> L.8.5: ...with playing chips. Everything is divided in the first fractions, the nine twelfths one...divide enerything into twelve parts, take twelve playing chips. From those twelve...take three and I keep nine. In the following fraction the total would be nine playing chips ... from the nine I take three and I keep six and the answer is three.

In this case, interfering notion of fraction linked to the part-whole interpretation, is appreciated at the time of utilizing a discrete interpretation. Another PT used rectangles in his explanations.

> A.6.2: ...An equivalent fraction to nine twelfths with a numerator of six ... twelve minus three is nine....
> A.6.3: ...the equivalent fraction would be six ninths....
> A.6.11: ...before I divided a rectangle into twelve (parts) and I took nine. And now I divide it into nine parts (he counts orally) and I take six ...I get the same parts, three.

To represent each fraction utilized he used a different rectangle, and justifies the similarities on the fact that the same amount of parts remains (additive relationship), even if the part sizes are different.

## DISCUSSION

The difficulties experienced by the PTs in this study originate in two situations:
(i) in relation to the type of item gave, when not asked to generate any fraction, but when the numerator (or denominator) is given, which can enter into conflict with their fraction generating procedure, and
(ii) in an attempt to explain with concrete referents the processes followed.

From the stand point of the connecting and developing processes identified by Hiebert, we can point out some PT's "weak" comprehension of the mathematical content. they recall a rule linked to determined tasks, but have problems when a modification of the task that leads to the necesity of generalizing the rule is proposed. In these situations, due to the bad connection between the symbol and the meaning, they vote for separating numerator and denominator (a/b is "seen" as 'a' and 'b'), and they establish additive relationships between numerator and denominator in each fraction separately, or between numerators and denominators. These additive strategies are often cited among erroneous strategies used by children in tasks on equivalence (Hart. 1980, 1989).

In other cases, they derived additive strategies at the difficulty of modeling the procedure with referents. They could create a correspondence between written symbols for fraction and its representation, but were unable to model the process of going from one fraction to another (the representation of the process).

Other PTs knew a procedure at the symbol level which allowed to resolve tasks, but explicitly showed the difficulty in modeling it with referents. Like in the previous case, they were capable of establishing a correct connection between the symbol of each fraction and the continous or discrete referents, but did not attempt to model the process that goes from one fraction to another one. A better understanding of the concept of fraction of these PTs leads them to not separating the terms (numerator and denominator) and to recognize their incapacity to generate explanations.

Another aspect which appeared is the influence of the type of referent on the connecting process. The difficulty of transfering general explanations in a representing system to another one proves that an association between a representing system and a symbol, even if appropriate, can be insufficiente to develop a complete significance.

If we consider that the pedagogical content knowledge generates upon PTs' understanding of mathematical content, the type of understanding showed in these cases can not be the most adequate at the time of using instructional representations to model mathematical process in elementary teaching. Instead, in cases where the mathematical concept understanding is erroneous, the inability to build upon pedagogical content knowledge is evident.

The Elementary Mathematics Teachers Education should concentrate on the PTs knowledge of the relationship between mathematic processes and the modelling of these processes with referents.

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# Empowering Prospective Elementary Teachers Through Social Interaction, Reflection, and Communication[1]

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A new mathematics course for elementary education majors at Indiana University (T104) was first implemented in Spring 1990. The goal of the course is not only to teach mathematics content, but also to develop teachers who have more mathematical power: that is, teachers who are reflective thinkers about how they learn and teach, who are cognizant of their own strengths and weaknesses, and who are thoughtful about their mathematical knowledge. Research on the effectiveness of T104 has been conducted, via document analysis, classroom observation, and interviews since Spring 1990. In 1992, interviews were conducted with two students who had previously been studied when they were in T104 in Spring 1990, and who are now in their final year of teacher preparation. This paper traces their evolving ideas about mathematics, learning, and effective teaching. The students observed that opportunities for social interaction, reflection, and communication in T104 were important components of the course.

In a recent bestseller, Toffler (1990) cogently makes the point that in today's society knowledge, rather than land or money or machines, is capital, and that the need for sharing this capital around the globe will be more and more important in the future. Citizens today need to be able to be critical of ideas, to make conjectures and decisions, and to be able to articulate their ideas clearly. Furthermore, minimal literacy to deal with information presented in today's media requires numeracy as well: that people be able to interpret numbers presented in context, analyze trends, read graphs, and interpret statistics (Paulos, 1988). How can we ensure that future citizens will be prepared to deal with this new information age? Numerous recent calls for education reform have proposed changes in our educational system, and in mathematics education in particular (National Research Council, 1989; National Council of Teachers of Mathematics [NCTM], 1989, 1991). But, unless teacher education itself is modified and revitalized, new citizens cannot be prepared according to this new perspective, and the desired changes in the educational system will never be realized.

The NCTM has addressed many of the competencies needed of future teachers (NCTM, 1991). Teachers in the 21st century must be knowledgeable about the content they will teach, must be able to make conjectures both about content and pedagogy, and to think critically about the appropriateness of their lessons for students' learning. They must also be sensitive in using assessment techniques to reflect on their own practice and on the progress of their students. Above all, teachers of the future must be intellectually engaged—open to exploring new ideas and methods. The competencies above make up what we conceive of as intellectual empowerment.

Teachers of tomorrow must also be confident in their own abilities, not only to teach effectively, but also to make more global decisions about curriculum and methodology. They must have high self esteem, and be sensitive, caring individuals, able to motivate students to engage in the excitement of learning. Teachers must be able to work effectively and to share information with other teachers, cognizant of the social needs of students, and above all, able to encourage students of varied backgrounds and cultures to work collaboratively toward common goals. Thus, emotional and social empowerment are also important competencies for teachers of the future.

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### Investigating the Empowerment of Prospective Teachers

It seems clear that teacher education programs must change if they want to empower the teachers they educate. Toward this goal, Indiana University, with three-year funding from the National Science Foundation, developed a new content course for prospective elementary teachers (Mathematics for Elementary Teachers via Problem Solving—T104). The developers of T104 envisioned it not only as a course where students would gain mathematical power, but also as one of the earliest collegiate experiences where prospective teachers' personal beliefs about content (mathematics) and pedagogy would be challenged and developed.

Two sections of this new course (50 students total) were taught for the first time in Spring 1990. During the first year-and-a-half of establishing the course, data were collected in an effort to evaluate the success of the course and to make recommendations for modifications and revisions. These data included samples of students' written work, interviews with students, interviews with instructors, and classroom observations. In Spring 1990, written work was collected from all 50 students in the course, and 8 of those 50 volunteered to be interviewed about their experience. Findings from these data were presented at the Chicago AERA meeting (Raymond, Santos, & Masingila, 1991) and in poster sessions at the PME XV meeting (Raymond & Santos, 1991; Santos, 1991).

It is now two years later, and most of the students who were enrolled in the first semester of T104 are now in their fourth, and final year of teacher preparation. Although T104 was a mathematics content course, it was one of the first courses where these students were challenged to think about issues related to teaching and learning. To see how T104 students remember the course, and how the ideas that they expressed in their written work and interviews two years ago may have been reinforced or changed by subsequent professional education courses, we interviewed two of the eight students who had been interviewed in Spring 1990. This paper reports insights from the 1992 interviews together with analyses of the two women's notebooks, journals, test papers, and interviews from 1990.

In the 1992 interview, we began by asking each of the women about her idea of the characteristics and behaviors of an effective teacher. Next we asked how their university coursework, in particular their work in T104, might have influenced the type of teacher they idealize and their own progress toward this ideal. We asked them to reflect back over how their ideas about teaching had changed over the past two years. Finally we asked them to explain their personal conceptions of teacher empowerment, and their views on the importance of communication, reflectiveness, and social interaction both in their own teacher education program and in the lives of practicing teachers in the field. Before reporting how the two students responded to our questions, it is important to describe the structure and goals of T104.

## Description of T104

T104 included problem-solving activities designed to involve students in cooperative work, to challenge them to communicate unambiguously, and to encourage them to be thoughtful and reflective about their own knowledge and skills. T104 students participated in small group problem solving during every class. Students were encouraged to make, test, and refine conjectures, and to solve each problem in a variety of ways. During whole class discussions students were expected to explain "why" as well as "how" they had solved problems and they made similar notes in their required weekly reflective journals. As a result of all if these activities, T104 students were challenged to rethink their notions about what mathematics is, about how they learn mathematics, and about the complexities involved in teaching it.

The non-traditional form of the T104 activities made it natural, and in fact necessary, for various forms of alternative assessment to be used, not only to provide data for grading, but also—and perhaps more importantly—to provide feedback to students and teacher alike. In the view of the course designers, perhaps the most important goal of the T104 assessment methods was to help students become more metacognitively aware: more cognizant of their own strengths and weaknesses; more thoughtful about what they know and about how they learn; and more reflective about the implications of these realizations for themselves as future teachers.

Group problem solving on tests aligned T104 assessment with instruction. Daily group problem solving provided an opportunity for students of every ability level to work together and to contribute to a common goal. Working cooperatively not only tended to enhance self esteem, improve confidence, and develop the social skills of the preservice teachers, but also helped to raise students' consciousness about nontraditional emphases of the course, such as that mathematics is not a solitary endeavor, that there are a variety of alternative approaches to problems, and that clarity and precision of mathematical communication is important. All of these elements of group work were further given emphasis and value through the inclusion on each test of a problem to be solved cooperatively and to be further explored individually. Thus, as recommended by theorists who write about the benefits of cooperative learning (Sharan, 1990; Slavin, 1983), T104 tests involved both individual and group accountability.

Throughout the semester, students collaborated on group presentations (for example, presentations on a variety of alternative approaches to proving the Pythagorean Theorem). These presentations enhanced students' mathematical power by demanding several class periods of independent exploration and by requiring clarity of communication. In preparing for group presentations, many students—for the first time—found themselves doing mathematics without continuous teacher monitoring. Moreover, in their presentations, students were expected, in a teacherly fashion, to explain concepts and to field questions about their mathematical thinking. In addition, these presentations added new layers to the instructors' subjective assessment of the students' understanding. The group presentations reinforced many of the expectations of the course for the prospective teachers, such as becoming better independent problem solvers, and enhanced students' confidence as doers, learners, and teachers of mathematics.

T104 students also kept a notebook containing a variety of assignments (daily activities, written assignments, problem-solving attempts, reactions to open-ended questions, and personal reflections). A few particularly effective assignments involved students in creating concept maps and writing a reflection about how their maps depicted the relationships in their minds among mathematical concepts. Notebook assignments were designed to force students to think about the "why" as well as the "how" of their mathematical work. Thus, notebooks displayed students' evolving thinking about mathematics and teaching, and provided a record that they could return to later to reflect on their learning growth.

Space does not permit here a more detailed elaboration for why all of the ingredients above were included in the design of T104. But the writings of Slavin (1983), Sharan (1990) and Vygotsky (1986) were important in conceptualizing the types of social interaction that would be beneficial for T104 students. Driving forces for the inclusion of activities designed to enhance verbal and written communication and reflectiveness included Connolly and Vilardi's (1989) book about writing and mathematics, Garofalo and Lester's (1985) seminal article about metacognition and mathematics, Novak and Gowin's (1984) studies about enhancing conceptual understanding through concept mapping, and Vygotsky's (1986) writings concerning the intricate connections between thought, language, and concept development. Finally, planning of T104 activities was influenced by writings concerning constructivism that appeared in publications about the time that the course was under development (e.g., Davis, Maher, & Noddings, 1990).

## Students' Evolving Conceptions about Mathematics and Teaching

In the paragraphs that follow we trace Marta and Helena's evolving ideas about mathematics, teaching, and learning by providing brief excerpts from our 1992 interviews along with selections from the data collected in the Spring of 1990 (when they were enrolled in T104). (Direct quotes from journals or interviews are italicized. More extensive examples will be shared during the PME presentation.)

### Marta

At the beginning of her T104 semester, Marta's journal entry shows that she was beginning to grow intellectually by having to think about "why" as well as "how."

Jan 8, 1990: *I feel that this class is going to be very helpful for me. I've always enjoyed math and have done fairly well. [But] I usually don't bother to take the time to ask why you do something the way you do. I always want to know the quickest and easiest way to solve a problem. I never take the time to think about why . . .*

Several weeks later, Marta elaborated further on the importance of understanding multiple approaches to a problem and being able to explain your thinking to others. She was beginning to see the benefits of cooperative work for her own learning, and we believe she was being empowered by the social interaction that the structure of the class required.

Jan. 24, 1990: *I really like working in groups to solve the problems. . . . It allows you to see another side to the solution because maybe one of your group members looks at the problem differently. It also gives you the experience of working with other people and explaining things to other people . . ..*

By the end of February, Marta and her group had spent a considerable amount of time working cooperatively, and they were becoming more independent problem solvers. In the journal excerpt that we quote next, Marta noted that her group was now able to explore multiple approaches to a problem on their own. For us, this independent investigative activity provided Marta and other T104 students with a context for being empowered socially, emotionally, and, most of all, intellectually.

Feb. 28, 1990: *My group outdoor activity [to measure the distance across a river indirectly] we did today was hard at first. We kept getting ideas that seemed like they'd work but there was always something wrong with them. I learned that it is possible to measure the length of something without actually measuring it. We discovered this after a period of trial and error. Using the theories [sic] of congruence of triangles (SSS, ASA, AAS) we were able to solve the problem.*

In her final journal entry reflecting on the entire course, Marta noted again that cooperative group work had helped her both in learning mathematics and in developing the ability to explain clearly.

May 1, 1990: *Now that the semester is over, I have a chance to think back over my classes and what I've learned. . . . I enjoyed working in groups and being able to talk things over with other people. This enabled me to get experience with explaining my thoughts and knowledge and also being able to get help from others when I needed it. . . . I don't know if I can tell where I am mathematically. I learned a lot . . . I feel more confident with my mathematical skills.*

Two years later, when Marta was asked to think back on her development as a prospective teacher, and what she could recall about T104, she emphasized group work, social interaction, and reflections as the features of the course that had helped her to approach her ideal of a good teacher. She began by focusing her attention (as two years before) on the importance of cooperative group work. She asserted that when emotional and social security are provided to students (as when they can collaborate on difficult tasks), students are able to learn more easily. However, Marta's focus seemed to have broadened: whereas in 1990 she had spoken more from the perspective of a learner within a group, now she also discussed the pedagogical implications of group interaction between students and teacher. She seemed more aware of the teacher's responsibility for choosing appropriate tasks, emphasizing the necessity to avoid excessive student frustration by *"challenging students with difficult problems which are not impossible tasks . . .."* And she said that teachers should be competent to explain the solutions to such problems in multiple ways.

Finally, Marta admitted that she felt she had only recently come to appreciate the benefits of introspection and reflection. During T104, she had not enjoyed being required to keep a reflective journal or having to explain the reasons behind her mathematical solutions, but in retrospect she recognized that:

Interview, 1992: *We learned to think more logically, to reason, to think about the process. With T104 and E343 [the mathematics methods course], I've seen different approaches to math. In high school, math was always just numbers. In T104 we had to explain why . . . I think math makes more sense now. . . . Some teachers don't bother to explain why, but I think that now that I know why, I'll use this for math teaching. . . . We [teachers] have to know the why first before going to how to teach to kids.*

### Helena

Helena had shown evidence of thinking from a teacher's perspective as early as her very first journal entry in T104 (Jan. 8, 1990), when she wrote: *"I will look forward to developing keen problem-solving skills. I hope to get a lot of information from this class to take into my own classroom in the years to come."* Helena found working in cooperative groups novel and useful, but she seemed to be more vocal than Marta about the benefits of allowing students to struggle to construct their own meanings for mathematical concepts.

Jan. 24, 1990 *Cooperative learning is a wonderful way to learn. . . . It helps because not everyone thinks the same way. This allows for new or different insights on a problem. It is a lot easier to remember concepts when you are able to figure them out yourself or with the help of your peers. Sometimes a teacher just can't communicate a concept so that it is understandable. . .*

Helena's written reflection on her group project was lengthy, including many important observations, both pedagogical and mathematical.

February 28, 1990: *This activity reinforced for me how important cooperation and communication is for cooperative learning. It took us a long time to actually carry out our plan—we weren't communicating with each other. Once we realized how important this was, we quickly got organized. But, now I can see we had another big problem—we set the points before the angles. BIG mistake because the angles' intersections determine the points, not the other way around. Another point that I feel kind of foolish for not recognizing was the fact that we were using 45°-45°-90° triangles and did not need to worry about making a congruent triangle because each of the legs are congruent. If we would have set angles and then* points, we only would have needed to measure the distance along the bank.

In the 1992 interview, Helena discussed how her T104 experience in Spring 1990, along with the next two years of her teacher education program had been important for her as a learner and prospective teacher, and in forming her conception of an effective teacher. She cited social interaction and meaning construction as two important benefits of the T104 group work. In fact, after initially seeing how her understandings of mathematical ideas were enhanced by being challenged by peers in T104, she now often seeks out fellow students in other classes to discuss and clarify ideas. Working with others showed her the importance not only of knowing multiple ways to solve a mathematics problem, but also of knowing appropriate questions to ask and multiple ways to teach. Helena said that she observed how skillful her T104 instructor was in posing questions to challenge learners who were at different points in seeking a solution for a problem. In T104 she had come to see that not everyone learns the same way. Furthermore, she observed that it was important that T104 (her mathematics course) came before E343 (her mathematics methods course) because she had struggled in T104 to learn elementary mathematics from an adult perspective, and to write cogent explanations of her understandings. These struggles laid the foundation for moving, in the methods class, to a more teacher oriented perspective.

Another important requirement of T104, in Helena's estimation, was the reflective journal. She mentioned three benefits of keeping such a journal. The first is primarily emotional: a reflective journal provides a place to release frustration, a place to record both feelings and thoughts. Secondly, the release of emotional tensions in a journal allows the writer to move more freely to intellectual reflections. Helena described the pride she felt in being able to compose a logically organized explanation of her group's struggle to solve a mathematical problem. She felt strongly writing explanations of problem solutions after discussing them with peers helped her consolidate ideas. Finally, Helena valued her journal as a record of personal growth. She indicated that in T104 she frequently reread her journal to see how her thinking had changed. She has continued this practice in subsequent courses.

One of the most important skills teachers need, according to Helena, is the ability to communicate clearly. She feels her university training, beginning with T104, has provided her with many experiences to develop this skill. Yet Helena recognizes that no university teacher education program can provide all of the experiences (content, pedagogical, psychological, emotional, social, etc.) that teachers need to be effective. Teachers must be life-long learners.

## Conclusion

Our talks with Marta and Helena gave us insight into their conceptions of good teaching, and our comparison of their recent interview responses with the earlier data we had collected from them helped us trace their professional development in the past two years. Both women stressed that it is important to have a firm knowledge base, especially math and science. Yet they also recognized that they could not know everything that they were going to teach, that good teachers must be life-long learners. Helena particularly mentioned that good teachers should show excitement in learning, and she felt that the T104 instructors had modeled such behavior. Two years ago, when asked similar questions, they primarily characterized good teachers as caring, understanding, and friendly. Now they stressed, however, that these affective characteristics are necessary, but not sufficient.

Both women mentioned that the stress in T104 on explaining "why" problems are solved the way they enhanced their understanding of mathematical concepts, which turned out to be an important point when they were confronted in mathematics methods courses with pedagogical issues. In all their classes they became aware of the importance of clear communication skills. Both women praised T104 as the first course in which they were really challenged to explain and communicate clearly. Finally, both women emphasized the need for teachers to be willing to engage in pedagogical introspection.

In the estimation of these two students, T104 planted the seeds for much of the intellectual and professional growth that they have experienced during their teacher education. It was the first of many classes in which they were expected to communicate their ideas in a clear way and to show reflectiveness. They believe they have grown in these areas over the past two years. Helena made an interesting suggestion for improving T104 and other courses emphasizing written reflections. She suggested that course instructors should not only write reflections (as T104 instructors did), but also should share these reflections with their students. She also thought it would have been beneficial for students to have been encouraged to share their reflections with one another.

As amply illustrated in our analyses above, Marta and Helena believed that cooperative work was another of the most potent aspects of T104. Although group work was also used in many of their other classes, they felt it was particularly useful in learning mathematics because working with others showed them alternative approaches and strategies. As prospective teachers they also recognized that group work provided opportunities for teaching each other, and forced them to share and negotiate meanings clearly. Cooperative work is a pedagogical technique that both women plan to use in their own classrooms.

Marta and Helena made no claims that T104 had provided all the benefits cited above, although they recognized that it was a firm beginning. Both thought that their other courses reinforced ideas that had been first encountered in T104, however they would like to have had even more experiences providing opportunities for social interaction, reflection, and communication.

Our talks with Marta and Helena certainly cannot provide a complete picture of how T104 students are integrating their mathematics experience with the rest of their teacher education training. But these conversations have served to pique our interest for further investigations. We would like to conduct more case studies like those reported here. And we also see the need for conducting a survey of a much larger sample of students from among the more than 500 students who have been enrolled in the class since its inception in spring 1990. Finally, we think it would be informative to conduct both types of studies with students at various stages of their professional development, including student teaching and beyond.

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# STUDENTS' UNDERSTANDING OF PROOF IN A COMPUTER ENVIRONMENT

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*This paper discusses the understanding of the need and utility of proof by students in two 10th grade classes using the program LOGO.GEOMETRIA, an open computer tool designed to deal with classical Euclidean Geometry problems. For some of the students, visual evidence was enough as a mean of validation. Others, made wide use of the random constructor capabilities of the software. Finally, for a small group of students, logical proof appeared as the more adequate way of certifying mathematical statements.*

## Introduction

This paper discusses the understanding of the need and utility of proof by students in two 10th grade classes. It is a partial report of a larger project, which major objectives were to promote in the students, in their study of Cartesian and Vector Geometry:

a) The construction of the relevant concepts on this topic.

b) The ability of fomulating and dealing with problem solving situations.

c) An understanding for the need and utility of the proof.

d) New attitudes and concepts about mathematics and about their role as students.

This experience used 8 PC type computers and the software LOGO.GEOMETRIA, which is an open computer tool designed to deal with classical Euclidean Geometry constructions (Veloso, 1989).

## Proof in Mathematics and in School Mathematics

The generality of the mathematicians knows that understanding a proof implies much more than verifying the correction of the deductions that it contains—there is an assemblage of ideas that supports it that must be felt by the reader. The importance of one proof is not only to convince us that the theorem is true—to confirm that the intuition did not nail a trick—but to show us why the theorem is true and in what conditions (Watson, 1980).

The learning of proofs by the students is done, usualy, through the memorisation. The teacher writes on the blackboard the theorem and its proof and the students copy them to their notebook with the intention and the need to reproduce them in an oral or written test. The student, by this way, is carried to a passive attitude in which he becomes a contemplate of a fininshed structure. This methodology hides to the students the proof "as a tool of proof, over increase the value of the proof as a kind of discourse where is done the essencial value to the structure" (Balacheff, 1988, p. 18). For the proof to have meaning to the students it is necessary that these feel it as a faithful tool of efficacy to establish the validity of one proposition. The students must feel that the proof "is necessary to establish the generality of the proposition we want demonstrate, that is, the possibility of its application to all the particular cases" (Fetisov, 1980, p. 18).

It is becoming widely accepted that the formulations of conjectures by the students, should have a bigger weight in the teaching/learning of mathematics in our schools. The conjecture will can take with it the conviction of the racionality (Watson, 1980). Of course that, afterwards, it is necessary to prove it, to prove its veracity, under the punishement that the conjecture never give up be. Through experimentation with one or various cases it is possible to support a conjecture, that, in turn, can be valid in still other cases. But we cannot be sure that it is always valid. Only a proof will give one reason that convince us definitively.

The elaboration of proofs, for the students, must be accompanied by the critical analysis of them, with a constant exploration of the mathematical objects, permanently questionated. By the other side, the teacher must stand, as much as possible, exterior to the taking of decisions about the validity of the proofs, allowing that these decisions be, fundamentaly, took by the students.

## Methodology

The students involved in the experience were in two 10th grade classes (aged 16 to 19). One class, with 27 students, was in the university track, taking a specialization in informatics; another, with 16 students, was in the technical-oriented branch, with concentration in agriculture.

The researcher provided the original proposal for the project and conceived most of the activities. These were afterwards discussed with the two secondary teachers (responsible for the two classes) and modified according to the indications got from the devolopment of the experience. In this activity the students worked from activities and situations proposed in worksheets, formulated conjectures and reflected on their strategies. Every week, students had a two-hour class in the computer room. They had also one-hour class in their regular room (2 for the agriculture and 3 for the informatics students). They were required to write what they had done, why they done it, and what were the difficulties they had met.

The students worked in groups of 2, 3 or 4 on the computer. All the classroom activities, with and without computers, were conducted by the regular teacher, although now and then there were some interventions of the other secondary teacher and the researcher.

The students answered to a small questionnaire (with 4 questions) at the end of the all classes, indicating their opinions about their activity in this experience.

The researcher made regists in a writting-pad -- during the classes with the computers -- about what he saw and heard (questions posed by the students; the originality of the processes the students followed in the solving of the situations; the doubts of the students). These regists were complemented with the discussion/reflexion had between the two secondary teacher school and the researcher after each class with the computer.

## Results

### a) The Visual Evidence as an Obstacle to the Proof

The students involved in this experience manifested, essencialy in the begining, a strong tendency to accept the evidence of the figures. "That we see, just look", said the students many times. They had the conviction that they were correctly solving the activities and they were assisted, in their conviction, for that they saw on the screen. It was due to the doubts posed by the teachers that the students did their verifications. This tendency was attenuated with the time but it did not desappear completly with it, as we can see in this report written in the eighth class with the computer (in which it was pretended that the students discovered the relation of equaly for the slopes of parallel straigh lines, justifying it):

> Activity: "Construct three parallel straigh lines r, s and t. What relation does exist between their slopes? What is its justification?"
>
> Report: "We begin by the construct of the three lines and with the comand DECLIVE (SLOPE) we verify that the slopes were equal. We ask to LOGO.GEOMETRIA for the equations of the lines and we could see one directing vector of each line—they were the same. So, being the same, with the same coordinates, the slopes are the same".

These students justified the relation of equaly with the equaly of the directing vectors (the LOGO.GEOMETRIA constructs the same directing vector for each straigh parallel lines family). These students "only" read what they saw in the ecran, they didn't felt the necessity to investigate if this equaly was the same if the three directing vectors were not the same, for example. The students just stayed with the results given by the computer.

b) Generalization

The students of the agriculture class were satisfied with the discovery of one mathematics law from the few experienced cases and the relations they founded. These students used these relations in other different situations as proved and learned.

The students of the informatics class, from the seventh class with the computer, began to do their experimentations with the random cases given by the random constructers of the LOGO.GEOMETRIA. These students passed to generalize from the analysis of these random cases. One example of this is the report about the same activity. The students wrote:

> "We construct three straigh parallel lines. R.ACASO "r to the first and PARAL to the other two. Afterwards we ask to the computer for their slopes and it gave us the same value. So we conclude that straigh parallel lines have always the same slope. One example of this is to consider one directing vector of one of the lines and to divide the ordinate by the abscissa and we have the slope. As the directing vectors of the three parallel lines are the same (or colinears) we can conclude that the quotient ordinate/abscissa will give, ever, the same value".

c) Test and Proof

To perceive out the insufficiency of the initial verification of a conjecture from a few examples and carry on to verify, yet, in another case, is an important step whatever the validation of the generalization done. It is an empirical answer that convinced some students.

In the activities proposed to the students we insisted very much on the test of the conjectures. We thought that the students would be satisfied with the discovery of the relations from the experimentation with one or more cases and, by other side, we had the intention to elevate the exigence of the students whatever the validation of one proposition. It would be to force the students to one transitory way from the generalization from a few cases to a proof for every, and all the cases.

The students that test their conjectures with the random case were satisfied themselves with it and did not do a proof to validate their conjectures. To these students, the veracity of the conjecture was done. To these students, this random example had the force of a total validation to the founded relation, as we can see in the following report (wrote in the eighth class with the computer):

> Activity: "Construct a straigh line r with general equation $2x - 5y + 200 = 0$
> Find a relationship of the line slope with the coordinates of its versor. Consider other cases of your choice. Test your conjecture."
>
> Report: "We constructed the line r and we found its slope (0.4). Afterwards we constructed one directing vector and we found its coordinates (-99.996,-40). The quotient ordinate/abcissa gave us the value of the slope of the line. To certificate that was the same to all the lines we construct one random line (R.ACASO "s) and we make the same quotient. We obtain the slope of the line".

Three groups of the informatics class presented a report after the sixth class with the computer (where we hoped that the students discovered the relation between the coordinates of the end and the middle points of a segment line, from a given segment and from others constructed by the students) whith a proof (not asked in the activity) of the relation they had obtained. The students believed that the relation was valid and they made a proof. That is the report:

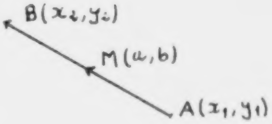
"$\overrightarrow{AM} = \overrightarrow{MB} \iff M - A = B - M \iff$

$(a,b) - (x1,y1) = (x2,y2) - (a,b) \iff$

$(a - x1, b - y1) = (x2 - a, y2 - b) \iff$

$a - x1 = x2 - a \ \wedge \ b - y1 = y2 - b \iff$

$a = (x1 + x2)/2 \ \wedge \ b = (y1 + y2)/2$".



This exigence of the students was observed in others situations.

During this experience the students were gaining the need to explain, to justify everything they did and all their conclusions. One example of this can be observed in this report (wrote as an answer to the first activity):

> "After the construction of the parallel lines r, s and t, we ask to the computer their slopes and this gave us the same value (7.692). To justify it we construct a directing vector to each line but the computer (maybe because it is lazy!) gave us the same vector (-1.3, -9.998). We divide the ordinate by the abcissa and we obtain the same value (7.692). Even the computer gave us three different vectors, they would be colinears, because the lines are parallel, and the quotient ordinate/abcissa would be the same value (7.692)".

These students "made counts without making them" to consider three different vectors, any one, without having construct them, even in the computer to visulize them, showing, by this way, that they had learnt the concept of the directing vector of one line as well as the concept of direction.

## Conclusions

The force of the evidence was, particularly in the begining of the works, and to the students of the both classes, an obstacle to the need and utility of proof. This real danger must be considered in mathematics education, specialy if we want implement the visual reasoning (necessary and useful) in mathematics education.

The students of the class of agriculture (with less knowledge of mathematics than their informatics colleagues) formulated conjectures but validated them immediately from the few concret cases they had experienced. However, many students of the informatics class worked with the random example given by the computer and felt it as a special case (not just another case), from the seventh class with the computer. They felt it as a representative of a class of objects of mathematics. Lots of these students validated their conjectures with the random cases. To them, these cases had the force of a proof. This study showed a clear relation between students' mathematics knowledge and the need and utility that they see to explain, to justify and to validate a mathematical proposition. By other side, this study carried the researcher to a bigger conviction of the importance and the utility of the students making their own proofs as the "heart" of the learning of mathematical proof.

The students felt that they gained power to solve more problems if they had the force of a proof of one proposition, because they could use it (it is valid for every cases) to solve another problem. Some students felt the necessity to proof, even when they were not asked for that. This study shows that the students must feel the utility of the proofs, on the contrary they look to them as something useless, sick and "just" to memorize.

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# UNDERSTANDING EQUIVALENCES THROUGH BALANCE SCALES*

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*We analyze the conditions under which 5 to 12 year old children believe that equilibrium in a balance scale is maintained and we attempt to show how this understanding is related to the algebraic procedure of cancelling terms appearing on both sides of an equation.*

Mathematical axioms are sometimes so convincingly obvious that it is tempting to treat them as a priori truths known to anyone with the power of reason. Leibniz for instance asserts with regard to equality that "if from two equal things the same quantity be taken away the things will remain equal; likewise .. if in a balance everything is equal on the one side and on the other, neither will incline, a thing which we foresee without ever having experienced it" (1702, p. 609). But the empirical investigation of mathematical thinking in children has repeatedly shown the limitations of such a view. Mathematical axioms are not psychologically axiomatic.

The rule, "when you move one element from one side of the equation to the other side you have to change its sign", provides a simple occasion for studying the development of the concept of equality. This procedural rule is actually a shortcut for adding equal terms to both sides of the equation. Do students spontaneously realize that adding or subtracting equal unknowns from each expression will not alter the equality, or is this recognition the result of a slow developmental process? Steinberg, Sleeman & Ktorza (1990) found that 13 to 15 year old students, when asked whether two linear equations had the same solution (for instance, $x + 2 = 5$ and $x + 2 - 2 = 5 - 2$), rarely justified their answers by appealing to the idea of equal operations on the left and right expressions. Instead they worked out values for the unknown in both equations and compared their results at the end. A failure to understand the conceptual basis of the rule may be at the root of children's mistakes documented by Kieran (1985), such as solving the equation $37 - b = 18$ through the addition of 37 and 18.

Studies on how children develop an understanding of equalities in physical models may help us better understand the kinds of conceptualization regarding quantities and unknowns that algebraic teaching can build on.

In this study we analyze the conditions under which 5 to 12 year old children agree that a given equality, established through the equilibrium in a balance scale, is maintained. Moreover, we analyze how this understanding is related to the adoption of rules for finding the value of unknowns through cancelling-out, which correspond, on the balance scale model, to removing equal quantities from each side.

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Balance scales have been used, among others, by Filloy & Rojano (1984), as a didactical situation, by Carraher e Schliemann (1987), to analyze the understanding of equalities and manipulation of unknowns among street sellers, and by Vergnaud & Cortes (1986) and Cortes, Kavafian & Vergnaud (1990), to analyze children's difficulties in learning algebra and the effectiveness of teaching programs. A two-platter scale is not a complete model of algebra since this physical model is limited. Subtraction as an unary operator can be represented through action, for instance, but not as a relation involving two quantities. Further, multiplication and division are difficult to interpret. Booth (1987) notes that situations in the scale are not always understood by students. However, as data by Filloy & Rojano (1984), Vergnaud & Cortes (1986) and Cortes, Kavafian & Vergnaud (1990) suggest, this is a meaningful situation to promote and analyze initial understanding of equations and of rules of algebraic manipulation.

## METHOD

Subjects: Subjects were 75 children divided into four age groups: 16 with ages 5 to 6, 19 of ages 7 to 8, 20 aged 9 to 10 and 20 aged 11 to 12. They were enrolled from kindergarten to 6th grade classes of a private school in Recife, Brazil.

Material and Procedure: A balance scale of the type commonly encountered in markets in Brazil was used throughout the tasks. Iron weights identified as weighing 1 kg, 2kg, and 5 kg, plastic bags containing cereals, and boxes were used to establish equalities on the scale.

Procedure: A first series of tasks served to *familiarize* the child with the scale. One bag of cereal was placed on one of the platters and the child was asked to determine its weight. Initially the weight of the bag corresponded to one of the identified iron weights available (1 kg); in the second example two iron weights (1 kg and 2 kg) were required to weigh the bag; in the final example a subtractive solution (see Carraher & Schliemann, 1987) was required: 1 kg had to be added to the platter with the bag and 5 kg was to be put in the empty platter.

In the second series of tasks the child was asked to *find the values for unknowns*, represented as cereal bags in the scale in equilibrium. In four tasks unknown weights appeared on only one side of the scale and the situations corresponded to the following four equations:

(a) $2kg + x = 3kg$
(b) $2kg + x = 4kg$
(c) $2kg + x = 5kg$
(d) $3kg + x + x = 5kg$.

In the other five tasks unknowns appeared on both sides of the scale and the corresponding equations were:

(e) $4kg + x = 3kg + x + x$
(f) $5kg + x = 3kg + x + x$
(g) $5kg + 2kg + x = 3kg + x + x$
(h) $5kg + y = 1kg + y + x$

(i) $5kg + 5kg + y = 4kg + 4kg + y + x + x$.

Tasks (e) to (i) aimed at analyzing whether or not children would adopt a cancelling out strategy, after demonstration and prompting by the examiner. Once the child had attempted to solve tasks (e) and (f), the examiner showed on task (f) that a solution could be reached if unknowns of equal values appearing on both platters were removed (this corresponds to crossing out variables on each side of an equation). Task (g) was then given and, if the demonstrated strategy was not used, the examiner recommended its use. Tasks (h) and (i) were given to 11 and 12 year olds in order to evaluate use of the strategy for other tasks since, as results will show, only at this age it was spontaneously adopted in the preceding task.

The third series of tasks analyzed the conditions under which children anticipate the ***maintenance of equilibrium*** of the scale when equal known weights or equal cereal bags or boxes of unknown weight were to be removed from both platters. Seven tasks were designed for this series. In the first five (tasks j to n) known iron weights and cereal bags were used and solutions could be straight-forwardly reached by hypothesizing values for the unknowns and testing to see whether these values led to the same result in each platter. The remaining two tasks (o and p) were designed in such a way that the hypothesis testing strategy would be more difficult to be applied. In these cases boxes of unknown weights and different sizes--where sizes were not correlated to weights--were the unknowns. The seven tasks correspond to the following equations:

(j) $2 + 1 + 1 = 1 + 3$
(k) $4 + x = x + 2 + 2$
(l) $3 + x = x + x + 1$
(m) $x + 3 = 3 + y + z$
(n) $y + x = x + x + z$
(o) $x + x + y = x + z$
(with x, y and z of same size)
(p) $x + x + y = x + z$
(with x larger than z, larger than y)

For each task children were asked whether the scale would still balance if the weights, of known or of unknown value, appearing in both platters, were to be taken away. They were also asked to justify their answers. Control tasks, not analyzed here (where the weight removed from one side was different from that in the other side and therefore equilibrium would not be maintained) were also included in order to avoid an acquiescent response bias ("yes, they all remain balanced").

Children were individually submitted to the three series of tasks, in fixed order, by an examiner and one observer who took notes about the actions performed. The interviews were audio recorded and transcribed for posterior analysis.

## RESULTS

In the *Familiarization* tasks all subjects were able to find the weight of a 1 kg bag and only three of those in the younger groups failed in combining weights of 1 and 2 kg to find the weight of a 3 kg bag. When asked to find the weight of a 4 kg bag, however, 5 and 6 year olds were not successful and 7 to 10 year olds only succeeded after hints from the examiner; but 11 and 12 year olds spontaneously found the solution.

In the *Finding Values for Unknown* tasks, when unknowns appeared in only one side of the scale (tasks a to d), only 5 and 6 year olds had problems in finding the correct answer and nearly all subjects in the other age groups immediately found the answers to the problems.

For tasks with unknowns on both sides (tasks f to i), results are shown in Table 1. These tasks were so difficult for 5 and 6 year olds that, after a few attempts, we decided not to collect data for this age group in these tasks. For the other ages, all tasks were correctly solved through means of different strategies. In task (e) the strategy chosen by all subjects was arithmetical computation, often combined with the hypothesis testing procedure where a value for a given unknown was hypothesized and, replacing all cases by the chosen value it was accepted if, at the end, the same result was obtained for each platter. The following protocol illustrates such strategy:

Subject: Daniela (9 years old). Equation: (e) $4 + x = 3 + x + x$
E (Examiner): How is the scale?
S (Subject): It is balanced.
E: What is the weight of each bean bag (x)?
S: 1 kg.
E: How do you know?
S: Because, don't you have 3 kg? (showing one platter) Then, 3 plus 1 makes 4, with 1 more makes 5. Then here (showing the other platter) you have 4, with 1 more, it makes 5. Then it is equal.

Results for task (f), analyzed after demonstration of the cancelling-out strategy, show that 11 and 12 year olds were more inclined than the younger subjects to use this new approach. For task (g) nearly all 7 to 9 year olds continued to rely on hypothesis testing and arithmetical procedures and, only after suggestion, adopted the cancelling out strategy. Contrasting to these data, most 11 and 12 year olds spontaneously adopted the strategy demonstrated in task (f) in order to solve task (g).

Results for the *Maintenance of Equilibrium* tasks are shown in Table 2, in terms of percentage of correct answers and type of justifications offered for the answers given. For tasks (j) to (n), except in the case of 5 and 6 year olds, most responses were correct. Tasks (o) and (p), however, were more difficult and only 11 and 12 year olds maintained their high rate of correct responses in all three tasks.

The difference in performance between the first five tasks and the last three, together with the analysis of the justifications provided, indicate that responses given by the majority of children in the older age group, as opposed to the younger ones, were based on different processes of reasoning. Three types of justifications were found: logical, computational, and wrong justifications. *Logical* *justifications* appealed to the fact that, if the weight to be taken from one side was equal to the one taken from the other, the equilibrium had to be maintained, as in the following example:

Subject: Raul (12 years old). Equation: (n) $y + x = x + x + z$.
E: How is the scale?
S: Balanced.
E: These bags (x) are equal. What happens if I take one from each side?
S: There will be the same weight.
E: Why?
S: Because you are taking the same amount from each side.

*Computational justifications* were based on the results of arithmetical computations on the values of weights left on each platter, hypothesizing values for the unknowns weights, if needed:

Subject: Fabiana (6 years old). Equation (k) $4 + x = x + 2 + 2$.
E: How is the scale?
S: Equal.
E: These two rice bags (x) are equal, they weigh the same. If I were to take one of these from each side, would the scale still be equal or would one side be heavier than the other?
S: Equal.
E: Why?
S: Because here there is 4 (in one platter) and 2 plus 2 is 4 (in the other platter).

Subject: Rosemberg (10 years old). Equation (m) $x + 3 = 3 + y + z$.
E: How is the scale?
S: Balanced.
E: If I take away these two weights of 3 kg, how will the scale stay?
S: It will stay balanced.
E: Why?
S: Because this one here... Let's suppose that this one (y) weights 2 kg, this one (z) weights 1 kg, and this one (x) weights 3 kg. 3 and 3 makes 6 (in one platter), 3 here (in the other platter) and these 3 (adding 2 plus 1) makes 6 too. If you take away, there will be 3 in one side and 3 in the other.

*Wrong justifications* were those given for wrong answers or nonsense justifications given for correct answers.

While at least 50% of the 11 and 12 year olds gave logical justifications for tasks where unknowns were to be left on the scale (tasks l to q) children in the other age groups clearly preferred the computational justifications. This explains why in tasks (o) and (p), where hypothesizing values for the unknowns left on the scale became more difficult, the percentage of wrong answers increased.

The relationship between the tendency to provide logical justifications for the maintenance of equivalences and the spontaneous use of the cancelling out strategy was evidenced through the analysis of each subjects' result in task (g) and task (p). While 56% of the subjects giving logical justifications spontaneously adopted the manipulation of unknowns approach, only 27% of those providing computational justifications and 11% for those giving wrong justifications justified their answers through the idea of equal operations.

## DISCUSSION

A systematic analysis was made of 5 to 12 year old's understanding of equalities. Children of 5 or 6 years of age were only able to understand the equilibrium of scales if, to a known weight on one platter corresponded a known weight on the other. By 7 to 10 years children determine the values for unknowns when items of unknown weights are located on one or on both sides of the scale by hypothesizing values and testing whether or not they lead to a balanced "equation". With prompting 7 year olds can adopt the cancelling out strategy for situations with unknowns on both sides. However, before age 11, they seem to do so as a mechanical routine. Instead of accepting as a necessary truth that the equilibrium will be mantained if the same amount is subtracted from both platters, they seem to work from the premise that one can only be certain that the scale will balance if one knows the total weights on each platter. From 11 years of age there seems to occur a shift in understanding and the cancelling-out strategy is spontaneously adopted after one demonstration. Moreover, many children in this age group believed that the equilibrium would be maintained if equal amounts were to be removed from both sides, *without needing to compute values left on the platters*. They argued that if the amounts to be taken away were equal the scale must stay balanced.

Understanding algebra requires preliminary work involving known and unknown quantities. If a solid foundation is not built it is unlikely that algebraic structures will firmly stand. If children are having trouble with relations among known and unknown quantities, they will most certainly have trouble understanding the relation between variables.

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Table 1: Strategies in the Finding Values for Unknown Tasks, when unknowns appeared on both sides of the scale

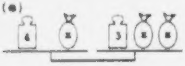
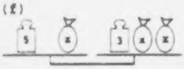
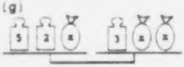
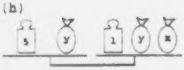
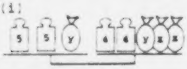
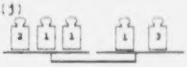
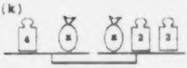
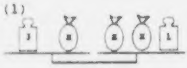
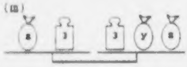
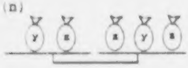
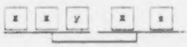
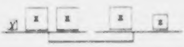
| Situation on the balance scale to find values for x | Age groups | Spontaneous cancelling out | Cancelling out after suggestion | Arithmetic computation |
|---|---|---|---|---|
| (e)  | 7/ 8 | 0 | 0 | 100 |
| | 9/10 | 0 | 0 | 100 |
| | 11/12 | -- | -- | -- |
| (f)  | 7/ 8 | 0 | 32 | 68 |
| | 9/10 | 0 | 15 | 85 |
| | 11/12 | 5 | 40 | 55 |
| (g)  | 7/ 8 | 5 | 74 | 21 |
| | 9/10 | 10 | 50 | 40 |
| | 11/12 | 70 | 10 | 20 |
| (h)  | 7/ 8 | -- | -- | -- |
| | 9/10 | -- | -- | -- |
| | 11/12 | 75 | 5 | 20 |
| (i)  | 7/ 8 | -- | -- | -- |
| | 9/10 | -- | -- | -- |
| | 11/12 | 80 | 5 | 15 |

Table 2: Justifications in the Maintenance of Equilibrium tasks

| Situation on the balance and values to be taken away | Age groups | Percent correct | Logical | Computational | Wrong |
|---|---|---|---|---|---|
| (j)  | 5/ 6 | 50 | 0 | 62 | 38 |
| | 7/ 8 | 100 | 5 | 95 | 0 |
| | 9/10 | 100 | 10 | 90 | 0 |
| (k)  | 5/ 6 | 75 | 0 | 62 | 38 |
| | 7/ 8 | 100 | 5 | 95 | 0 |
| | 9/10 | 100 | 0 | 100 | 0 |
| (l)  | 5/6 | 56 | 0 | 50 | 50 |
| | 7/ 8 | 90 | 5 | 75 | 20 |
| | 9/10 | 95 | 5 | 90 | 5 |
| | 11/12 | 90 | 65 | 25 | 10 |
| (m)  | 5/ 6 | 60 | 6 | 47 | 47 |
| | 7/ 8 | 80 | 5 | 75 | 20 |
| | 9/10 | 90 | 5 | 85 | 10 |
| | 11/12 | 95 | 65 | 30 | 5 |
| (n)  | 5/ 6 | 56 | 6 | 44 | 50 |
| | 7/ 8 | 80 | 5 | 65 | 30 |
| | 9/10 | 100 | 15 | 85 | 0 |
| | 11/12 | 90 | 60 | 30 | 10 |
| (o)  | 5/ 6 | 27 | 0 | 20 | 80 |
| | 7/ 8 | 60 | 15 | 35 | 50 |
| | 9/10 | 80 | 10 | 70 | 20 |
| | 11/12 | 95 | 55 | 35 | 10 |
| (p)  | 5/ 6 | 20 | 0 | 7 | 93 |
| | 7/ 8 | 30 | 5 | 20 | 75 |
| | 9/10 | 60 | 25 | 30 | 45 |
| | 11/12 | 85 | 50 | 35 | 15 |

# KNOWING AND USING THE PYTHAGOREAN THEOREM IN GRADE 10

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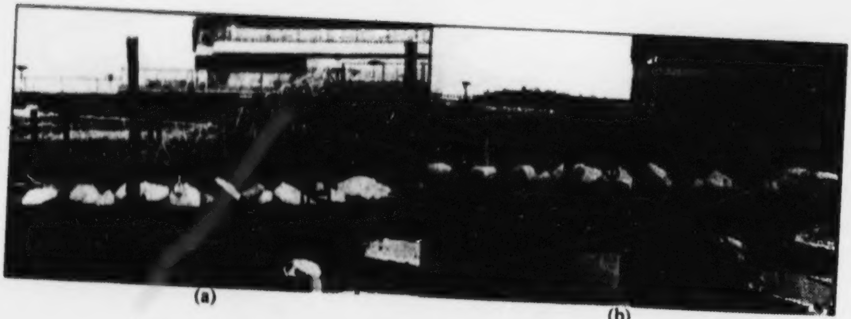
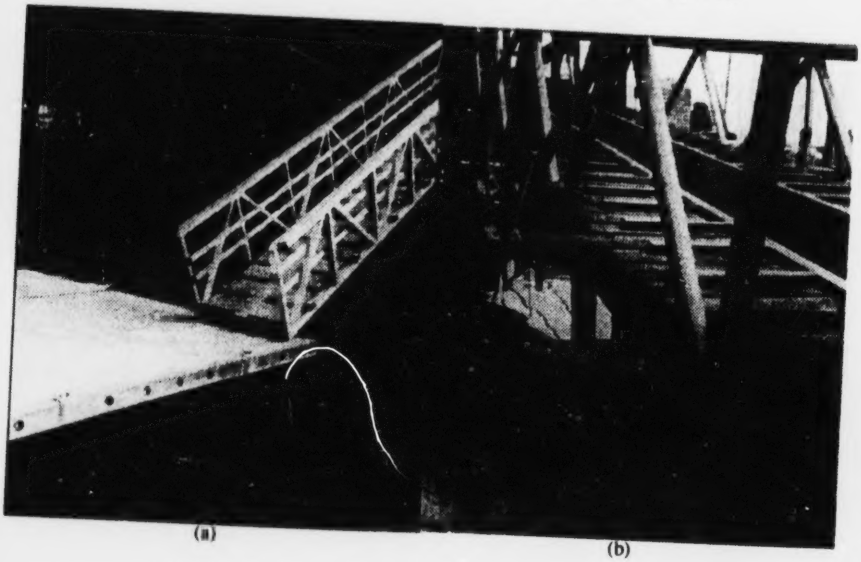
*The purpose of this study was to provide a qualitative assessment of students' problem solving; its focus was on the nature of students' thinking, their problem-solving strategies and heuristics, the mathematical approaches that they selected, and the ways they monitored their progress. The data include interviewers' field notes and students' written responses to a non-routine problem presented orally and with photographs in 15 task-based interviews with individual students and pairs of students in Grade 10. Although the problem can be solved most directly by applying the Pythagorean theorem, many students initially tried using trigonometry. The main difficulties students encountered seemed to be metacognitive, having to do with identifying and representing the salient features of the situation in diagrams and deciding what mathematical processes to apply. Implications are discussed.*

In countries all around the world, mathematics curricula for the lower levels of secondary education include the Pythagorean theorem, usually with intended learning outcomes stating that students should be able to apply the Pythagorean relationship both to determine whether a given triangle is a right triangle and to calculate the length of a third side of a right triangle given the other two sides. Current curricula often emphasize practical problem solving situations (e.g. how far up a wall does a ladder reach?) and indicate that a calculator should be used as appropriate. The results discussed in this paper are part of a project (Schroeder, in press) that was conducted in conjunction with a large-scale assessment (Robitaille, Schroeder, & Nicol, 1991) that used achievement test items in multiple-choice and constructed-response format. The goals of the small-scale study were (1) to develop a series of non-routine problems that would be challenging yet accessible, that would demand planning and reflection, that would permit a number of different methods of solution, and that would embody a variety of familiar mathematical processes and operations, but not in obvious, routine ways; and (2) to use those problems in interviews with individuals and pairs of students to describe and assess the mathematical processes and operations they apply and the problem solving plans and strategies they adopt. One of the seven problems developed for Grade 10 students had content related to the Pythagorean theorem. This task, referred to as the "dock" problem, was presented to students orally by an interviewer who explained the problem situation with the help of photographs.



(a) (b)

*Figure 1: A waterfront restaurant and its dock at low tide and at high tide.*



(a) (b)

*Figure 2: The lower end of the ramp at present and as proposed.*

**Interview Task.** Students were told that the waterfront restaurant shown in Figure 1 has a dock, one part of which rises and falls with the tide. Access from the fixed part of the dock to the floating part is by means of a ramp which is relatively steep at low tide (Figure 1a), but less steep at high tide (Figure

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1b). Currently, the lower end of the ramp rests on the floating dock (Figure 2a). As the floating dock rises and falls, the end of the ramp scrapes across it causing scratches in the surface. In order to prevent this damage, the owners of the dock plan to mount wheels on the lower end of the ramp and tracks on the floating dock for the wheels to run in – an arrangement similar to the one shown in Figure 2b, which shows another dock nearby. The problem is to determine how long the track needs to be. The data provided are that (1) the ramp is 18 m long, (2) when the tide is at its highest, the floating dock is 1 m below the fixed dock, and (3) when the tide is at its lowest, the floating dock is 6 m below the fixed dock.

It was anticipated that students would approach the problem by drawing a diagram and by recognizing the importance of two right triangles, both having the 18 m long ramp as the hypotenuse, one 1 m on its vertical side, the other 6 m tall. The difference between the lengths of the horizontal sides of these two triangles is the required length of the track. Thus the simplest method of solving the problem is to use the Pythagorean theorem to determine the two unknown sides and then subtract. It was anticipated that students might use other means of solving the problem such as constructing a scale drawing or applying trigonometric ratios to calculate the unknown lengths.

**Procedures.** Seventeen volunteers (11 females and 6 males) took part in interviews based on this problem; thirteen were interviewed individually, the remaining four in two same-sex pairs. The students were provided with the photographs, a summary of the data, a scientific calculator, tables of square roots and trigonometric functions, squared paper and plain paper, and a geometry set (ruler, protractor, set square, and compass). In the introduction to the interviews students were informed that the activity was not a test and that their performance would not affect their standing in their mathematics course. They were told that the interviewer was more interested in how they went about solving the problem than in whether they got the correct answer, and that they could ask questions of the interviewer at any time. Students were urged to think aloud as they solved the problem so that the interviewer could understand how they worked on the problem, and they were told that the interviewer might ask them questions for clarification or give them hints or help if they wished.

As the students worked on the problem, the interviewer observed them closely and made field notes. At the conclusion of each interview the interviewer completed a record sheet designed to be a convenient and standardized way of summarizing and reporting students' work on the problem. The sections of the record sheet headed "Understanding," "Strategy Selection," and "Monitoring" reflect Polya's description of the phases of problem solving and the importance of metacognitive activity in problem solving; they list a number of anticipated features of students' work which can be checked off as appropriate, and provide spaces in which to describe students' work. In the final section headed "Overview" the interviewer is to indicate whether student(s) solved the problem essentially on their own, or solved the problem with needed help from the interviewer, or did not solve the problem even with help from the interviewer. The amounts of time spent reaching a solution, extending the problem or looking back, and using different approaches are also to be noted.

**Results.** In the discussion which follows, the unit of analysis is the interview, of which there were 15. Because of the small numbers, results are not reported separately for males and females nor for students interviewed individually or in pairs. The overall results showed that in five interviews (33%) the students solved the problem on their own, and that in ten interviews (67%) the students solved the problem with help from the interviewer; in none of the interviews did the students fail to solve the problem. The total length of time spent in each interview varied from 22 to 50 minutes with a median of 38 and a mean of 39, and the time taken to reach a solution ranged from 9 to 50 minutes with a median of 18 and a mean of 22. Interviews in which the students solved the problem on their own tended to be somewhat shorter overall and noticeably shorter in time to solution. The median time to solution for students who solved the problem on their own was 10 minutes as opposed to 24 minutes for students who received help; the means were 13 minutes and 27 minutes respectively. All these time measures include about three minutes spent exchanging introductions, recording facts such as names and birthdates, summarizing interview procedures, and obtaining students' consent to participate. An additional two or three minutes was taken showing students the photographs and presenting the problem orally. Although these gross measures give a sense of the extent of the interviews and an idea of how well the students performed, they were not the focus of the analysis; the qualities of students' work was the main concern.

It was anticipated that students would draw diagrams both as a means of understanding the problem and to facilitate their work on it. Although all students eventually solved the problem using diagrams that included two right triangles, there were wide variations in their initial drawings, some of which are shown in Figure 3. In these early diagrams different features of the problem are prominent, and in some of them critical features of the problem are misrepresented. For example, in two interviews students first represented the situation as in Figure 3a with three parallel lines. Two students initially drew diagrams similar to the one in Figure 3b with the track in the plane of the ramp rather than the plane of the floating dock. The student who drew the diagram in Figure 3c focussed on the rotation of the ramp about its upper end. As she examined the photos she rotated her pencil holding the upper end stationary just above the paper and allowing the point to trace out an arc on the paper.

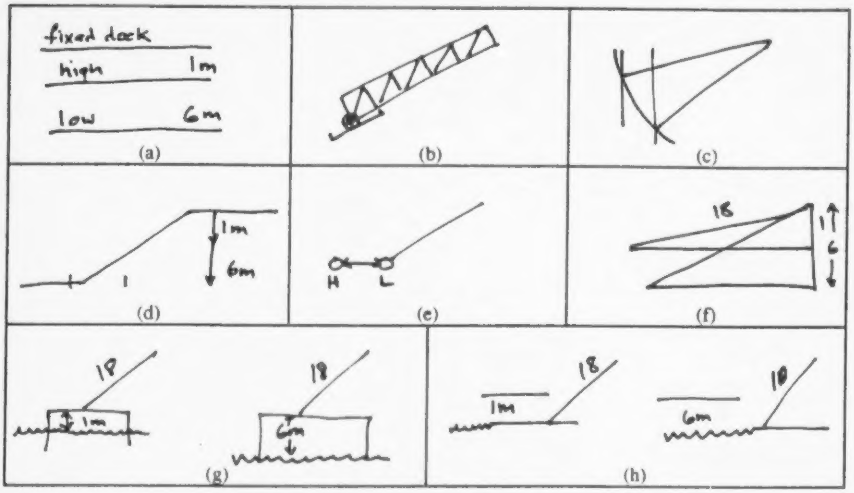

*Figure 3: Initial diagrams drawn by students.*

The five students who solved the problem on their own quickly produced appropriate diagrams in which the two needed right triangles were prominent. In four cases the triangles were drawn in two separate figures; in one case they were overlapping as in Figure 3f. In five of the ten interviews where the problem was solved with help from the interviewer, students produced diagrams on their own which they used to make progress toward a solution; the help they received was unrelated to representing the problem in a diagram.

In the remaining five interviews students received help that was related to representing the situation in diagrams and identifying the relevant parts of diagrams they had drawn. In one case, after the student had spent some time studying the photographs and appeared to be stuck, the interviewer suggested that it might help to draw a diagram. In two cases the students had drawn appropriate overlapping diagrams (similar to Figure 3f), but after several minutes had not made progress using them. In one of these cases the interviewer asked, "Are there any triangles in the diagram that you could use?" and in the other he said, "Would it help to draw two separate diagrams?" to which the student immediately replied, "You mean one for high tide and one for low tide?" Each of these hints led immediately to progress toward a solution. The difficulties experienced in the remaining two interviews seemed to be related to misconceptions regarding the names and relative positions of the fixed dock, the ramp, the floating dock, and the linkages between them. These students produced the initial diagrams shown in Figures 3g and 3h. Their difficulties were resolved in question and answer exchanges between the interviewer and the students which focussed the terms, the photos, the data, and the students' diagrams.

It was anticipated that students would use the Pythagorean theorem to find the horizontal dimensions of the two right triangles formed by the floating dock, the vertical, and the ramp, and in all 15 interviews students obtained a solution in this way. In 11 of the interviews (73%) students used the Pythagorean theorem without being given a hint that they should do so, and without receiving any help in applying it to the figures they had drawn. The helps and hints given in the remaining four interviews (27%) ranged from the fairly oblique, "Is there any way you could relate the side you want to the sides you know?" in one case, to the quite direct, "Would Pythagoras's theorem help?" in another. A third student wondered aloud whether the angle was a 90° angle, and was asked by the interviewer, "What if it was 90° and what if it wasn't?" to which she replied, "If it was, I could use Pythagoras." In the fourth interview, the two students had spent more than 40 minutes trying various approaches without success, when one of them asked, "What's the square root table for?" The students decided on their own that it could be a hint to use Pythagoras, and before long they reached a solution by this method.

All of the students seemed to be quite familiar with the Pythagorean theorem, although one student referred to his method as "using a theory," and another referred to it as "Mr. X's method," presumably because that teacher had taught or reviewed it. In cases where the students received hints related to the

Pythagorean theorem, the hints were mostly vague questions rather than direct hints, and they concerned whether to use the Pythagorean theorem, not how to use it. There were no instances in which students made errors using the Pythagorean theorem that they did not detect and correct by themselves (e.g. failing to square or take the square root, adding the squares rather than subtracting them, making computational errors, etc.).

Application of the Pythagorean theorem was not, however, the first approach adopted in all the interviews; in seven interviews (47%) students began by using or proposing to use trigonometry. One student produced his first solution using trigonometry, but most of the students abandoned this approach either because they ran into difficulties with it or because they noticed that applying the Pythagorean theorem would be simpler. In all cases where there was time available, students who had found the solution were asked if they could solve the problem in another way. In six interviews (40%) students solved the problem using trigonometry. In two interviews the students produced a trigonometric solution without help from the interviewer, but assistance of various types was given in the other four. Two of the students commented that they were just starting to learn trigonometry in their mathematics class; they thought trigonometry could be used, but they weren't sure they could do so successfully. The fact that the students had only recently begun studying trigonometry probably accounts for the large number of students who thought of using it and for the difficulties they encountered in doing so.

Before the interviews were conducted it was anticipated that some students might use scale drawing as a means of solving the problem, and for that reason a geometry set was provided. None of the students proposed solving the problem with a scale drawing, and two students thought it would not be possible when the interviewer suggested it.

**Discussion.** One of the most remarkable findings of this study was the amount of time that the students spent working on the problem. By comparison with multiple-choice test items, which students are expected to answer at the rate of about one per minute, or constructed-response items, which take on the order of five minutes, this task was quite time consuming, and there is a question whether the time required is justified by the information obtained. The amount of time that the students spent is a measure of their perseverance with the task and their willingness to reflect on and extend their work. One student, when asked whether she could solve the problem in a different way, commented that solving problems in more than one way was not something that was ever done in her mathematics class.

In the large-scale assessment students responded to five multiple-choice items involving the Pythagorean theorem. On two items about two-thirds of the students showed that they could find the hypotenuse of a right triangle with integral sides. About half of the students were successful answering two routine application items, but less than a third of the students were able to solve the non-routine problem of finding the length of the diagonal of a rectangular solid. An interpretation panel called these results "disappointing" and concluded that "students did not have an understanding of the Pythagorean theorem" (Taylor, 1991, p. 158). This sweeping generalization does not seem apply to the students who took part in this study, but they may not be representative of the whole population tested in the large-scale assessment. It is clear that the students who solved the dock problem did not lack understanding of the Pythagorean theorem nor related computations skills. Their difficulties had to do with general problem-solving skills and metacognitive abilities such as representing the problem, identifying relevant features, and planning what to do.

Students solved the dock problem without help from the interviewer in only 33% of the interviews. One way of interpreting this result is to say that the problem was relatively difficult for them, but an analysis of the nature of the hints and help provided by the interviewer suggests that what they needed was not direction about what to do, but encouragement and help thinking about their plans for proceeding. The hints in the form of questions which were described earlier which resulted in progress are typical of the internal dialogue that many researchers have identified as crucial for success in problem solving. The interviews suggest that students' cognitive monitoring needs to be developed, and this problem may provide an appropriate context in which this development can take place.

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# SOCIAL DIMENSIONS OF PROOF IN PRESENTATION: FROM AN ETHNOGRAPHIC INQUIRY IN A HIGH SCHOOL GEOMETRY CLASSROOM

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*Though social processes have been recently recognized as important aspects in teaching and learning of proof, empirical investigation of the topic is still limited. The present study explored social processes of mathematical proof by ethnographic methods in a high school geometry classroom in the United States. From the analysis of classroom episodes complex faces emerged of proof as presented in the classroom. Problematic aspects of proof format and communicative elements in proof explanation are identified. It is argued that research needs to pay more attention to the context where the proof is situated.*

Mathematical proof has taken important part in mathematics curriculum, and many efforts have been done on its teaching and learning. The concepts of proof underlying those efforts were often that of formal proof. The teaching of formal proof, however, poses several difficult problems. First, it is hard to relate formal proof to students' other experiences. Also, recent van Hiele theory (van Hiele, 1986) indicates that students need more experience in which they reason in less formal mathematical systems. Furthermore, a stress on formal proof may distort the dynamic processes of mathematical inquiry (Hanna, 1983). Recent studies in the philosophy and history of mathematics have proposed a sociological approach to mathematics (Bloor, 1976; Lakatos, 1976; Tymoczko, 1985; Wittgenstein, 1978). They contend that mathematical knowledge does not grow as traditionally conceived, that is, by an accumulation of deductive truths. Instead, scholars stress the importance of informal mathematics in mathematical inquiry and picture mathematics as an endless interaction among conjectures, informal proofs, and criticisms (refutations) in social situations.

Nevertheless, these alternative--informal and social--aspects of proof have been little explored. Furthermore, very few studies (e.g., Balacheff, in press; Lampert, 1988; Moore, 1991; Tinto, 1990) have been conducted on them in the actual classroom, which is an important social setting. Without understanding the informal, social aspects of proof, the isolation of proof from students' experiences and the rest of mathematics may not be resolved, and students' critical and creative thinking might be discouraged. The present study was an exploratory attempt to understand the informal and social aspects of proof in the classroom.

## Description of the Study

The present paper is based on my dissertation study (Sekiguchi, 1991). The study explored the social nature of proof in the mathematics classroom. The focus was the relation of proof to "refutation"--to counterargument. The underlying assumption was that when mathematics is construed as an interpersonal, dialectic practice, the activity of refuting another's or one's own conjecture or proof is given a significant status. The study investigated (1) social practice of proof in the mathematics classroom, (2) that of refutation in the mathematics classroom, and (3) the relationship of refutation to

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proof. The present paper, however, will discuss some of the results concerning only (1). The main research questions dealt with are as follows: What does proof as social practice in the classroom look like? Are there differences between proof as social practice and proof appearing in formal mathematics? How can those differences be explained? How do social contexts influence proofs?

The study used an ethnographic approach. It enabled the investigator to obtain data on the actual practice of proof in the classroom, the context where the proof was situated, and the participants' perspectives. One Advanced Geometry class in a high school in the United States was chosen for the study. The participants were the teacher, the student teacher, and the students in the class. The class was observed every day throughout the second half of the school year 1989-1990. Interviews of the participants and collection of their writings were often conducted to supplement the observational data. The lessons were audio- and video-recorded every day. The interviews were usually audio-recorded. During the period of data collection those records were used to write detailed field notes, perform informal analyses of the data, and decide on a focus for the next data collection. The records of lessons were also occasionally used in an interview to help an interviewee recall an incident that had occurred in class. After the data collection ended, the records were partially transcribed and used for formal analysis. As a main strategy of data collection and analysis, the constant comparative method (Glaser & Strauss, 1967) was used. Categories and hypotheses relevant to the research questions were developed by applying this method during and after the data collection.

## On the Format of Proof

It is well-known among mathematics educators that there are several different formats for writing a proof (Martin, 1990). In the classroom of the present study also various formats for proof were identified, which seemed perceived as different with each other by the participants. In terms of writing style, I found three different forms:

| | |
|---|---|
| *T-form* | well-known as "two-column form," writing the statements in the left column and the reasons in the right one. |
| *paragraph form* | a writing style using ordinary sentences and paragraphs. |
| *informal form* | a much more sketchy form than paragraph form, often omitting reasons. |

Also, in terms of the logic of reasoning, direct and indirect forms, in terms of content area, geometric and algebraic forms were found (see Sekiguchi, 1991).

It was distinct that various elements in the classroom situation seemed to influence the use of a particular form: the standard set by the textbook, the teacher's objective for the lesson, the students' difficulty in understanding, the content of the procedures used in the proof, the students' learning in elementary and middle school, and so on. The T-form, direct, and geometric forms were standard, but there appeared several other combinations depending on the context. For example, in the class of this study, when students were asked to "show" an algebraic relation on geometric figures, they often used the informal form in writing proofs. The students' past experience in learning algebraic problems seemed to have had influence in that behavior.

The preceding classification of proof format mainly concerned the written proof. When a proof was situated in the context of classroom presentation, the classification, however, needs to add another dimension. The following episode illustrates this point. A student, Monica, presented a proof of the Pythagorean-theorem to the class. The teacher had already discussed one proof in class. Monica presented another proof taken from the teacher's resource book for bonus credit:

Monica (M) began to draw a triangle on the OHP (Figure 1).
M: "Okay. Start off with right triangle HAB. Okay, I'm gonna draw this line; it'll be equal to *a*. Okay, I'm gonna extend, make that a C right here. We're gonna extend this line, make that a D. Okay, C, point C, wait, through point C there is only one line, there's exactly one point perpendicu-, perpendicular to this line [pointing at points D and C alternately]." (Figure 2)

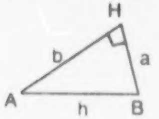


Figure 1. Monica's drawing (1)

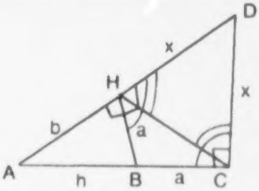


Figure 2. Monica's drawing (2)

M drew a segment DC and a box at the angle C.
M: "Okay. Since this [angle AHB] is perpendicular, you know this [angle DHB] is perpendicular also. So, I'm gonna draw a line from here [H] to here [C]. Okay, this [triangle BHC] is an isosceles triangle since these both lengths [BH and BC] equal *a*. These angles [angle BCH and BHC] right here are congruent. So if you subtract these angles from these angles [pointing at angle DHB], what will these angles be right here? They'd [drawing arcs at angles DCH and DHC] be congruent, okay. Let's say this is *x* [labeling both DH and DC by 'x']."

M: "Okay, now that you know all this, you can say that triangle AHB is similar to triangle ACD [writing "$\Delta AHB \sim \Delta ACD$"] 'cause they both have this angle [angle A] and they both have a right angle. So if we make some proportions ... "
M wrote on the screen this way:

$$\frac{AH}{AC} = \frac{HB}{CD} = \frac{AB}{AD}$$

M: "Okay. What does AH equal? Right here [pointing at segment AH]."
S1: "*h*."
S2: "*b*."
S1: "Oh, yeah."

M wrote "$\frac{b}{\quad}$".
M: "AC [pointing at segment AC] equals?"
S3: "*h* + *a*."

M wrote "h + a" under "$\frac{b}{\quad}$".
Monica continued to ask to the class what HB, CD, AB, and AD were, making proportions and working on them. Reaching an equation $b^2 + a^2 = h^2$, she finished the presentation.

On the OHP Monica wrote only the "statements" part of proof but no "reasons." As a written proof, her proof is to be classified as an informal proof. However, during the presentation Monica was communicating reasons in oral form: "Through point C there is only one line, there's exactly one point perpendicu-, perpendicular to this line," "since this [angle AHB] is perpendicular," "since these both lengths [BH and BC] equal *a*," "subtract," "'cause they both have this angle [angle A] and they both have a right angle," and so on. If those reasons were written down, the proof would be very close to a paragraph-form proof. Therefore, when a proof is put in the context of oral communication, the distinction between an informal proof and a paragraph-form proof is sometimes blurred at least with respect to the amount of information communicated. The same thing was the case on the distinction between a paragraph-form proof and a T-form proof. When the oral explanation of a proof was transcribed, the transcription often appeared to be very close to a paragraph proof.

We can extrapolate here an extreme type of proof presentation in which a presenter talks about his or her proof but does not write anything down. It is not difficult to imagine that type in the geometry class because in geometry if you use a diagram, you can often explain a full proof without writing down any statement or reason. When an oral communication is incorporated into the classification of proofs, thus, we can locate proofs in a two-dimensional plane with the axes of information in written and oral forms in Figure 3.

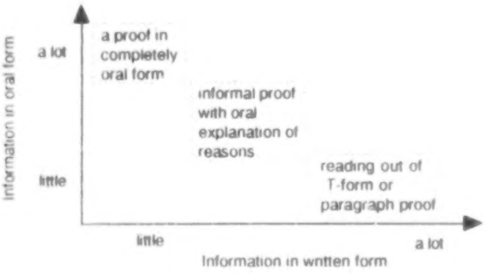

Information in oral form
a lot
little
a proof in completely oral form
informal proof with oral explanation of reasons
reading out of T-form or paragraph proof
little
a lot
Information in written form

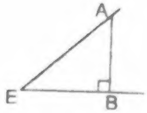
Figure 3. The amount and form of information in proofs

## On Communicating a Proof

When a proof was situated in the context of classroom presentation, it was colored by various elements of communication. Proof was not simply a practice of establishing the validity of a proposition. Presenting a proof involved communicating the underlying ideas and the emphasis. A written proof constituted only a part of the presentation. The other part of the information might be suppled in oral form either by the presenter or the audience--note that in the preceding episode, several students gave inputs to Monica's presentation.

In next episode, a student, Fred, volunteered to present to the class his proof of a homework problem: "Prove: Where $E$ is any acute angle, $(\sin E)^2 + (\cos E)^2 = 1$. (*Hint*: From any point on one side of ∠, draw a perpendicular to the other side.)" (Jurgensen et al., 1988, p. 276). The teacher asked Fred to explain his proof. On the board the proof in Figure 4 was written. Fred stood next to his writing.

Fred: "Angle E is acute, and AB is perpendicular to BE, that's given. And number two ... "



Prove $(\sin E)^2 + (\cos E)^2 = 1$

| | |
|---|---|
| 1 $\angle E$ is ACUTE, $AB \perp BE$ | 1 GIVEN |
| 2 $\sin E = \frac{AB}{AE}$ | 2 DEF OF SIN |
| 3 $\cos E = \frac{BE}{AE}$ | 3 DEF OF COS |
| 4 $(\sin E)^2 = \frac{AB^2}{AE^2}$ $(\cos E)^2 = \frac{BE^2}{AE^2}$ | 4 MULT |
| 5 $AB^2 + BE^2 = AE^2$ | 5 PYTHAG THEM |
| 6 $\frac{AB^2}{AE^2} + \frac{BE^2}{AE^2} = 1$ | 6 DIV |
| 7 $(\text{SIN } E)^2 + (\text{COS } E)^2 = 1$ | 7 SUBST |

Figure 4. Fred's proof of $(\sin E)^2 + (\cos E)^2 = 1$

He continued to explain his proof almost just by reading it line by line from left to right.
T (the teacher): "Beautiful. Beautiful proof [Ss clap their hands]. Fred. Stay up there for a second please. Someone has a question. Wonderful."
Mary: "How did you figure that out?"
T giggles.
Fred: "I just wrote, uh, I just, uh, I just wrote down sine and cosine, what they are equal to, and then I set this [pointing at AB in the diagram] plus this [pointing at BE] equal to this [pointing at AE] was Pythagorean theorem. And, then I put them all together in proof, squared them, and then I divided everything by $AE^2$ because this is $AE^2$ down here [stress by Fred], divide by $AE^2$, which equals 1. So, and I substituted in from here [pointing at Step 4 statement] to here [Step 6]. They give you this [Step 7]."
T and students expressed a surprize at Fred's idea and praised it.

In this episode Fred wrote up a complete--at least from the teacher's view--T-form proof on the board. He followed a common method of proof presentation in which the presenter reads his or her proof line by line from left to right, inserting a phrase "that's," "that is by," "because," "because of" or whatever before the reason, and occasionally referring to the diagram. I call this type of presentation of a T-form proof the statement-reason type (S-R type).

As indicated in the teacher's applause, the S-R type was a perfectly acceptable method of presentation for the class. However, it seems not to be the best method to serve another purpose of presentation: to communicate to the audience the idea or plan that was the basis on which the presenter generated the proof. Mary could not see Fred's idea of proof when she asked, "How did you figure that out?"

Responding to Mary's question, Fred explained his proof a different way. His explanation indicated at least the following processes:

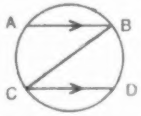
1. Steps 2 and 3 compose one set of actions.
2. Step 5 consists of another set of actions.
3. The idea of squaring in Step 4 actually came after Step 5, probably inspired by the squaring operation in Step 5.

4. The idea of dividing by $AE^2$ came from a comparison of the right side of the equation in Step 5 with that of the equation he wanted to prove.

His explanation thus contains rich information about the process of generating his proof.

The S-R type of presentation does not seem to be very generative. Though it shows explicitly that each step is justified, the whole idea underlying the generation of proof seems to be lost in the forest of detailed steps. However, it was the predominant type of presentation in the class.

There was another type of presentation that was different from S-R type. It is a kind of opposite of the S-R type. It is the reason-statement type (R-S type) where in each step the reason is mentioned first and the statement follows it (the statement is sometimes just pointed at by a finger, or omitted). A student, Liz, several times followed this pattern in her presentation of a T-form proof (Figure 5):



If 2 chords of a ⊙ are ∥, the two arcs between the chords are $\cong$

Prove $\overset{\frown}{AC} \cong \overset{\frown}{BD}$

| | |
|---|---|
| 1 $\overline{AB} \parallel \overline{CD}$ | 1 Given |
| 2 $m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$<br>$m\angle BCD = \frac{1}{2} m\overset{\frown}{BD}$ | 2 The m of an inscribed $\angle$ is = to $\frac{1}{2}$ m of intercepted arc |
| 3 $\angle ABC \cong \angle BCD$ | 3 If ∥, alt int $\angle$s $\cong$ |
| 4 $m\angle ABC = m\angle BCD$ | 4 Def of $\cong$ |
| 5 $\frac{1}{2} m\overset{\frown}{AC} = \frac{1}{2} m\overset{\frown}{BD}$ | 5 Subst |
| 6 $m\overset{\frown}{AC} = m\overset{\frown}{BD}$ | 6 Mult |
| 7 $\overset{\frown}{AC} \cong \overset{\frown}{BD}$ | 7 Def of $\cong$ |

Figure 5. Liz's proof

> Liz: "So you can say that the measure of angle ABC is one half the measure of arc AC [she traces angle ABC and arc AC on the diagram], and the measure of angle BCD is equal to one half the measure of BD [she traces arc BD on the diagram]. And from that, because the lines are parallel, you can use alternate interior angles to say that angle ABC is congruent to angle BCD. And definition of congruency to say that measure of angle ABC equal to measure of angle BCD. Then you substitute, and you find that these half measures are equal, and you multiply and find the measures of this arc and this arc are congruent. Use definition of congruency to find they're congruent."

In the preceding presentation, Liz mentioned the reason before the statement in almost all the steps. In terms of semantics, there is no difference between sentences (1) "*p* because *q*" and (2) "Because *q*, *p*." In that sense there is no difference between the S-R type and the R-S type. In terms of communication, on the other hand, they seem to convey different meanings. When a proof is presented in the S-R type, the stress of one's voice is put on the statement part and not on the reason part. This pattern of stress seems to indicate that the presenter put emphasis on the statement part. The reason part sounded like a *justification* of a conclusion already reached. On the other hand, when a proof is presented in the R-S type, the emphasis seems to be put on the reason part. Through the presentation the process of reasoning in the proof stands out: "Use [a theorem, definition, postulate]," "you substitute," "you multiply," and so on. The reasons are treated as tools to find relations useful for making a proof.

## Concluding Remarks

There are many limitations in this study because the data were taken from just one classroom for only a half of the school year. The classroom was not particularly ideal one that mathematics educators aim to actualize. The study was rather driven by the investigator's theoretical interests in social aspects of proof. Some of the conclusions drawn here, though tentative, are as follows: The use of particular format of proof is influenced by the context and situation where the practice of proof is located. When a proof is presented in the classroom and the oral communication is taken into account, some of the distinctions among proof formats may lose significance. The activity of explaining a proof to the others involves elements of communication: communicating the underlying idea and emphasis.

Research on proof indicates that students use various kinds of proof or justification (Balacheff, 1988; Bell, 1976; Fischbein & Kedem, 1982; Galbraith, 1981; Vinner, 1983). During the present study also I observed similar kinds of proof--for example, empirical justification, genetic example (Balacheff, 1988), and informal proof. Unlike these previous studies, the classroom context had a distinct influence on the kinds of proof the students used. The students' justification was influenced by the textbook, the teacher's demands, the context of the justification, and the interaction with the other students. Fischbein and Kedem (1982) stated that from the standpoint of common sense the validity of a statement is judged according to whether it is self-evident or has empirical confirmation, but not by formal proof (p. 128). This position may need to be modified so as to include social context as one of the bases of people's judgment on validity. Vinner (1983) observed that (1) students in the classroom did not think of a computation as a proof and that (2) when asked to prove a mathematical statement about a particular case, students tended to repeat the general proof in terms of the particular case (p. 289). These kinds of observations need to be qualified by more specific classroom contexts because what form to use in a proof seems to be influenced by the social context of the classroom.

The conclusions drawn here of communicative elements in proof presentation should be considered working hypotheses for future research. More systematic and intensive studies on classroom discourse in proof presentation are necessary. The present paper focused on the presenter and presentation of proof, but closer look is needed as to how the audience perceived the presentation and how much they shared the understanding of the proof with the presenter.

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# METACOGNITION: THE ROLE OF THE "INNER TEACHER"(4)

## Contrasts Between Japan and the United States

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## ABSTRACT

*The nature of metacognition and its implications for mathematics education are the main concerns of our investigations. We argued in the last three papers that "metacognition" is given by another self or ego which is a substitute for one's teacher and we referred to it as "inner teacher". To assess this "inner teacher", we administered a questionnaire to preservice teachers in Japan and the United States.*

*Results suggest that in Japan, teachers usually emphasize making a drawing as a strategy of problem-solving and the algebraic expression as the final representation of a problem, but teachers, in the United States, don't so emphasize them. In the United States, teachers emphasize estimation and students accept it positively but teachers, in Japan, don't so emphasize it and students don't respond as positively.*

## AIM AND THEORETICAL FRAMEWORK OF THE RESEARCH

### 1. Definition of "metacognition" and "inner teacher"

We are often inclined to emphasize only pure mathematical knowledge in education. And we fail to enact it in students. Consequently, they fail to solve mathematical problems and forget soon after paper and pencil tests.

Recently, "metacognition" has come to be noticed as an important function of human cognitive activities among researchers of mathematics education as well as among professional psychologists. But even so, the definition of "metacognition" is not yet firmly settled, and results from the research have been of little use to the practice of mathematics education.

The ultimate goal of our research is to develop clear conceptions about the nature of "metacognition" and to apply this knowledge to improve methods of teaching mathematics. This paper is one of a series of studies in pursuit of this goal.

Roughly speaking, we could regard "metacognition" as the knowledges and skills which make the objective knowledges active in one's thinking activities. There are a few proposals on the categorization of "metacognition" in general, but here we will follow the suggestion of Flavell and adopt four divisions of metacognitive knowledge of:

(Metaknowledge)

1. the environment 2. the self
3. the task 4. the strategy

and three divisions of metacognitive skill:

(Metaskill)

1. the monitor 2. the evaluation
3. the control

Our unique conception is that this "metacognition" is thought to be originated from and internalized by the teacher him/herself. Teachers cannot teach any knowledge per se directly to students but teach it inevitably through his/her interaction with students in class.

We start from a very primitive view that teaching is a scene where a teacher teaches a student and a student learns from a teacher. In the process of teaching, a phenomenon which is very remarkable from a psychological point of view will soon happen in the student's mind; we call this the splitting of ego in the student, or we may call it decentralization in a student, using the Piagetian terminology. Children, as Piaget said, are ego-centric by their nature, but perhaps as early as in the lower grades of elementary school, their egocentrism will gradually collapse and split into two egos: the one is an acting ego and the other is an executive ego which monitors the former and is regarded as the metacognition. Our original conception is that this executive ego is really a substitute or a copy of the teacher from whom the student learns. The teacher, if he/she is a good teacher, should ultimately turn over some essential parts of his/her role to the executive ego of the student. In this context, we refer to the executive ego or "metacognition" as "the inner teacher" .

The advantage of this metaphor is that we could have the practical methodology to investigate the nature of metacognition; that is, we may collect many varieties of teachers' behaviors and utterances in lessons and carefully examine and classify them from some psychological view-points.

## 2. Positive and Negative Metacognition

For Metacognition, we think that there are two types. One is a positive metacognition that promotes positively students' problem-solving activities. The other is a negative one that obstructs their activities. For example, most students believe that statements like questionnaire item Ⅲ.19 "When you get lost while solving the problem, please think of other strategies." help them and have a positive effect on problem-solving. This item shows metacognitive knowledge of strategy for problem-solving. This works according to the monitor "I have lost my ideas for the next step". A metaskill of control "Please think of other strategies." works successfully according to a logical conclusion of modus ponens from two premises; item

Ⅲ.19 and the monitor. (See Hirabayashi & Shigematsu, 1987, for more detail.) Other statements, like item Ⅳ.18 "Can't you solve this easy problem?", are believed to make students do worse and to have a negative influence on problem-solving.

## 3. Metacognitive Framework of Problem-solving

A classroom lesson includes varieties of activities of students and among them we notice the so-called problem solving activities are the most preferable phenomena to think over the nature of metacognition, because there we may observe many features of this complicated concept. Thus, we are exclusively concentrated on these learning situations in our research on metacognition.

At first we introduce the classification framework for teachers' utterances, which has two dimensions: one may be referred to as the problem solving stages and the other as metaknowledge categories, and so we have 20 sections in all as is shown in the following table. The former dimension is suggested from that of Schoenfeld(1985) and the second from that of Flavell(1976) and both of them were slightly modified by us.

(Table 1) Metacognitive framework in problem solving

1. GENERAL STAGE
   11) environment 12) task 13) self 14) strategy
2. ANALYSIS STAGE
   21) environment 22) task 23) self 24) strategy
3. DESIGN/EXPLORATION STAGE
   31) environment 32) task 33) self 34) strategy
4. IMPLEMENTATION STAGE
   41) environment 42) task 43) self 44) strategy
5. VERIFICATION STAGE
   51) environment 52) task 53) self 54) strategy

Some comments will be needed about this framework. To Flavell's metacognitive knowledges we add the 'environment', because we think that there are some metacognitions which control the situation of students' problem-solving. For example, item Ⅲ.22 "After you finish solving the problem, you may leave the class." To Schoenfeld's stages we add the 'general stage' in the beginning, because we think that there are some metacognitions which can not belong to the specific stages but have influences on all stages; for example, item Ⅲ.8 "Don't be afraid of mistakes; you may make mistakes." would be made in any stage of students' activities.

## METHODOLOGY OF THE RESEARCH

We first collected teachers' utterances through lesson observations and from the recorded teaching-learning processes. From these records, we made the questionnaire. We classified these items into 4 classes according to the types of teachers' behaviors in the lesson.

### The questionnaire

1) Categories of Items

| | | | |
|---|---|---|---|
| I. explanation | 28 items, | II. questioning | 25 items |
| III. indication | 47 items, | IV. evaluation | 21 items |

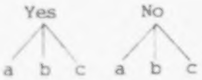
Sample items include:

I. explanation
  5. If you can draw a picture, you can solve the problem.
  22. I (teacher) make mistakes, too.
II. questioning
  10. Can you use that strategy at any time?
  16. Can you explain the reason for this?
III. indication
  2. Read the problem carefully.
  47. Please give me an example for that.
IV. evaluation
  1. That's right.
  14. You could have grasped the important idea.

2) Responses to Each item

For each question, students indicated which of the following two-part responses best reflects their experience.

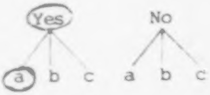
My mathematics teachers have made this comment often



This kind of comment: a. helps me.
b. doesn't help me.
c. makes me do worse.

For example,

I. 1. You already have the experience of solving a problem similar to this.



## Data collection

We have used the questionnaire (originally in Japanese) to analyse university-students' impression of their teachers' utterances in two countries. This is because, as we argued, teachers' utterances would have become the important components of students' metacognition.

We collected the data not only from students in a mathematics course for elementary school teachers in Nara, Japan, but also a mathematics course for similar students in San Diego, USA. The numbers of each were as follows:

| | Japanese | American |
|---|---|---|
| | 1990 | 1991 |
| Preservice teachers | 76 | 25 |

## RESULTS AND DISCUSSION

We want to pay attention to the more interesting contrasts between students in the two countries.

### 1) Categorization of items according to Yes-No responses

Contrasting responses from Japanese students and the United States students, we classified them into categories according to the frequency of responses to "Yes" and "No".

1) category A

In this category each item is responded to by 'Yes' by over 80% of Japanese students but not by the United States students. Data are reported in per cent for each country, Japan(J) and the United States(U).

Ⅰ. 9. Whenever we can write an expression, we can solve the problem. (J 83.9, U 36.0)

Ⅲ. 1. Try to make a drawing of it by yourself. (J100.0, U 56.0)

In Japan, teachers usually emphasize making a drawing as a strategy for problem-solving and the expression as the final representation of a problem. According to these data, teachers in the United States don't emphasize these ideas so much.

2) category B

In this category each item is responded to by 'Yes' by over 80% of the United States students but not by Japanese students.

Ⅰ.23. For each step in solving the problem, we have a reason. (J 27.6, U 80.0)

Ⅱ.10. Can you use that strategy at any time? (J 34.2, U 80.0)

Ⅳ.21. It's an excellent idea. (J 34.2, U 80.0)

In the United States, teachers usually emphasize a logical process of problem-solving and praise students (Hess & Azuma, 1991). According to these data, teachers in Japan don't emphasize these ideas so much.

**2) Categorization of items according to positive("a") and negative("c") responses**

Contrasting responses from Japanese students and the United States students, we classified them into categories according to the frequency of responses to "a. helps me" and "c. makes me do worse".

1) category C

In this category each item is responded to positively by over 70% of Japanese students and by over 70% of the United States students. Data indicate per cent of students who choose response "a".

Ⅰ. 5. If you can draw a picture, you can solve the problem. (J 84.2,U 76.0)
Ⅱ. 6. What is the thing you must look for? (J 73.7,U 88.0)
Ⅲ. 3. Try to reduce the problem to a simple and similar problem.(J 77.6,U 84.0)
35. I(teacher) will try to explain using these objects. (J 88.2,U 80.0)
Ⅳ. 1. That's right. (J 84.2,U 91.7)
7. Good idea! (J 88.2,U 84.0)
8. Good! (J 90.8,U 92.0)
9. It's a rather good strategy. (J 97.4,U 84.0)

We note that students in each country find teachers' expression of support to be very helpful.

2) category D

In this category each item is responded to positively by over 70% of Japanese students but not by the United States students.

Ⅲ.12. Solve the problem by yourself without any help if possible.(J 77.6,U 32.0)
34. Check some special cases for this problem. (J 81.6,U 36.0)
Ⅳ.14. You could have grasped the important idea. (J 90.8,U 12.0)

In Japan, teachers usually emphasize solving the problem slowly and carefully by oneself, but in the United States teachers put less stress on this kind of independent work (Hess & Azuma, 1991).

3) category E

In this category each item is responded to positively by over 70% of the United States students but not by Japanese students.

Ⅰ. 4. Please give an estimate for the solution. (J 36.8,U 80.0)
8. Can you understand? (J 43.4,U 72.0)
14. Can you understand the meaning of the problem? (J 36.8,U 84.0)

Ⅲ.16. Please explain your solutions when the time is up. (J 5.3,U 72.0)

In the United States, teachers emphasize estimation and students accept it positively but teachers in Japan don't emphasize it and students don't have the experience to solve the problem successfully using it (Shigematsu, Iwasaki & Koyama, 1991). In Japan, students see items Ⅱ.8, Ⅲ.16 as negative, but students in the United States don't think so (Hess & Azuma, 1991).

4) category F

In this category the responses to each item are not so intense (<70%), but the data show some typical features.

Ⅱ. 7. Is that right?
a. helps me (J 18.4,U 29.2)
c. makes me do worse (J 60.5,U 12.5)

17. Can you give a reason for it using other expressions?
a. helps me (J 19.7,U 50.0)
c. makes me do worse (J 48.7,U 0.0)

Ⅲ. 7. You can understand.
a. helps me (J 11.8,U 52.0)
c. makes me do worse (J 61.8,U 16.0)

In Japan, teachers try to encourage students using these utterances, but students think that they are always wrong when they hear it. Students in the United States don't think so.

## CONCLUSION

Contrasting responses of strong("Yes") and weak("No") from Japanese students and the United States students, we classified them into categories according to the frequency of responses. According to these data, in Japan, teachers usually emphasize making a drawing as a strategy for problem-solving and the algebraic expression as the final representation of a problem, but teachers in the United States don't emphasize these so much. In the United States, teachers usually emphasize a logical process of problem-solving and praise students, but teachers in Japan don't emphasize these so much.

In contrasting responses of positive("a. helps me") and negative("c. makes me do worse"), Japanese teachers usually emphasize solving the problem slowly and carefully by oneself, but in the United States teachers stress working quickly in the lesson. In the United States, teachers emphasize estimation and students accept it positively but teachers in Japan don't emphasize it so much and students don't have the experience to solve estimation problems. In Japan, students viewed some teacher question negatively, Ⅱ.8 "Can you understand?", Ⅲ.16 "Please explain your solutions when the time is up.", but students, in the United States, don't think so. Lastly in Japan teachers try to encourage students using the utterances such as Ⅱ.17 "Can you give a reason for it using other expressions?", but students think that they have made a mistake when they hear it. Students in the United States don't think so.

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# Metacognition in Cooperative Mathematical Problem Solving : An Analysis Focusing on Problem Transformation

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## ABSTRACT

This paper reports the results of a study on metacognition in problem solving processes of secondary school students. We present a framework for analyzing the paired mathematical problem solving protocols with a focus on solvers' metacognition. And its application, to the protocol of two students working on a construction problem in pairs, is given. From this application, both findings about metacognition of the students and some considerations on the framework are discussed.

## INTRODUCTION

In mathematics education community, much attention has been directed at identifying and understanding the role of metacognition during problem solving. However, the recent explosion in research on metacognition is not without its problem. One of the issues in research on metacognition left to investigate is the difficulties associated with identifying and assessing metacognitive processes of solvers. Although the pioneering work by Schoenfeld(1985) has provided useful framework for this issues, some difficulties in the analysis arise, as described in the following section, in the case of paired problem solving.

In this paper, we describe a framework for analyzing the paired mathematical problem solving protocols with a focus on metacognitive processes. Then we provide, as an example, its application to the protocol of two secondary school students working on a construction problem in pairs. From this analysis, findings on metacognitive processes of the students are discussed. And considerations about the framework we propose are given.

## A FRAMEWORK FOR ANALYSIS

One of the most useful analytical framework for studying the metacognitive processes of problem solvers is those proposed by Schoenfeld(1985). His work on the macroscopic parsing of verbal protocols has suggested a method for analyzing problem solving protocols with a focus on metacognitive processes. The protocols are parsed into major episodes, namely, periods of time when the solver is engaged in a single set of like actions : reading, analyzing, exploring, planning, implementing, or verifying. According to Schoenfeld, it is during the transitions between such episodes that problem solvers make managerial decisions.

Some modification of that framework will likely be needed, however, when we apply it to protocols obtain from the sessions of secondary school students working in pairs. Consider the following reasons.

First, we often observe, in the protocols of paired problem solving, that individuals in pairs explore in different directions concurrently. Indeed, Hart & Schultz(1985) has reported the case of three-person group problem solving, "members of the group were actually operating in different episode types"(p.109). Thus, "cooperative problem solving" can often proceed like "parallel". Kroll(1988) has reported the existence of the "parallel episodes" of college students in pair, and she introduced the possibility of parsing parallel, but different episodes for two individuals. My own subjects, secondary school students, have occasionally showed the same behaviors. In such cases, each solver engage "in a single set of like actions" individually.

Second, many studies have applied Schoenfeld's schema to the protocols of adults, namely, college student (Schoenfeld,1985), pre-service teachers (Hart & Schultz,1985;1986; Kroll,1988), or working mathematicians (Schoenfeld,1985; Silver & Metzger,1989). Modification of framework will likely be needed when it is extended to explain the data obtained from different subjects, for example, junior high school students. In the case of "naive" solvers like junior high school students in this study, because their behaviors are often "event-driven" trial and error type, when new ideas come to their mind, they would likely to try many problems "related" to the original one.

Third, from an instructional point of view, it would be useful to identify the point where students' metacognition might take place in real time. In other words, we would like to give student the feedback about their behaviors during, or after, problem solving from the viewpoint of metacognition. And if we could intervene students at the point where their metacognition arise, it would more effective to ask them the reasons why they were doing such behaviors. Episodic analysis suggests some ideas about these issues.

*Framework for analysis : problem transformation*

By those reasons described above, the author has been investigating methods for analyzing students' metacognition in paired problem solving. It is likely that another framework would be needed to take the solution in different directions made by each individuals in pairs into consideration. So, the author propose a task-oriented analysis.

Several attempts had been made in this line. Hillel(1983), for example, proposed the method taking the "task demands" into account for analyzing problem solving protocols. Although this approach is interesting in describing solvers' solution space, it does not aim to elucidate the role of metacognition during problem solving.

The author will focus on "problem transformations" in problem solving processes. By the term problem transformation, we mean such activities as follows. It is often the case that solvers, when working on a problem, transform the problem at hand to easier ones or to related ones, for example, to an auxiliary problem. In other words, to solve a problem, solvers often restate the goal they are trying to attain to another goal. These activities are also known as problem formulating or reformulating in problem solving processes (Kilpatrick,1987). It should be noted that in the process of problem solving incorrect and/or unsuccessful transformations will sometimes be done.

Thus, we regard problem solving processes as repeated transformations of the original problem. In transforming the problem, what processes does the solver engage ? Knowledge of the mathematical facts and procedure will be used. And as Silver & Metzger (1989) reported, aesthetic monitoring also shapes behavior of expert solvers. And, more importantly, it is between two transformed problem where problem solver either should be, or are likely to be, engaging in metacognitive behaviors.

It would be more interesting to explore with students not only what they thought they had done in transforming a problem, but why they thought this to be important. It also would be interesting to explore attempted and unsuccessful transformation, namely, those which do not lead, for one reason or another, to a solution. We would be able to elucidate the role of metacognition by examining the verbal data between transformation. It is the advantage of this framework that we can anticipate in some degree problem transformations of a solver by examining the mathematical structure of original problem like a "problem space".

Schoenfeld has described about his framework that "the idea behind the framework is to identify major turning points in a solution"(pp.314-315). This framework makes it possible to identify the turning points during problem solving processes, like episode analysis but in somewhat different manner. We can trace "problem space" of each student, which is rich source of information about the solvers' thought and, in particular, metacognition.

## APPLICATION OF THE FRAMEWORK

In this section, we will apply the framework to a protocol obtained from videotaped problem solving sessions, in which junior high school students working on a problem in pairs. Students' behaviors described in the followings illustrate that having students work in pairs will not necessarily make them to collaborate in the same direction of solutions, in particular, when they are trying to solve a challenging problem.

*Subjects*

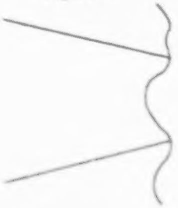
Six junior high school students (9th grade) are asked to solve a problem working in pairs. In this paper, we focus on one of those pairs, student M.A. and S.O., they showed several interesting behavior and produced interesting solutions.

*Problem*

Task used was the following construction (angle-bisector) problem in plane geometry. The situation of the problem highly motivate students and it allows many, more than ten, solutions.

> *You are given a paper on which an angle was drawn. As it happens, the paper was ripped, and the vertex of the angle is torn off (See Figure 1). Using a compass and straightedges, construct the bisector of the angle. As a natural result of the accident, you cannot draw on the torn-off part of the paper.*

*Figure 1*



*Methods*

Three pairs were asked to work on the task together. These entire solution processes were videotaped and then transcribed to submitted to the analysis. Follow up interviewing were conducted to ask them explanations of the their solution(s) and their impressions on the problem. These explanations were used to support identification of problem transformations.

*Results*

Student M.A. and S.O. worked on the problem about 32 minutes and 40 seconds. Although they worked on the problem together, they proceeded into the different directions. They often asked the partner to explain the approach, listened partner's comments and exchanged some ideas. After final "solutions" were obtained, they explained about them each other. Their final solutions were obtained by each solver as follows *(See Figure 2 & Figure 3)*.

Solution by student M.A.

1) Construct line $l_3$ parallel to $l_2$ (let the intersection of $l_3$ and $l_2$: A).
2) Take the point B and C so that AB=AC.
3) Draw the line BC (let the intersection of BC and $l_2$: D)
4) Construct the perpendicular bisector $l$ of BD, which is the angle bisector.

"Solution" by student S.O.

1) Connect points A and B, which are "ends" of the original lines.
2) Bisect angle A and B.
3) Using point C, the intersection of the angle bisectors, construct two triangles ACD and BCE such that AD=CD and BE=CE.
4) Bisect angle DCE.

*Figure 2 : Solution of student M.A.*

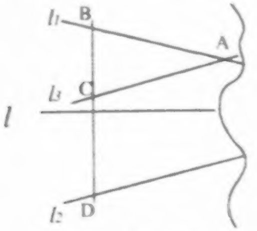

$l_1$
B
A
$l_3$
C
$l$
$l_2$
D

*Figure 3 : Solution of student S.O.*

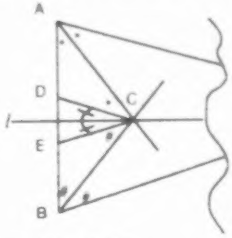
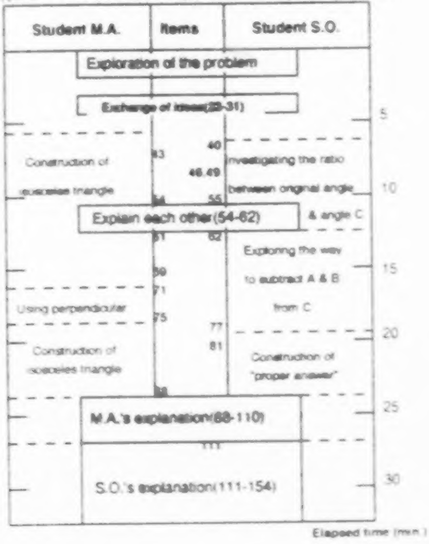


*Figure 4 : Problem transformations of M.A.& S.O.*


Student M.A.
Items
Student S.O.
Exploration of the problem
Exchange of ideas(23-31)
5
40
43
Construction of
isosceles triangle
46.49
Investigating the ratio
between original angle
& angle C
10
54
55
Explain each other(54-62)
61
62
Exploring the way
to subtract A & B
from C
15
69
71
Using perpendicular
75
77
20
81
Construction of
isosceles triangle
Construction of
"proper answer"
88
25
M.A.'s explanation(88-110)
111
S.O.'s explanation(111-154)
30
Elapsed time (min.)

### Problem transformations of M.A.& S.O.

Figure 4 Shows the *Problem transformations* of MA & SO. This diagram describes both each solver's problem transformation and their verbal exchanges, and the latter are shown using boxes when it continues more than thirty minutes. The "items" (central row of the diagram) shows the number of the verbal items in entire protocol to be interpreted as a sign of metacognition. (Although more descriptions about this diagram are needed, the details are omitted here.)

### *Some findings on metacognition*

From the analysis of the protocol, some findings on students metacognition were obtained. We describe some of them ; S.O.'s conception about mathematics, cognitive monitoring of M.A., and metacognition raised by interaction with peer.

### S.O.'s conception about mathematics

One of the components of metacognition related to mathematical problem solving is knowledge and beliefs about oneself, for example, as doer of mathematics, and beliefs about school mathematics (Lester & Garofalo,1985). Student SO's behavior illustrates the influence of his conception about mathematics on his behaviors.

About six minutes past after the problem given, he tried to find any rule among the angles in his figure. So, he were investigating the ratio between original angle and angle C (*See figure 3*). Here is an excerpt from the protocol. He mentioned about a rule behind the problem.

*46.SO: Surely, here is certain rule underlying this figure.*
*47.MA: Is this angle the same as this one...., isn't it ?*
*48.SO: This angle is 35 ° .so*
*49.SO: wait a moment, 40 ° ....., certainly, there must be certain rule.*

His search for the "rule" was obvious from the beginning of his first transformation of the problem. He had calculated, for more than five minutes, the ratio among the numbers of angles generated by drawing some triangles. These observation shows that he has such a conception that he can find some rules behind a problem when he is doing mathematics and that this conception shapes his behaviors.

### Cognitive monitoring of M.A.

Student MA's behaviors showed that he was frequently engaging to monitor his own behaviors. For example, when he decided to give up his first approach, "construction of a isosceles triangle", he said as follows.

> *65.MA: Isn't it work well ? The condition of this figure is.....so....it might be impossible.*

And he also commented on the likelihood of difficulties of the transformed problem.

> *69.MA: So, it will be more troublesome.*

These comments indicate he assessed his own approach. And as a result, he selected another approach, "using a perpendicular", to the problem. Thus the assessment of his own state were used to decide to explore another approach.

On metacognition raised by interaction with others.

As Kroll(1988) reported, we could observe the monitoring move made by individuals in a pair. This is the case in my subjects. Each student solved the problem using different approach, and their solutions don't seems to be influenced by their partner. However, in asking, replying, and explaining, they were urged to reflect their own solution by their partner.

Student SO, for example, explained his idea, "investigating a ratio to find a rule", to MA. This explanation began to say "*It's my reason why I take this approach....*"(item 56). But this explanation turned to make himself to reconsider why he had been taking this approach. Similar observations could made when they compared their idea with partner's.

Thus, we can say that interaction with others raise one's metacognition and such situations as one explain to others and as one compare one's idea to others' make one to be reflective .

## DISCUSSION

Several studies on metacognition have clarified the role of metacognition in mathematical problem solving. The findings of these studies are needed to compare with the findings in the case of various subjects, various tasks, and different settings. This paper suggests the importance of metacognition in the processes of problem solving in the case of secondary school students working in pairs.

A few observations on the influences of the conception of mathematics to problem solving behaviors and on the cognitive monitoring during problem solving are described in this paper. As to the former, the influences of "mathematical beliefs", from the observations on SO's behaviors, it could be discussed that constructing some kinds of reasonable beliefs in students would possess them powerful bases of their mathematical performance. And MA's success suggests the importance of assessing the likelihood of the difficulties before jumping in. Another message to be consider is that peer interaction of students has the effect to raise metacognition, by each student taking the role of monitor.

The framework presented in this paper is in its formative stages. So, there are both issues to be resolved and research questions that might turn out to be fruitful.

One of issues to be investigated is about establishing formalism of the analytical methods. Another issue is about the comparison of this framework to the Kroll's framework, which is the modified version of the Schoenfeld's. It seems that these frameworks aim to elucidate the same phenomena from different angles.

Kilpatrick(1987) posed the following research question about problem formulating. "What metacognitive processes are needed for problem formulating ?" (p.143) As to this question, it would be interesting to explore with students attempted and unsuccessful transformation, namely, those which do not lead, for one reason or another, to a solution. We can learn some lessons from two students' formulating of the problem. MA's success suggests the importance of the assessing the likelihoods of difficulties for problem reformulating.

It could be argued that, from a viewpoint of instruction, identifying the points where metacognition might work has some advantages for teaching mathematical problem solving. Observing students' behaviors with a focus on problem transformations will suggest to teachers when it would be appropriate to intervene the students. In such situations, when problem transformations might occur, asking students "why" would makes it possible them to reflect their own selection of the direction at significant points in entire solution processes. Thus, although some modifications might be needed, it will be fruitful to use the analytical framework in this paper for instructional purpose. Research in this direction might allow us to reconsider the "Socratic method" from metacognitive perspectives.

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INTERNATIONAL GROUP FOR
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# PROCEEDINGS of the SIXTEENTH PME CONFERENCE

University of New Hampshire
Durham, NH (USA)
August 6 - 11, 1992

Volume III

INTERNATIONAL GROUP FOR
THE PSYCHOLOGY OF
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Volume III

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# Research Reports (continued)

# CHILDREN'S APPROACHES TO MATHEMATICAL PROBLEM SOLVING

Dianne Siemon
Phillip Institute of Technology, AUSTRALIA

*A response mapping technique was developed to represent individual problem solving episodes in terms of Flavell's (1981) model of cognitive monitoring. The consistency of the patterns observed in the individual models suggested a relationship between conceptual and procedural knowledge of a cognitive and metacognitive kind. A meta-model was proposed to describe and explain the nature of this inferred relationship. The paper provides, an outline of the response mapping technique, a summary of the meta-model, and an example of the response map data. The analysis supported the view that children's approaches to mathematical problem solving can be described in terms of the four generic approaches identified by the meta-model.*

The work reported here derives from two teaching experiments involving upper primary grade children and their teachers. The first, conducted from an information-processing perspective, was designed to investigate the effects of metacognitive training on children's mathematical problem solving performance over a ten week period. Two questions prompted the first study, (1) to what extent was metacognition a "driving force" in children's mathematical problem solving, and (2) to what extent could children's cognitive awareness and ability to monitor and regulate their actions be improved through training. A ten-week teaching experiment involving three grade four classes and one grade six class was designed to address these questions (Siemon, 1986). While the results of the initial study supported the view that metacognition as it was defined, is a "force" governing mathematical problem solving behaviour, and that for some students enhanced metacognition can be achieved by training, the complexity of the problem solving behaviour observed suggested that metacognition is not a single or even bi-valued "driving force", but a multiplicity of complex "forces"and relationships not all of which may be operating in the same direction at the same time. The second, and more major of the two studies was conducted from a constructivist's perspective. A year long teaching experiment was designed to explore the role of metacognition in children's mathematical problem solving, specifically, the sufficiency and viability of a theoretical meta-model of children's approaches to mathematical problem solving derived from a *post hoc* analysis of the individual interview data obtained from the first study. A summary of the meta-model and the response mapping technique which helped generated it will be addressed in this paper.

The uniqueness and significance of children's existing knowledge, beliefs, goals, and motivations, in relation to their subsequent learning and problem solving attempts was powerfully demonstrated by the first study. As a result, problem solving was seen as an exercise in negotiating meaning, not as a "process" to be taught and exercised in schools because it was the "focus" of the decade. It was apparent that a theory of learning as opposed to a theory of instruction was needed, and a view of metacognition which recognised the importance of existing metacognitive knowledge, beliefs and experience in the construction of mathematical meanings and solution strategies. The underlying theory of learning was found in *constructivism* (for example, Cobb, 1987). The more encompassing view of metacognition was found in Flavell's (1981) Model of Cognitive Monitoring, in which he includes the two recognised aspects of metacognition, *metacognitive knowledge* and *metacognitive experience*, and adds two other components, *cognitive goals* and *cognitive actions*. From a constructivist's perspective of problem solving, this model seemed to provide a better explanation of what had been observed in the protocols than the two component model so prevalent in the problem solving literature conducted from an information processing perspective (for example, Schoenfield, 1983; and Garofalo and Lester, 1985).

### *The response mapping technique*

On the basis of this re-evaluation of research goals and meanings, a protocol analysis scheme was developed which recognised the interactive role of metacognitive knowledge, metacognitive experiences, cognitive goals and cognitive actions. The scheme was adapted from the diagrams used by Biggs and Collis (1982) to characterise the various levels in the SOLO Taxonomy. While the basic system of recording used to generate the maps is similar to that used by Collis and Watson (1989), there are some important differences. For instance, the response mapping technique does not assume that information or procedures not given in the problem stem but essential for solution, are available to the solver if they have been taught or can be deemed to be in the "universe of discourse" (Collis and Watson, 1989, p.182). The technique is not concerned with identifying the particular SOLO level of a child's response nor with classifying problems in terms of SOLO levels on the basis of written test data. It was developed to accommodate clinical interview data obtained from 9 to 10 year olds. A major difference in the technique, at least in its formative stages, is the inclusion of a summarised record of interview which can be used to support an inference or to elaborate a response. The following symbols are used to construct the response maps developed for this study: **o** - information provided in the problem stem or by the researcher, which may or may not be relevant to solution, and which may or may not be attended to by the solver; and **o** - information observed or inferred on the basis of transcript evidence or the child's actions in the course of a problem solving attempt, that is, information provided or generated by the solver which may or may not be relevant to solution.

In this context, *information* includes: numerical data, problem conditions, questions, cognitive goals, items of cognitive or metacognitive knowledge (including beliefs, concepts, skills, processes and strategies), and the products of cognitive actions and metacognitive experiences. Two forms of information were proposed: *presented* information and *generated* information. Two types of presented information are listed on the left hand side of the response map. The first type is information provided in the problem stem or by the researcher (indicated by the symbol ●). The second type is information provided by the solver in the form of an inferred *cognitive goal* or item of *cognitive* or *metacognitive knowledge* (indicated by the symbol o), for example, the decision to represent a problem by drawing a diagram, or the belief that because it is a mathematics problem, an algorithm is required. Generated information is information provided by the solver in the course of the problem solving enterprise as a result of a *cognitive action* or *metacognitive experience*, for example, a drawing, an answer to the first step of a multiple step problem, or suddenly recognising that a cognitive goal or action is inappropriate. Items of generated information are also represented by the symbol o, but they appear in various locations to the right of the list. Classifying cognitive goals and items of metacognitive knowledge as items of presented information recognises that the solver's knowledge, beliefs and prior experience play a powerful role in interpreting and utilising the result of any cognitive action or metacognitive experience. That this distinction is worth making is strongly supported by the transcript data from the first study which suggests that differences in children's approaches to problem solving are related to differences in the ways and extent to which cognitive goals and cognitive actions are generated, retrieved and monitored.

The maps are constructed from the transcripts of interviews and any written record that the student might have made during the problem solving episode. In general, a seperate map is constructed for each problem attempted. Where necessary, an abbreviated transcript is presented with the map. "Expert" maps may be generated for the purposes of discussion or elaboration, but no assumptions are made about the uniqueness or "appropriateness" of such maps. An example of an expert map for the *Incy Wincy* problem, more widely recognised as the frog in the well problem, appears in Figure 1. The items of presented information are listed on the left hand side of the map. They include the data and question from the problem as well as the items presumed to be brought to the situation by the expert. That is, the recognition (metacognitive experience) that this problem is similar to the frog in the well problem (metacognitive task knowledge), and that it can be solved by representing the situation in some way and by acting it out (metacognitive strategy x task knowledge).

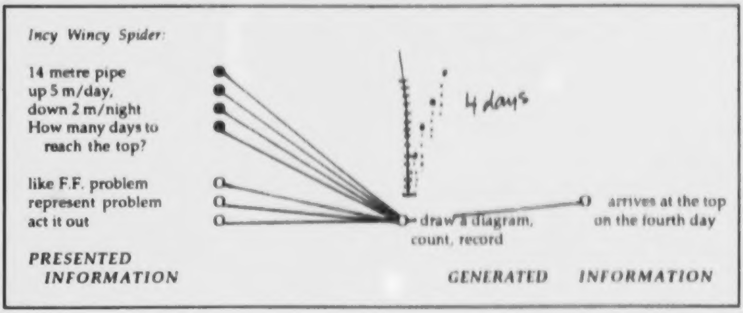


Figure 1. An 'expert' response map for the *Incy Wincy* problem

The decisions, to represent the situation and to act it out, are cognitive goals. The implementations of the decisions are cognitive actions which produce items of generated information, a diagram in the first instance, and a solution in the second. These appear to the right of the list. Items observed or inferred as being responsible for prompting a particular cognitive action or item of generated information are linked by lines to the generated item(s) concerned. As each response map was constructed by the researcher in an attempt to explain the child's observed and inferred behaviour, no claim is made for the independence and reliability of this technique. Rather, a claim for generalisability is made on the basis of tne number of interviews conducted and analysed in this way and on the basis of the broad patterns that were observed over time and problem type.

### *A meta-model of children's approaches to problem solving*

The relatively consistent patterns observed in the individual models of problem solving provided by the response maps suggested there was some value in identifying and comparing individual children's problem solving attempts on the basis of the actual solution attempt made rather than on the basis of some pre-set criteria about the perceived nature of the problem or task itself. For example, observable patterns in Julia's response maps suggested that it was her knowledge and beliefs, both metacognitive and cognitive, that was driving her problem solving efforts more so than the particular demands of the task. Cobb's (1987) observation that our "primary objective is to develop explanations that either account for novel, unanticipated observations or resolve conceptual inconsistencies and contradictions within the theory or model." (p.31), prompted the decision to construct a meta-model that was consistent with a constructivist's perspective of problem solving.

From the *post hoc* analysis of the individual interview data it was apparent that different children appeared to attend to and value qualitatively different aspects of the problem solving experience. Julia, for example, seemed overly concerned with procedures and algorithms (procedural knowledge of a cognitive and metacognitive kind), as opposed to understanding and representing the problem meaningfully (conceptual knowledge of a cognitive and metacognitive kind). Julia's response maps revealed that she was more likely to implement a range of cognitive actions on a trial and error basis than to stop and think about what she was attempting to do or what her actions might mean. While she demonstrated considerable skill in monitoring the implementation of her actions, she did not display anywhere near the same commitment to the formulation and evaluation of cognitive goals in relation to the problem's conditions and/or questions. What appeared to be driving her actions was her beliefs about the nature and purpose of school mathematics (prior metacognitive knowledge), for example, her view that doing mathematics is not concerned with meaning: "we don't do meanings in maths". Other case studies suggested a predisposition to value and attend to the construction of meaning. For example, Annie and Nancy (Siemon, 1991), demonstrated a preparedness to thoroughly analyse problem statements before making any decision about what needed to done. This valuing was reflected in their response maps which indicated that both girls tended to formulate and evaluate cognitive goals which in turn informed their selection and execution of a range of cognitive actions. The apparent role of conceptual and/or procedural knowledge, both of a cognitive and metacognitive kind, in the formation and operation of these predispositions, suggested that Julia's predisposition or approach to problem solving could be described as *low conceptual/high procedural*and Annie's approach as *high conceptual/high procedural*. Further analysis suggested that Annie's earlier approach might be classified as *high conceptual/low procedural* which led to the conjecturing of a fourth possible approach, *low conceptual/low procedural*.

It was postulated that a *high conceptual/high procedural* approach would be typified by a high instance of effective goal setting linked to the implementation of an appropriate range of cognitive actions. Instances of cognitive monitoring, both with respect to goals and actions, would also tend to be associated with this approach. A *high conceptual/low procedural* approach was hypothesised to be typified by a relatively high instance of appropriate goal setting, but a correspondingly low level of implementation of appropriate procedures and actions. Where there were instances of cognitive monitoring they would be more likely to be directed at cognitive goals than at cognitive actions. By contrast, a *low conceptual/high procedural* approach was believed to be typified by infrequent instances of appropriate goal setting, but a correspondingly high level of implementation of appropriate procedures and actions. In this case, where there were instances of cognitive monitoring, they would be more likely to be directed at cognitive actions than at cognitive goals. A *low conceptual/low procedural* approach would be typified by low levels of appropriate goal setting and implementation and few, if any, instances of cognitive monitoring.

A systematic examination of all of the response maps generated from the first study transcripts and records of interview revealed that virtually all of them could be described in terms of either one or other of the four approaches hypothesised. The transcripts were then examined qualitatively with respect to such notions as impulsiveness and reflectivity, field dependence-independence, and approaches to learning based on motivation and type of strategy usage (Biggs and Telfer, 1987; Marton and Saljo, 1976). The meta-model which resulted from this empirical and theoretical analysis is shown in Figure 2. It reflects the major dimensions along which the response maps varied and lists the attributes believed to be characteristic of each approach as a result of the qualitative analysis of the individual interview data with respect to the literature on cognitive style. The suggestion of orthogonality is deliberate as the approaches are not meant to imply discrete, mutually exclusive entities, but *tendencies* towards particular behaviours rather than others, in relation to a specific task at a given point in time. To simplify further discussion, it was decided to name the four approaches as the Solver's Appproach *(high conceptual/high procedural)*, the Diver's Approach *(high conceptual/low procedural)*, the Player's Approach *(low conceptual/high procedural)*, and the Survivor's Approach *(low conceptual/low procedural)*.

### *The sufficiency and viability of the meta-model*

The second study was similar to the earlier study in that the teacher conducted at least one problem solving session per week which was video-taped. The program explored a variety of problem types and strategies and provided an accessible model of cognitive monitoring in terms of the ASK-THINK-DO problem solving cycle (see Barry, Booker, Parry and Siemon, 1985) . Key components of the program were (1) the mathematics content of the particular problems considered was selected by the teacher to support her curriculum objectives, (2) the problem solving process was specifically talked about in terms of the ASK-THINK-DO cycle and modelled by the teacher whenever a strategy was reviewed or introduced, and (3) key questions, strategies and observations about problem structure were discussed and recorded on a large ASK-THINK-DO problem solving chart which was on constant display in the classroom. Problems were worked on individually, in small groups or as a class. Twelve children were selected to be interviewed on a regular basis. The interviews consisted of a reflective review of a problem considered in class followed by an attempt at a similar or related problem. Response maps were constructed for all interviews conducted (155 interviews over 15 tasks). The meta-model was judged to be sufficient and viable if the observations of each child's problem solving attempts could be explained by the meta-model without excessive contradictions or glaring omissions.

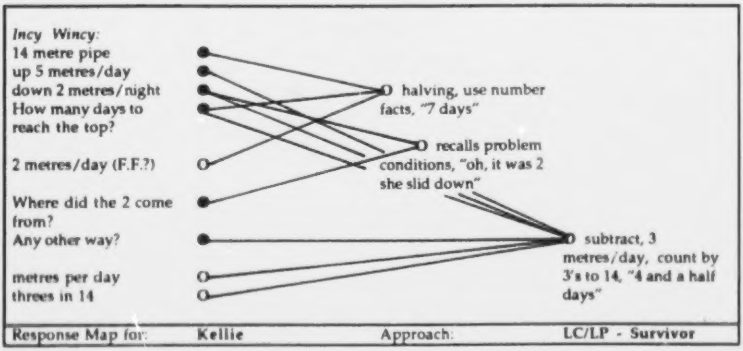
**CONCEPTUAL KNOWLEDGE:** metacognitive knowledge and cognitive goals

| | |
|---|---|
| cognitive goals attended to more than cognitive actions<br>comprehension strategies: checks, monitors, plans, predicts, links, reflects, more on knowledge than on actions<br>access to a variety of strategies, not always well used<br>identifies goals<br>tends to synthesis and analyse data<br>some tendency to conceptually-driven premature closure<br>questions directed more at goals than actions<br>some undirected actions<br>uses labels, understands structure<br>deep approach to learning<br>extended locus of control for actions | attends to cognitive goals and actions<br>comprehension and regulation strategies: checks, predicts, monitors, plans, links, introspects, reflects, on knowledge and actions.<br>uses a variety of strategies knowingly<br>identifies goals and appropriate actions<br>synthesises and analyses data<br>strong tendency to persist until reasonable solution obtained<br>questions directed at goals and actions<br>directed manipulation<br>uses numbers, labels, structure<br>deep-achieving approach to learning<br>internal locus of control |
| *Diver: High conceptual/ Low Procedural* | *Solver: High Conceptual/ High Procedural* |
| tendency to remember and replicate, but experiences difficulty<br>strong tendency to premature closure<br>tends not to check, monitor, reflect or predict on actions or goals<br>experiences difficulty identifying goals<br>experiences difficulty identifying appropriate actions<br>more likely to synthesise than analyse<br>little or no questioning<br>surface approach to learning<br>external locus of control for knowledge and actions | remembers and replicates, often quite effectively<br>tendency to procedurally-driven premature closure<br>regulation strategies: checks, monitors, predicts, reflects, more on actions than on goals<br>tends to assume goals<br>tends to try a range of actions<br>more likely to synthesise than analyse.<br>questions directed more at actions than at goals<br>surface-achieving approach to learning<br>external locus of control for knowledge |
| *Survivor: Low Conceptual/ Low Procedural* | *Player: Low Conceptual/ High Procedural* |

**PROCEDURAL KNOWLEDGE:** metacognitive experiences and cognitive actions →

Figure 2. A meta-model of children's approaches to problem solving

The analysis of the response map data indicated that while there were important qualitative differences in individual response maps, both within and among children, there were some fundamental similarities in the ways in which some children approached problem solving. Essentially, the patterns observed in different children's responses to the same problem and in each child's response to a range of problems could be described by the nature and extent to which the child was observed to be actively engaged in monitoring his or her cognition, that is, in terms of the four approaches identified in the meta-model. For example, some children were more likely to monitor their cognitive actions than their cognitive goals (Player's). Others were more likely to monitor their cognitive goals and their cognitive and metacognitive knowledge than their cognitive actions (Diver's). An example of a response judged to be characteristic of a Survivor's Approach, that is, *low conceptual/low procedural*, is provided below for the *Incy Wincy* problem referred to earlier. The response map is supported by an abbreviated transcript of interview (Kelly's comments are in bold).


*Incy Wincy:*
14 metre pipe
up 5 metres/day
down 2 metres/night
How many days to reach the top?
2 metres/day (F.F.?)
Where did the 2 come from?
Any other way?
metres per day
threes in 14
halving, use number facts, "7 days"
recalls problem conditions, "oh, it was 2 she slid down"
subtract, 3 metres/day, count by 3's to 14, "4 and a half days"

Response Map for: **Kellie** Approach: **LC/LP - Survivor**

. . (problem presented orally ). .**7 days** . . . How did you get that? . . . **half of 14 and I thought that might be the answer because if you count by twos to 14 . . you'll get 7** . . . Do you remember the problem? . . . **Yes** . . . . Where did the 2 come from? . . . **Oh, it was 2 she slid down** . . . Do you still think it is 7 days? You'd be fairly happy about that? . . .Is there any other way you could do it? . . . **You could go by 3's to 14, but then you'll have to say when you get to 12 . . if you add another 3 that would be 15, so I'd have to do an extra** . . . 3 from? . . . **oh, here . . from 5 down to. . there was 3 left** . . . How many days will it take Incy Wincy to get to the top of the pipe? . . . **4 and a half days** . . . How did you do that? With the threes was it? . . .**yes** . . so how did you get 4? . . .**well, there are 4 threes in 12, and when I said "and a half", that was including, that meant that he had 2 more jumps to go** *(note link to Freddie context, "jumping")* . . . Would you use a diagram? . . . **I might.** . . but in your head is the first way you would do it? . . . **yes, and then, to check it I might do a picture** . . . Do you have a picture in your head like you told me you had for Freddie? . . . **yes** . . .What did you have . . imagine in your head? . . . **well . . I was thinking every night he went back 2 . . 2 of the steps went away** . . .So how many steps did he go up each day? . . **4 each day . . but in 24 hours . . 2 steps** . . .That was for Freddie was it? . . **yes** . . .What about Incy Wincy?. . . **there were 14 steps and then going up 5 and sliding back down 2 and that meant . . it took him one day, then up 5 and down 2 that took him 2 days, and he kept on going like that until he got to the top** . . So, how many did he do in each day? . . **Altogether 3**

## *Discussion*

Quite clearly, the link between the conceptual and procedural knowledge is a dynamic and complex one affected by contextual setting, specific content knowledge, beliefs, motivations, and values. The orthogonal relationship proposed in the model was useful in describing and explaining the children's problem solving behaviour. This feature of the model highlighted the fact that many problem solving efforts failed not necessarily because of a lack of monitoring ability *per se* or even a lack of knowledge as to what strategies to use and when, but because of a lack of access to specific content, and the inability to apply monitoring strategies to what is known as opposed to what is done. For example, many children knew that a diagram was useful for the *Incy Wincy* problem but failed to implement this strategy, not because of their inability to draw, but because they did not have access to a well structured knowledge base which could generate and withstand the negotiation required to establish *what* diagram was needed. This confirms an almost identical result reported by Resnick and Nelson-Le Gall (1987) and Lesh's (1983) observation that "the activities that facilitated successful solutions were those which focussed on conceptual rather than procedural considerations" (p.6). As an exploratory study, the work reported here is believed to have important theoretical and methodological implications for future research and classroom practice.

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# UNDERSTANDING MULTIPLICATIVE STRUCTURES: A STUDY OF PROSPECTIVE ELEMENTARY TEACHERS

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*This study focused on the development of understanding of the area of a rectangular region as a multiplicative relationship between the lengths of the sides. The analysis of data from a whole class teaching experiment involving prospective elementary teachers resulted in the development of an explanatory model of the quantitative reasoning involved. Key aspects of this model involve the anticipation of a rectangular array of units as the structure of the area quantity and the subsequent constituting of the units based on the linear measures.*

## Background

This report focuses on the development of understanding of the area of a rectangular region as a multiplicative relationship between the lengths of the sides. The research described was part of a larger study[1] on the development of prospective elementary teachers' abilities to distinguish multiplicative situations. Hart (1981), Inhelder and Piaget (1958), and Karplus, Karplus, Formisano, and Paulsen (1979) have demonstrated with different populations of learners the high incidence of those who apply additive strategies to multiplicative situations, particularly in situations involving ratio.

Our earlier informal observations of prospective elementary teachers revealed that, although they computed areas of rectangles by multiplying, the choice of multiplication was often the result of having learned a procedure or formula rather than the result of a solid conceptual link between their understandings of multiplication and their understandings of area. We, therefore, chose area as the initial context for studying prospective elementary teachers' development of multiplicative relationships.

Quantitative Reasoning: Our work builds on Thompson's (in press) theoretical work on quantitative reasoning. Multiplicative reasoning is one aspect of quantitative reasoning. Thompson defines quantitative reasoning as thinking about a situation as a set of quantities and the relationships between them. Quantities are measurable qualities of an object. Thompson underscores the key distinction between quantitative reasoning and numerical reasoning (reasoning about particular numbers to evaluate a quantity). "It is important to distinguish between constituting a quantity by way of a quantitative operation[2] and evaluating the constituted quantity. One can conceive of the difference between your height and a friend's without giving the slightest consideration to evaluating it." This distinction helps us to understand the study of area as involving both constituting area as a quantity and evaluating that quantity.

## The Study

Subjects: We conducted this study with a class of elementary education majors who were in their junior year at the University. These students had volunteered for a pilot program which they knew was part of a research project on prospective elementary teachers' learning of mathematics and mathematics education.

They took consecutively a course in mathematics and a course in the learning and teaching of mathematics.[3]

Twenty-six prospective elementary teachers participated in the first course (mathematics). This report is based on the first eight classes of this course.

Methods: The CEM project used a whole-class teaching experiment design (Cobb & Steffe, 1983), used productively by Cobb, Yackel, and Wood (1988) in their second grade problem solving project. The research team constructed models to explain the empirical data from the classroom. These models, which were derived in relation to extant work in the field, were tested during ensuing class sessions to refine the models' power to explain students' problem-solving actions and explanations. The first author was the teacher; the second author, who observed all class sessions, collaborated with him on the data analysis, model-building, and the generation of instructional interventions. All classes were videotaped and small-group work was videotaped or audiotaped. Following each ninety-minute class session, the researchers separately analyzed the class. They then met to negotiate shared perspectives on the mathematical development which took place in that class and to determine instructional interventions for the next class based on their analysis. The development and modification of explanations of the students' conceptualizations led to the construction of a more comprehensive model which is outlined below.

## Data Collection and Results

Instruction in the first class began with a small cardboard rectangle being given to each of the small groups of students seated at the classroom's six large rectangular tables. The problem was to determine how many rectangles, of the size and shape of the rectangle that they were given, could fit on the top surface of their table. Rectangles could not be overlapped, could not be cut, nor could they overlap the edges of the table. Students were instructed to be prepared to describe to the whole class how they solved this problem. (This will be referred to subsequently as "problem 1.") The students worked collaboratively and generally had little trouble solving the initial problem.

The universal strategy was to use the given rectangle as a measure to count the number of rectangles along the length of the table, and the number of rectangles along the width of the table and to multiply these two numbers. However, a question arose in a few of the groups as to whether the rectangle should be kept in the same orientation (see Figure 1) as it is moved along the length and along the width or whether the rectangle should be turned so that the measuring is always done using the same edge of the rectangle (see Figure 2). The groups eventually agreed that the former, holding the rectangle in the same orientation, was appropriate.

When the small groups came back together for whole class discussion, the teacher asked them how they solved the problem. They described their method. The teacher then asked, "Why did you multiply those numbers together?" This proved to be difficult for the students to answer. Some asserted that "it seemed like the easiest way," or "in previous math classes you learned the formula for areas." Some responded that they did it because it works; they had seen examples of how the product was the same as the counting up of all the rectangles. The teacher pressed them to consider whether there was any reason to expect that it would always work. Most of the students seemed puzzled by this question. However, Molly offered the following:

Figure 1

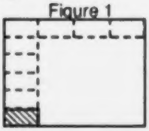
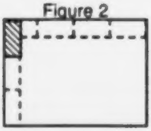


Figure 2



Molly: Well, it would work because, umm, multiplying and adding are related in that multiplying is, is like adding groups, and so it would always work because you add them up to see how many is in the square and to multiply the groups that go like that, that'll always work. You would get the same number, I'm saying if you added them or if you multiplied that side times that side. Because you're adding, I mean, you're multiplying the number of groups by the number in the groups which is the same as adding them all up.

Molly went on to demonstrate on the chalkboard how in a rectangular array [our language] of rectangles each row was a group and the number of rectangles in a row was the number in each group. She observed that summing the rectangles in each row (repeated addition) was equivalent to multiplying the number in a row by the number of rows.

The other students' subsequent comments suggested that Molly's explanation was understood by only a few of them. "Sally said, I don't think we're multiplying by the number of groups because the number of groups is two." Further discussion revealed that the "two groups" for Sally were the number of rectangles in the length and the number of rectangles in the width. The teacher asked students to paraphrase Molly's explanation without using the word "groups." Georgia responded with an explanation which asserted that you multiply the number across the top by the number down the side. When the teacher again probed, "Why multiply?", responses included "Cause that's the way we've been taught." and "... it's a mathematical law." Only after working with these ideas in an additional problem (problem 2 below), were the majority of the students able to articulate a multiplication-as-repeated-addition perspective (similar to Molly's) on finding the number of rectangles.

Problem 2: Bill said, "If the table is 13 rectangles long and 9 rectangles wide, and if I count 1, 2, 3...,13 and then again 1, 2, 3..,9, and then I multiply, 13 x 9, then I have counted the corner rectangle twice." Respond to Bill's comment.

After providing students the opportunity to articulate why they multiplied, the teacher raised the issue about whether to turn the rectangle (the question that had been raised by students during the small-group work). He demonstrated holding the rectangle one way to measure the length and turning it 90 degrees to measure the width (as in Figure 2). Most of the students responded that this method was not correct, that it did not give the number of rectangles that could fit on the table. After some discussion, he asked whether the number obtained when he measured this way and multiplied the two numbers had any meaning. "Does it tell us anything about this particular table?" (We refer to this extension of problem 1 as "problem 3.")

Students were not only persuaded that the number generated was not useful, they also developed consensus that when you turn the rectangle you generate a set of "overlapping rectangles." They reasoned that since the rectangles are overlapping, the number that is obtained is nonsense. We anticipated that students might connect a visual sense of area (perceived amount of surface) with an abstract concept of multiplication. However, no one in the class seemed to make this connection.

## Analysis of the Data: Model Building

How can we explain the thinking of these students on this set of problems? The initial model that we developed is based on the data described above and the following background data that we had on this population of learners. These students were competent with traditional multiplication problems (represented by example 1 below) where a multiplier and a multiplicand are clearly distinguished. Also, those who could solve example 2 (below), generally did so in the following way. They considered one pair of pants with all the different shirts (12 outfits) and then multiplied the number of outfits generated with one pair of pants by the number of pairs of pants (12 outfits/pants x 8 pants) in order to get the total number of outfits.

Example 1: Kim bought 4 cookies. The cookies sold for $.75 apiece. What was the total cost of the 4 cookies?

Example 2: I have 8 pairs of pants and 12 shirts. Assuming I can wear each shirt with each pair of pants, how many different outfits can I create with these clothes?

We generated the following set of hypotheses to explain the students' treatment of examples 1 and 2 and problems 1 and 3:

In example 1, the quantitative question is "How are the number of cookies and the price per cookie related to the total cost?" The thinking here is fairly straightforward for these students. They identify a structure, each cookie associated with the given cost, and a quantitative operation, combining these unit costs to produce the total. The structure and operation fit with a repeated addition approach to evaluating the quantity in question.

The students think of multiplication as repeated addition. Repeated addition requires an asymmetric pair of quantities; one must serve as the multiplier (number of iterations) and the other as the multiplicand (iterable quantity). In example 1, the number of cookies is the multiplier and the cost per cookie is the multiplicand. Repeated addition then is the iteration of the multiplicand, the multiplier giving the number of iterations ($.75 per cookie used as an addend 4 times).

In example 2, which we classify as a Cartesian product problem, the quantitative reasoning that seems to be in evidence is more complex. Similar to example 1, the quantitative question is "How are the number of pairs of pants and the number of shirts related to the number of possible outfits?" The students know that outfits are created by pairing a pair of pants with a shirt. The problem is to devise a structure and operation that will generate an exhaustive list without double counting. The students accomplished this by considering the outfits generated by one pair of pants with each of the shirts. This leads to a collection of equivalent subsets of outfits. The number of subsets is determined by the number of pairs of pants and the number of outfits in each subset is determined by the number of shirts. However, the connection of the given quantities to the total number of outfits involves reconstituting the number of shirts as the number of outfits per pants. This reconstitution of the quantity represents a significant step from what was required in example 1. Whereas dollars per cookie can be interpreted as the price of each individual cookie, the number of shirts is not as readily seen as the numerosity of each of the subsets of outfits. The structure, operation, and quantities described lead to evaluation by a repeated addition model of multiplication.

To summarize, the problem solver encounters a problem that demands that a connection be developed between some initial quantities and some target quantity. The solver's first step is to anticipate a structure for the target quantity. This structure gives rise to subgroups of the target quantity (which are themselves quantities) and to a quantitative operation which constitutes the target quantity out of the subgroups. At this point it may be necessary to reconstitute one of the initial quantities in order to relate them to the groupings that have been determined. We now examine to what extent this model needs to be modified to account for problems 1 and 3.

Problem 1, like example 2, involves two quantities, the number of rectangles along the length and the number of rectangles along the width. The students must wrestle with the question of how these two quantities are related to the total number of rectangles. We see Molly's explanation as involving the same three steps, as outlined above, although with some added difficulty. The students do not see immediately a connection between the two initial quantities and the quantity to be constituted (as they did for example 2, pants paired with shirts result in outfits). The first step is to anticipate a structure (organization of the units) for this quantity. The anticipated structure is a rectangular array of the given rectangles. The question remains, "How are the number of rectangles along the length and the number of rectangles along the width related to the rectangular array of rectangles?" This is resolved by reconstituting these quantities. (Note that this time both quantities must be changed.) The number of rectangles along one side became number of rectangles per row and the number of rectangles along the other side became number of rows. The total number of rectangles could be constituted by the quantitative operation of combining the rows of rectangles. Once again the structure, operation, and units lend themselves to a repeated addition model of multiplication for evaluating the quantity in question.

How do we explain students' difficulty with problem 3 in which the teacher measured both dimensions using the same side of the rectangle? The fact that students who could clearly articulate an explanation for problem 1 were having difficulty in solving this extension problem (problem 3) suggests that additional conceptual complexity was involved in its solution. Our a priori analysis of problem 3 revealed an important difference with respect to the problems discussed to this point. This difference was found in the unit for the product.

Whereas area could be characterized in problem 1 by the number of rectangles of the size given, the area in problem 3 lends itself to measurement in squares with side-length equal to the side of the rectangle which was used for measuring. Unlike example 2 in which outfits was given in the problem statement and problem 1 in which rectangles was given in the problem statement, the unit of the product, squares (of the appropriate size), does not appear anywhere in the statement of problem 3. Thus, squares must be constituted by the problem solver.

The students were unable to anticipate squares as the appropriate unit for the product. What is more, the fact that problem 3 grew out of problem 1 and that it continued to use the rectangle as the measuring tool set up a situation that invited the students to misanticipate the unit of the product. They assumed that the rectangle was the unit and were unable to question that assumption. In their minds, they continued to hold images of an array of rectangles (not squares). This misanticipation of the units of the product provides some indirect evidence of the importance of anticipating appropriate units and seems to be consistent with our perspective that the learner is anticipating a structure for the product quantity.

How does one come to understand the relationship between area units and units of length and width? This question brings us to a second point revealed by our a priori analysis of problem 3. In order to solve problem 1, students only needed to be concerned with how to count the total number of rectangles. Thus they were working only in "one dimension."[4] In divising a strategy for counting the number of rectangular pieces of cardboard, they did not have to consider the rectangle as a two-dimensional region generated by the product of one-dimensional measures. Their solution processes were not much different from those that might be generated if the groups of rectangles (rows) were arranged end to end. Problem 3, on the other hand, demanded an understanding which is at the heart of area as a multiplicative entity. In order to solve problem 3, the student had to constitute the rectangular units (in this case squares) on the basis of the particular linear measures given.

We assert that thinking about area as being generated from linear units is not the thought process by which most learners make such a connection. Rather, learners must first understand the area of a rectangular region as a quantification of surface and envision it as measurable by a rectangular array of units (anticipation of structure). They then come to see the linear measurements of the rectangular region as providing information about the shape and the size of that array. The misanticipation of the product unit by our students suggests that they had not connected linear and area measures in this way and/or did not take note of the linear measures employed.

To summarize, through analysis of these four problems, we have developed a four step model to characterize the quantitative reasoning that is involved in the students' solutions:

1. Identify a structure (organization of the target quantity).
2. Constitute subquantities of the target quantity (based on this structure).
3. Identify a quantitative operation that constitutes the target quantity from the subquantities. [Note that #2 and #3 are likely not sequential, occuring essentially simultaneously.]
4. a. Identify a unit for the target quantity.
   b. Reconstitute units of the initial quantities to connect with the structure and units of the target quantity.

## Conclusions

Several questions can be posed and several conclusions can be drawn from this work. First it seems that many prospective elementary teachers do not have a well-developed concept of area nor an understanding of why the relationship of the length and width of a rectangle to its area is appropriately modeled by multiplication. This is in contrast to their having memorized that "area equals length times width," which is often over-generalized beyond the rectangle. What we see with prospective elementary teachers is likely only the tip of the iceberg. What is it that students in elementary and middle school understand about the area of a rectangular region as they measure length and width with a ruler (a small rectangle itself) and plug their measurements into the formula?

More tentatively, we conclude that the quantitative reasoning involved in making sense of Cartesian products and areas of rectangles may be usefully represented by our model. The model must be investigated in a variety of contexts. We believe that a model of this type, which provides a guide for unpacking component reasoning processes, will prove useful for the design of instruction. We are pursuing the question of how this model may help us to think about proportional reasoning.

The point at which students are able to see how the linear measures determine the shape of the rectangular array of units seems to be the transition to conceiving the area of a rectangle as a multiplicative object. Our position is that instead of learners building up area from linear measure, they develop the ability to connect area with linear measure and to move back and forth between the linear measures and the rectangular array that characterizes the area. Our inadvertent creation of a situation where students misanticipated the units for the area provided us a unique view of this process.

Our research was done with prospective elementary teachers for whom the equation, "area = length x width" was well-known. Their challenge was to connect area and multiplication conceptually. The model that we have begun to articulate will be of greater importance if it has explanatory power with respect to the development of elementary school children's concepts of area, students who have not prematurely learned this notorious formula. This should be investigated through teaching experiments with children.

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Footnotes

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[2] According to Thompson, a quantitative operation is a mental operation that creates a new quantity from previously conceived quantities.

[3] The project also followed a target group, selected from these students, through their student teaching semester which immediately followed the two courses.

[4] "One dimension" in the sense that any field is one-dimensional over itself.

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# MISCONCEPTIONS AND THE CONSTRUCTION OF MATHEMATICAL KNOWLEDGE

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This paper presents a brief, critical examination of the "misconceptions" view of learning mathematics, which leads in turn to a call for more detailed, content-specific constructivist theories of learning. The key notion in this proposal is modeling students' mathematical knowledge as complex systems. The misconceptions perspective has focused on the flaws in students' ideas, advocated their replacement by accepted knowledge, and failed to identify students' productive resources for learning new ideas. Efforts to elaborate constructivist theories must search out the variety and interrelation of student ideas, attend to their productive application, and study their roles in supporting further learning.

Much concern was expressed in the 1970s and 80s about students' misconceptions (Confrey, 1990). Extensive data showed that students came to mathematics and science classrooms with ideas that were relevant to the concepts in the curriculum; that those ideas were different from, and often contradictory to what was taught; and that they were hard for teachers to dislodge, even when instruction was directed specifically at them. Developing roughly in parallel to the empirical record of student misconceptions, constructivist notions came to dominate theoretical discussions of how people learn mathematics (e.g. Resnick, 1987). But this perspective was not developed into a detailed theory and often reduced to the slogan, "students actively construct their knowledge of mathematics."

My thesis, which is set forth in detail elsewhere (Smith, diSessa, & Roschelle, in preparation), is that the misconceptions emphasis on the mistaken character of student ideas cannot be reconciled with the constructivist thesis that students construct mathematical knowledge. On the one hand, we cannot look away from the empirical reality of misconceptions. The path toward an adequate mastery of elementary school mathematics is difficult for many students (and teachers) to achieve and is strewn with many flawed and incomplete understandings. But neither must we accept, on the other hand, the current rough, unelaborated, and internally inconsistent versions of learning as knowledge construction. The central task of the "post-misconceptions" period should to articulate constructivist models of learning that do justice to both the achievements and the flaws of students' mathematical conceptions.

A Sampler of Mathematical Misconceptions: Since the term "misconceptions" is more frequent in discussions of learning science than learning mathematics, we first consider some examples. The following list, though far from complete, is an attempt at a representative selection.

1. "Multiplication makes numbers bigger; division makes numbers smaller." Numerous studies have shown that students' knowledge of multiplicative operations (* and ÷) is governed by their experience of the effect of those operations on whole numbers (Fischbein et al., 1985; Tirosh & Graeber, 1989). This faulty generalization leads to problems conceptualizing and estimating the effects of multiplication and division on other numbers, particularly rational numbers between 0 and 1.

2. "Proportional relationships preserve the difference between (not the ratio of) corresponding components." This additive proportions strategy suggests that the same shape rectangular figure 5 units by ?? units can be produced from one that is 3 units by 4 units simply by adding 2 to 4, because 3 + 2 = 5 (Hart, 1988).

3. "Fractions with larger differences between their numerators and denominators (D - N) are greater than those with smaller differences." This strategy for comparing fractions leads to errors when the size of the fractions components differ widely (e.g. 3/5 and 91/99) and when the D - N differences are equal (e.g. 3/5 & 5/7) (Smith, 1990).

4. "Decimals with the more digits to the right of the decimal point are smaller numbers than those with fewer" because digits farther to the right indicate smaller size decimal parts. Based on a faulty extension of fraction knowledge, this ordering strategy also works in some cases and not others (e.g. 3.2 ~< 3.156) (Resnick, et al., 1989).

5. "Even small numbers of events should be consistent with the relevant theoretical probabilities." Application of this "representativeness heuristic" (Tversky & Kahnemann, 1982) leads to many

predictive errors, such as the expectation that a coin flipped 6 times will more likely turn out "HTHTHT" than "THTTTH."

6. "The graph of a line is governed by 3 entities: the slope, x-intercept, and y-intercept and each entity should be represented in the equation of that line." This "3 slot schema" leads to serious difficulties in interpreting linear equations (Schoenfeld, Smith, & Arcavi, in press; Moschkovich, 1990).

Shared Properties: If this brief list as representative, we can draw some general conclusions about the nature of mathematical misconceptions. First and foremost, these notions are different from the concepts that we want students to learn. More to the point, they contain clear conceptual flaws. Second, they appear as students begin to study concepts that are both elementary and substantially complex. Multiplicative operations, rational numbers, and linear functions – though considered "elementary topics"– are qualitatively more difficult for students than additive relationships between whole numbers. But in spite of their flaws, each indicates a measure of genuine mathematical insight. The difference between a fraction's numerator and denominator does capture, though imprecisely, the part-whole relationship expressed by that number. All three entities in the 3 slot schema are important graphically, though they are not all parameters in a single linear equation form. It is the case that the digits farther to the right of the decimal point represent smaller decimal parts, but the composite value of all decimal places must be considered in judging decimal size. Thus misconceptions represent students' attempt to cope with new mathematical concepts by adapting their existing mathematical knowledge. Even the brief descriptions given above show that misconceptions are often thoughtful efforts to make mathematical sense, even if they are not completely successful.

The proposal I will outline shortly is that misconceptions provide the main raw material for building constructivist theories of the learning mathematics. What have been termed "misconceptions" represent many aspects of successful learning, and the task of the learning researcher is to explain, in substantial detail, the initial generation of and the eventual evolution of misconceptions into more sophisticated forms of mathematical knowledge.

The Core Theory of Misconceptions: But the dominant view of misconceptions understands the learning process quitely differently. Misconceptions are things that "interfere" and students must "given them up" for proper learning to take place. The key propositions that underlie this perspective are listed below. Though you will not find any single researcher who has expressed this view explicitly and completely, there is substantial evidence that it represents shared view of large community in mathematics and science education (Smith, diSessa, & Roschelle, in preparation). Very often, particular parts of this perspective are only tacitly expressed.

1. Students have definite ideas about mathematical concepts prior to classroom instruction (i.e. they have misconceptions), but their ideas are different from accepted forms of knowledge.
2. The source of students' misconceptions is their prior knowledge, learned either from classroom instruction or interaction in the physical and social world.
3. Misconceptions can be stable and widespread among students (and adults). Those that are stable and widespread are often strongly held and resistant to change.
4. Due to their strength and flawed character, misconceptions can interfere with learning.
5. Learning mathematics means replacing students' existing misconceptions with accepted forms of knowledge.
6. To neutralize them, instruction must confront students with the disparity between their misconceptions and accepted knowledge. When this disparity is made explicit, students will appreciate the advantages of the accepted view and give up their misconceptions.

Problems with the Core: When this "core theory" of misconceptions is presented in explicit terms, a number of deep problems are manifest, particularly for the constructivist. First, the emphasis on fundamental differences between misconceptions and the knowledge accepted by the mathematical community widens the chasm of the learning paradox (Bereiter, 1985). If there is any principled problem in conceptualizing how more adequate knowledge structures can emerge from more primitive ones, proposition 1 deepens the problem. Secondly, the root of the proposed learning process– that misconceptions are replaced by "expert" ideas– does not stand up to empirical test. A number of studies have shown that students can learn the "correct" mathematical concept without "giving up" their misconceived versions (Fischbein, et al., 1985;

Schoenfeld, et al., in press). Third, while having students articulate their conceptions may be an initial part of all deep and substantive learning, the instructional model of "confrontation" needs to be reexamined. It is tied conceptually to learning as a process of replacement; in the final analysis, the student's conception needs to disappear. The possibility that the student and the more expert teacher may not be comparing the validity of their ideas across the same range of application is not considered.

Overall, the core view fails to put forward any productive role for students' conceptions. Little attention is given to what useful resources they might bring to the learning task. The focus instead is on the antagonistic relationship between what is inside their heads and what is outside, in the instructor's head and in the textbook. The failure to articulate a productive role for students' prior ideas is especially problematic for constructivists because learning is in essence the reformulation of existing ideas. Thus if constructivists accept the core view of misconceptions, they are caught between contradictory models of learning, between the need "replace" what is faulty and to principled requirement to "reformulate" existing knowledge.

Knowledge as a complex system: The task of a constructivist theory of learning mathematics is as much to explain the evolution of misconceptions into more sophisticated and productive forms of knowledge as it is to explain their genesis. The required Gestalt shift is from regarding misconceptions only as learning failures to appreciating the partial but developing insights they embody. The theoretical development of constructivism will require an expanded vision of what counts as knowledge and learning. The key part of this reconceptualization is the notion of knowledge as a complex system; the major methdological commitment will be to detailed studies of learning specific mathematical ideas over extended periods of time (i.e. years).

A root problem in modeling students' developing mathematical knowledge is the strong impact of the polished, codified forms of knowledge characteristic of the discipline. Mathematical knowledge (as articulated within the professional community) is known for its expression in compact, general, deductively organized sets of unitary propositions. We can easily forget the gradual, historical evolution of mathematical ideas and "default" to these overly rationalized views of knowledge. A substantial body of empirical evidence indicates that mathematical knowledge as applied by both the sophisticated and the more naive students contains many more elements, organized in more complex ways than disciplinary models have suggested. For example, in two numerical domains where a single procedure is sufficient, the most competent students know and apply a wide variety of methods (e.g. Siegler & Jenkins, 1989; Smith, 1990).

In our investigations of complex systems we cannot restrict ourselves to the general principles and procedures emphasized in textbooks. That they are known and applied by students is not evidence against the possibility that other, quite different forms of knowledge are playing co-equal, if not more important roles. Some elements of complex systems are only applicable in restricted contexts– a fact already evident in some of the "misconceptions" listed above. More generally, mapping the applicability of knowledge elements (and the changes in their applicability) will be as important as describing their content. Features of problem situations, e.g. specific numerical features, may cue clusters of elements that apply in combination. Finally, we need to attend to the generativity of knowledge and to the issue of whether and how students "constructed" knowledge may make the learning of the general principles of more advanced knowledge forms possible (e.g. diSessa, in press).

Two Epistemological Alternatives: While the research program of complex knowledge systems is one proposal for moving the study of mathematics learning beyond misconceptions, there are alternatives that differ dramatically in their epistemological foundations. One "post-misconceptions" program centers on the design of learning environments that mitigate against misconceptions ever appearing. Nesher's (1989) analysis of "learning systems" suggests that carefully constructed representations of mathematical ideas and guided activities within them can support more or less errorless learning. While the existence of such "perfect" representations is certainly open to empirical demonstration, this proposal is questionable on two counts. Representations never have a single interpretation. The meaning of any instructional representation, though transparent to its designer, is rarely so obvious to the student. Equally important, sealing knowledge of the world out of learning systems leads students even more toward learning "inert" mathematical knowledge that is largely irrelevant to their lives.

A quite different and increasingly influential theoretical program shifts the analytical focus from isolated individuals to groups of students who know and learn from the activities, social relations, and discourse within a specific classroom community (Cobb et al., 1992). On most issues, I understand social constructivism and the particular form of constructivism advocated here to be consistent, though maintaining different emphases. But if there is one weakness in the social constructivist proposal it is with respect to learning, specifically with knowledge change within individuals. Even if we grant the utility, if not the necessity of a social constructivism, both the science and the practice of mathematics learning will require theories that provide detailed accounts of the evolution of particular concepts. While knowledge, its sources, and its criteria of validity are all socially negotiated, those negotiations, in classrooms and "outward" into larger communities, lead inexorably to shared judgments that some forms of knowledge are more adequate, more useful, and yes, more "correct" than others. It is for the particular task of characterizing the changes in individual students' mathematical ideas that the notion of knowledge as a complex system looks promising.

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# THE ACT OF TEACHING MATHEMATICS: A CASE STUDY

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*The instructional activities of the 3rd-grade teacher described in this case study are considered within a framework designed to account for self-monitoring among the multiple senses of the complex act of teaching. The examples of this teacher's self-monitoring activities were gleaned from interviews, from observations of her class, and from her writings. We learn that even though this teacher might be considered exemplary, she continues to need support. The kind of support she needs, however, changes as she herself grows as a teacher.*

> It requires skill and art to select from the total circumstances of a case just what elements are the causal conditions of learning, which are influential, and which secondary or irrelevant. It requires candor and sincerity to keep track of failures as well as successes and to estimate the relative degree of success obtained. It requires trained and acute observation to note the indications of progress in learning, and even more to detect their causes--a much more highly skilled kind of observation than is needed to note the results of mechanically applied tests. (Dewey, 1974, p.181)

Skill and art, candor and sincerity, and the ability to observe--these develop when a teacher continually reflects on her/his instructional activities, monitoring them closely and judging their fit to the ideals she/he holds. This case study portrays a teacher for whom self monitoring is an integral part of her professional life.

Kilbourn (1991) has claimed that self monitoring is one of the single most essential aspects of teacher professionalism. Building upon previous work by Komisar (1968), Kilbourn (1991) distinguished among self-monitoring activities in each of three different senses of teaching: occupation, enterprise, and act. Occupation refers in the most general way to the job one does. Examples of occupational norms of teaching include coming to work on time, arriving sober, and behaving ethically on the job. The enterprise sense of teaching includes all the things a teacher normally does while teaching, including taking roll, assigning grades, preparing overheads, and supervising in halls. Finally, the act sense of teaching involves those things a teacher says and does while actually teaching students. Kilbourn provided an example highlighting the distinction between the enterprise and act senses of teaching:

> Thus, one might say at two o'clock (somewhat peculiarly, but without contradiction): "I am teaching [enterprise] this afternoon, but I am not teaching [act] now; I am on hall-duty." In other words, there are times when we use the term "teach" to indicate things we are doing that *are* centrally concerned with the whole point of the matter. So, a teacher might say, "I wish they didn't interrupt with P.A. announcements; can't they see I'm trying to teach?" And here the term would be used to mean that the teacher was, at the moment, actively engaged in doing things intended to foster learning in one or more students. (Kilbourn, 1991, pp. 724-725)

The *act* sense of teaching consists of supporting acts and intellectual acts. Supporting acts refer to things done by a teacher who is actually teaching, but are not normally seen as the point of teaching. Actions designed to reduce anxiety, arouse interest, or focus attention are supporting acts. Note that these acts are more central to the point of teaching than actions that fall into the enterprise category, because they involve actions taken while in the process of teaching. Intellectual acts refer to those acts that involve the teacher helping a student understand something. Intellectual acts comprise the fundamental goals of teaching. Questions concerning intellectual acts include: "Does this make sense? Is this intelligible? Is this logically lucid?" (p. 732).

These three types of self-monitoring activities provide a framework for the discussion of the teaching behaviors and reflections of the teacher selected for this study.

### Alyssa: A Case Study of a Professional

Alyssa is one of four teachers presently participating in a research project designed to study the teaching and learning of middle school number and quantity concepts. As part of this project, she took a written assessment of content knowledge, participated with three other teachers in 6 three-hour seminars, and was interviewed about her background, her beliefs about teaching and learning, the manner in which she plans and carries out instruction, and the manner in which she assesses learning and uses assessment information. She was visited three times in her classroom, and post-interviews accompanied all the classroom visits.

Alyssa's road to mathematics teaching has been indirect. Shortly after graduating from college with a degree in political science, Alyssa accepted a job team-teaching a combination 3rd-4th grade at a private school. After completing her first year of teaching, she went back to school at night to secure her teaching credential. It was at that time that she took her first mathematics course since high school, a course for prospective elementary school teachers. After two years at the private school, she left to attend school full-time to finish her credential and begin work on a master's degree in education with an emphasis in mathematics. As part of her coursework, Alyssa completed a problem solving course and a graduate seminar on the elementary school mathematics curriculum. Following the completion of her master's degree two years later, Alyssa spent one year as a mathematics resource teacher in a small suburban school district. Although this job required her to visit many classrooms and to give demonstration lessons, she did not teach her own class. After one year in this position, Alyssa lost her job due to district budget cuts, and she accepted a position teaching third grade in a private school. At age 28, she is currently in her first year at this new position and is very happy with the job.

Alyssa engages in self monitoring at all levels of Komisar's senses of teaching. At the level of occupation, she, like most teachers, behaves ethically in her role as teacher, but she engages in many other occupational activities as well. She attends professional conferences and sometimes makes presentations. She has been involved in a variety of extra projects, including one in which she helped student teachers who led an after-school math activity, another in which she worked with inservice teachers, and the project described earlier involving the four teachers. She is familiar with the latest documents in the field of mathematics education and refers to them often. One example of her self-monitoring at the level of occupation involved her preparation for a recent conference presentation. She explained that she was making a concerted effort to complete the unit that was the subject of the presentation because it was important for her to see for herself how her students responded to the unit before she shared it with other teachers.

Alyssa's strongest display of self-monitoring at the level of occupation came when she described in writing her initial hesitation about participating in the current project:

> I have always been aware of my weaknesses in this [mathematics] content area, and I was somewhat hesitant to participate for fear of embarrassment of my 'ignorance' [in front of] other teachers. But I am such an advocate for personal growth, and I knew this would be an incredible opportunity for growth, that I wanted to struggle through problems in order to experience growth.

This statement reflects the qualities of candor and sincerity mentioned in the opening quote by Dewey. Alyssa is willing to take personal risks in order to be the kind of teacher she can respect.

The enterprise sense of teaching includes those things a teacher normally does while teaching, even though they may not be centrally related to "actual" teaching. Some of the tasks included under the enterprise sense are planning lessons, preparing overheads, supervising halls, contacting parents, putting up bulletin boards, and assigning grades. In describing how she plans lessons, Alyssa explained that she begins by plotting out the months and "brainstorming ideas revolving around the broad units of study" she considers important for the third grade, based on her past experience with second, third, and fourth graders, and on recommendations from the *Standards* (National Council of Teachers of Mathematics, 1989) and the *Framework*. (California Department of Education, 1985). For each unit, she sorts through her resource materials and puts all relevant materials together to use in daily planning. Another example of Alyssa's self-monitoring at the enterprise stage of teaching relates to the role assessment plays in planning. She stated:

> I find myself in battles, ideological battles, about how much attention do I pay to standardized testing, and how much do I continue doing what I believe is correct? But it does bother me. I think a lot of us feel this way. We think that the way we're teaching is certainly different from what is assessed on standardized testing. Perhaps we're giving the big picture, and we think that the fine details, for example the skills, will follow along.

### The Act of Teaching Mathematics

The act of teaching is made up of supporting acts and intellectual acts. Alyssa is very attuned to the supporting acts of teaching. She encourages and praises her students for giving explanations and maintains a supportive atmosphere in the classroom. At times she appears to bend over backwards to assure that all students will feel good about their classroom contributions, even though these comments may reflect incorrect or incomplete mathematical thinking. Reflecting upon the classroom atmosphere, Alyssa talked about her childhood experiences in which she did not feel she was good at mathematics, did not enjoy it, and generally had negative feelings toward it, and how those feelings changed when she transferred to a school where mathematics was taught in a more informal manner. She remembered those negative feelings when describing to us how important a child's emotional state is in a mathematics classroom.

The intellectual act of teaching mathematics denotes those acts that involve the teacher helping students develop understanding about the mathematics they are studying. Alyssa demonstrated self-monitoring at this level on repeated occasions. One hallmark of reflecting on intellectual acts involves a teacher's ability to listen to her students and make instructional decisions based on what she hears. Alyssa consistently displayed this orientation. On one occasion, she taught a lesson in which students represented numbers and addition of numbers using base ten blocks. During the lesson, she linked the meaning of standard and nonstandard procedures to the blocks. Students explained what they discovered and Alyssa accepted all of their explanations. Her lesson incorporated a variety of representations, including using manipulatives, real world situations, students' explanations, and paper-and-pencil algorithms. Throughout the lesson, Alyssa adjusted the pace to fit the students.

After the lesson, Alyssa was asked why she chose to focus the instruction on two-digit addition and subtraction, since children seemed to know the standard algorithm. Alyssa said that their knowledge was not based on understanding. For example, they insisted on writing:

$$\begin{array}{r} {}^{1}\cancel{1}7 \\ \underline{-\ 8} \\ 9 \end{array} \qquad \text{and} \qquad \begin{array}{r} {}^{1}8 \\ \underline{+\ 8} \\ 16 \end{array}$$

On the basis of these observations, Alyssa had concluded that her students did not understand place value and she designed the lesson above to link the algorithms to place value understanding.

Alyssa's awareness of her students' strategies for solving problems is another example of her reflectiveness at the level at which students engage in the act of learning. She described a lesson that she included in a unit taught on the development of the concept of multiplication as follows:

> I gave them the problem: "If everybody in this classroom went out, how many chopsticks would we need?" And so they talked about that and they explained and wrote how they solved the problem. They had various different ideas. Some of them said, "There are 17 people in the class, so I counted 17 twice." Some people said, "I counted by twos to 17." Some said--I can't remember, but we just talked about all the different strategies.

On another occasion she described a lesson she had taught on solving subtraction story problems. Alyssa described a student's approach to the following problem: Carl ran 14 laps around the track. Doug ran only 6 laps. How many more laps did Carl run than Doug?

> For this one, she said, "I know 7 plus 7 is 14, and I took one away to get to 6, and I know 8." That's an example. So for these, we had a great discussion of all the thinking strategies they used to do the computation. We talked about all the different ways that you could do it. And there were great discussions.

A teacher genuinely interested in the strategies used by her students, who bases a classroom discussion around the sharing of those strategies, and who uses that information to make curricular decisions is rare indeed.

In one class we observed, Alyssa used the first 50 minutes of her 60 minutes allocated mathematics time to work two problems she had selected from a problem-solving book. During those 50 minutes, students explained how they had solved the two problems. One of the problems involved finding the number of possible paths from a house over a bridge to a lake, given that there were two paths from the house to the bridge and two more paths from the bridge to the lake. During the class discussion, in an attempt to focus the students on their reasoning instead of on their answers, she asked, "Who wants to tell me, without telling me the route, how you were thinking." When asked about that question afterwards, Alyssa responded:

> Yes, the motivation was going back to the recurring theme of "just tell me the answer" kind of thing. And I always sort of have that on the top of my head--"We're not just going to tell the answer; we're going to explain thinking." And so that's kind of why the question was asked, but the question really wasn't fitted that well to this particular situation. Because it really was hard to explain what your thinking was without really drawing or explaining the roads for them. I think for this particular problem it didn't work."

Notice first that Alyssa's focus of discourse had been on the students' reasoning processes, not their answers. Second, notice that Alyssa came to recognize that her question did not fit this situation. Finally, note that Alyssa did not conclude that such questions are unimportant, but rather that they work better in some circumstances than in others.

## What Have We Learned?

Excellent teachers evolve, they don't become. Alyssa is involved in the process of evolving. She does not behave as if she has "arrived" nor does it appear to us as if she will ever feel that way. Her self-awareness provides her with occasions to reflect on her teaching and, as a result, to grow. For example, Alyssa is facing a tension between the supporting act of encouraging her students to share their responses and the intellectual act of knowing how to respond to student comments which reflect incorrect or incomplete mathematical thinking. Alyssa has not yet resolved this issue, but we are confident that she will.

There are missed opportunities in every classroom, and such was the case in Alyssa's room too. The earlier mentioned lesson on problem solving was interjected in the middle of her unit on multiplication. One of the two problems discussed during that lesson involved finding how many possible paths there were from a house, over a bridge, and to a lake, given that there were two paths from the house to the bridge and two more from the bridge to the lake. Even though this problem solving lesson took place in the middle of a unit on multiplication, Alyssa had no plans to have the students pursue the multiplicative relationship inherent in this situation. She seemed to think of this lesson solely as a problem solving lesson and did not consider how it might have been integrated into her multiplication unit. We do not know why this opportunity was missed. Was Alyssa unaware of the multiplicative relationship inherent in the problem? If so, was it because she did not reflect upon it? Or was her knowledge structure such that she was not poised to see it?

Alyssa is young and relatively new to the profession. It is heartening to see how this young teacher is creating for herself an image of teaching so closely aligned with that envisioned in the *Professional Standards for Teaching of Mathematics* (NCTM, 1991). She is emerging as a leader in the local mathematics education community and will be able to lead other teachers to embrace her vision of teaching. But she will continue to need support. She appreciates, benefits from, and almost craves interaction with others who will challenge her intellectually and offer her moral support to undertake change. In describing the impact of her involvement with our current project, she wrote:

> The greatest impact on me has been an increase in my reflection on both my mathematics teaching and my own mathematical understandings. I have become more aware of all that I don't understand.

We know that our project did not fundamentally influence Alyssa's teaching--she was already an exemplary teacher, exhibiting the skill and art, candor and sincerity, and power of observation that are hallmarks of a teacher who will continue to learn (Dewey, 1974). Our project was simply one more occasion for learning. We offered her opportunities to interact with similarly minded people and to reflect on her teaching, thereby enabling her to uncover some of those blind spots we all have but need help to see. As teachers assume leadership positions, there is the tendency to think of them only as "givers" and not as "needers." Alyssa reminds us that the need for support structures does not disappear as one becomes a better teacher, even though the *kind* of support needed might change.

One of the assumptions upon which the *Professional Standards for Teaching Mathematics* (NCTM, 1991) is based states that teaching is a complex practice not reducible to recipes or prescriptions. In order for teachers to draw upon the many facets of this complex task, they must reflect upon their knowledge of mathematics, their knowledge of their students, their understanding of how students learn, and their goals for instruction. Self-monitoring of these intellectual acts aims at the very heart of teaching and is indispensable to the success of the current mathematics reform. We need to find, cherish, and nourish our Alyssas.

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# Sources of Certainty and Uncertainty in Mathematical Problem Solving

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*This study investigates the certainty and uncertainty that students feel as they work on a mathematical problem. It is hypothesised that the over-confidence in decisions that characterises reasoning in many fields of human endeavour is also exhibited in mathematical work and that it may partly explain why students generally are reluctant to check their work. Students who feel certain that their work is correct would see little reason to check it. In the problem used in this study, uncertainty arose in making a generalisation, but also from carrying out straightforward calculations . Students with wrong methods that gave easy arithmetic were, in the end, almost as certain that their answers were correct as students with the correct method. These observations may help to explain why students with "obviously" wrong answers do not check, why students more often check arithmetic than reasoning and the tendency for groups to choose a simple wrong solutions even when a correct solution been proposed.*

Investigations into the strategies and methods students use to solve mathematical problems have found that generally students do not check their work spontaneously (Davis, Jokusch & McKnight, 1978; Kantowski, 1977; Lee and Wheeler, 1987; Stacey, 1989). When they do, it is often an incomplete check or simply a repetition, rather than a re-assessment, of what they have just finished. Indeed, Stacey and Groves (1985) have noted that with many students, a verbal instruction to check their work is interpreted only as an instruction to repeat it. Several reasons have been advanced in the literature to explain why students do not check their work. The "looking back" phase is generally recognised as the most neglected, both in actual problem solving and in teaching emphasis. Schoenfeld (1985, p 316) notes that "lack of plan assessment and absence of review" are major factors contributing to failure in problem solving. When an answer has been obtained to a problem the natural, but immature, response is to want to move on quickly to the next task, rather than reflecting on what has been achieved. There is also a substantial body of evidence that some students do not know how to check their work. The research reviewed by Bell, Costello and Kuchemann (1983) concerning understanding of proof showed widespread misunderstanding both of the role of counterexamples and of the inadequacy of examples for proving that a generalisation is true. Understanding of both content is also involved here. Lee and Wheeler (1987), for example, point out that students can only use substitution of numbers as a strategy to check algebraic manipulations when they have a reasonably clear understanding of the relationship between arithmetic and algebra.

In this paper we examine another factor which underlies students' checking behaviour, which was suggested to us by Fischbein's (1987) analysis of the role of intuition in mathematical thinking. A series of research studies, reviewed by Fischbein (1987), into the subjective evaluation of certainty has established that for a reasoning endeavour to continue, a person must feel confident about the decisions they make during it. A steady tendency has been found for people to be over-confident in the accuracy of their own knowledge, decisions, interpretations and solutions. This natural tendency to over-confidence in the decisions that they make whilst solving a problem may therefore contribute to students' lack of checking, particularly of their reasoning.

Fischbein brings together experimental findings from various sources to develop his theory that a high degree of certainty in a reasoning process, a fundamental need of the human mind, is produced by reliance on self-evident, intrinsically certain, persistent intuitions. "During the very course of our reasoning, of our trial-and-error attempts, we have to rely on representations and ideas which appear, subjectively, as certain, self-consistent and intrinsically clear" (p x). Although most of Fischbein's analysis concerns conceptual structures and clusters of beliefs, he also recognises "anticipatory" intuitions which are specific to the problem solving process. Observations of students solving mathematical problems by Galbraith (1986) also point to a desire for high levels of certainty. He noted that students wanted to "close upon a definite result rather than maintain and open mind in the absence of conclusive information" (p 430) and observed a tendency to redefine or amend data so that it supports a particular inference or conjecture.

Galbraith also noted that when students were asked to select the better of two explanations, many students made their choice of the grounds of simplicity. This observation has also been made by Stacey (1990) who observed that when groups of students were solving problems together, they often chose a simple incorrect solution, even when a correct solution had been proposed by a group member. The crucial factor in whether the group solution was correct seemed not to be having the ideas, so much as choosing between them. Misplaced confidence in a simple idea and lack of adequate checking strategies were common faults.

## Aims

With the background outlined above in mind, data was collected to explore the following questions:

(i) What factors cause certainty and uncertainty in problem solving amongst high school students?

(ii) How does the certainty of students who answer a question correctly compare to the certainty of students who answer it incorrectly?

(iii) Is there a relationship between certainty and checking? In particular are students who do a problem in only one way more certain of their results than students who search for other approaches?

## Method

One problem (see Figure 1) was given to 227 Year 8 students (average age 13 years) at two girls' schools during class time. After completing the problem, the students answered a questionnaire. About 15 students (exhibiting different responses) were interviewed within two days of the problem solving to further elucidate the reasons for the responses. The questionnaire asked students to rate their certainty in their answers to the three parts of the problem on scales from 0 to 10. The zero, five and ten positions on the scale were annotated with descriptive comments, such as "completely sure you answer is wrong" at zero. This measurement of certainty was adapted from calibration studies reviewed in

Lichenstein, Fischhoff and Phillips (1982). Then students were asked to indicate, separately for the 10x10 and 50x50 blankets, whether they had done the question in only one way or in more than one way and whether they had obtained one or more answers. This was the criterion selected for judging whether students had checked their work. Finally each student was presented with one of six additional pieces of information and then asked to re-assess their certainty for the 50x50 blanket. The purpose of the additional information was to investigate how supporting or contradictory evidence affected certainty. Space precludes reporting the results from this part of the study, but comparison of the changes in certainty due to each version indirectly gave insight into the methods that students used to check their work. In particular, it was found that students tended to use the gross features of the additional information as a check of reasonableness of answers, rather than to directly compare predictions made from their rules with the independent, additional evidence. It is very difficult, even in interview, to establish what checking a student is actually doing (Lee and Wheeler, 1987) and so the further refinement of this as a research tool is indicated.

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*Imagine you are going to make a patchwork doll's blanket by sewing together some tiny squares of material measuring 1cm by 1cm. You want to know how much sewing you will have to do to make the blanket.*

*To make a square blanket measuring 3cm by 3cm, you need 9 squares of material and it takes 12 cm of sewing. (Diagram given of 3x3 blanket with sewing between the squares, but not along the outside edge of the blanket, clearly shown)*

- *How much sewing is needed to make a square blanket measuring 5cm x 5cm?*
- *How much sewing is needed to make a square blanket measuring 10cm x 10cm?*
- *How much sewing is needed to make a square blanket measuring 50cm x 50cm?*

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Figure 1: Making a Blanket

## Results and Discussion

About half of all students and three quarters of the students who were correct checked their work in some way. Of those who did check, 42% were correct whereas only 18% of those who did not check were correct. This difference is statistically significant (chi-squared=5.3, d.f.=1, $p<0.02$). Students who checked their work and found the same answer in two different ways had significantly greater certainty than students who did not check ($p<0.05$). These in turn had higher certainty than students who checked but found two different answers.

In order to relate changes in certainty to mathematical behaviour, responses were classified according to the solution method used. Some students followed one general rule for all three answers. Some began by drawing the 5x5 blanket and counting the number of centimetres of sewing but used a general rule for the 10x10 and 50x50 blankets. Others used the generalisation only for the third blanket. The number of students using the most popular methods are given in Table 1, which uses algebraic notation to indicate the rules even though the students themselves only indicated their generalisations by the calculations they performed. Thus, for example, the third row of the Table indicates that 20 students found the answers to all the parts of the question by calculating $n^2+n$ for $n = 5$, 10 and 50, although they did not write the algebraic notation. This rule (and also the perimeter rule) were popular because they fitted the given information that a 3x3 blanket required 12cm of sewing. Table 1 also shows the mean certainty on each part of the problem for the students using each rule. In all cases, students were most certain about their answers for the 5x5 blanket and their certainty dropped for the 10x10 blanket and again for the 50x50 blanket. The higher certainty for the 5x5 blanket was associated with use of drawing and counting to obtain the answer. The only two groups that did not count (the perimeter and the $n^2+n$ rule) had the lowest certainties for this part. These groups of students seemed to settle immediately on a solution that happens to fit the given data without exploring the situation in any depth, but they did not do this because they were very certain of the answer. In Fischbein's terms, they did not have an anticipatory intuition of which they were very sure. Instead, they seemed merely to accept the uncertainty and not do anything about it - although a few of them drew the 5x5 blanket, they did not even count.

Table 1
*Mean Certainty of Users of Different Rules for 5x5, 10x10 and 50x50 blankets.*

| Method | N | 5x5 | | 10x10 | | 50x50 | |
|---|---|---|---|---|---|---|---|
| | | rule | mean | rule | mean | rule | mean |
| correct rule | 46 | count | 8.89 | $2n(n-1)$<br>or count | 8.17 | $2n(n-1)$ | 7.20 |
| perimeter | 51 | 4n | 7.10** | 4n | 6.94** | 4n | 6.82 |
| $n^2 + n$ rule | 20 | $n^2 + n$ | 6.60** | $n^2 + n$ | 6.55** | $n^2 + n$ | 5.45** |
| $2n^2$ rule | 3 | $2n^2$ | 7.67 | $2n^2$ | 6.33 | $2n^2$ | 5.67 |
| count and $2n^2$ | 2 | count | 10.00 | $2n^2$ | 9.00 | $2n^2$ | 8.50 |
| count and $2n^2$ | 4 | count | 9.25 | count | 9.25 | $2n^2$ | 7.25 |
| multiples of previous answer | 7 | count | 9.57 | twice 5x5 answer | 8.14 | 5 times 10x10 answer | 7.29 |
| multiples of previous answer | 14 | count | 8.79 | count | 7.79 | 5 times 10x10 answer | 6.43 |

** sig. difference between this mean and corresponding mean for the correct rule ($p<0 .01$)

*Uncertainty on moving to a generalisation.*

Naturally certainty dropped when students moved from counting to a general rule. The mean drop in certainty when students counted on two consecutive parts of the problem was 0.78 (averaged over 18 unambiguous instances from Table 1) and when students used a rule for two consecutive parts, the mean drop was 0.32 (157 instances). However, there was a much larger mean drop in certainty for students moving from counting to use of a general rule (drop of 1.47 averaged over 27 instances). Students with the correct rule were excluded from this analysis because it was sometimes unclear whether they had counted or used the rule or both in the second part of the problem.

### *Uncertainty when calculating.*

Some of the interviews indicated that part of the drop in certainty from the 5x5 blanket to the 50x50 blanket could be attributed to uncertainty about the results of arithmetic calculations involving larger numbers. For the purpose of this investigation, a calculation was defined as hard if it involved multiplication of two numbers over ten, even though this skill is taught at least four years lower down the school than Year 8. The mean drops in certainty from the 10x10 to the 50x50 blankets for students using the same rule on these parts of the problem were calculated separately for methods involving hard calculations ($n^2+n$, $2n^2$ rules requiring $50^2$) and easy calculations (perimeter, multiples requiring multiplication by 4 or 5). The mean drop for rules involving hard calculations was 0.96 (25 instances), which is significantly greater than the mean drop of 0.21 for rules involving only easy calculations (58 instances). The size of this effect had not been expected. Over three years after students have first learned how to perform multiplications such as 50x50=2500, the uncertainty involved with it is still comparable with the uncertainty they feel when making a generalisation. This may partly explain why students' checking concentrates so much on repeating arithmetical calculations, where teachers see a greater need to concentrate on reasoning.

### *Certainties of students with a simple or complex rule.*

A comparison of students with the correct rule and the perimeter rule shows the practical implications of the drops in certainty due to generalising and harder calculations. The perimeter group began with an immediate generalisation, based on little evidence. They did not subsequently have to make the transition to a generalisation and their calculations were at all stages very simple. Thus, although the perimeter group were significantly less certain of their 5x5 blanket answers ($p<0.01$) than the correct group, there was no significant difference between their certainties for the 50x50 blanket and those of the correct group. On the challenging part of the problem, the certainties of students who are correct and those who have grabbed the simplest wrong rule apparently without any investigation, are comparable.

## Conclusion

The use of certainty ratings has proved to be a promising research tool for understanding students' thinking during problem solving. Students who jumped quickly to generalisations based on little evidence did not seem to do so because of stron certainty, an intuition in Fischbein's sense. However, their certainty in the correctness to the final part of the problem is comparable to that of students who are correct. Thus, in a group discussion, both a simple wrong rule and the correct rule may be propounded with equal conviction, leading a group to choose the wrong answer over the right answer. Both making a generalisation and doing arithmetical calculations caused students to lose certainty in their work. Students who have made wrong assumptions at the start of their work which happen to lead to simple processing will be no more likely to check their work at the end than students who are correct. Their simple (wrong) rule has "proved itself" in the ease of calculation of the answers it produces.

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# The Description of Solids in Technical Drawing

## - Results from interviews of experienced draughtsmen

*Twenty experienced draughts(wo)men were asked to sort and classify a set of 16 technical drawings using criteria they themselves had to choose and explain. Approximately half of the drawings were perspective drawings, while the others gave 1, 2 or 3 orthogonal views of the machinery drawn. Additional data on the professional biography, the experience in technical drawing and the use of advanced technology (esp.: CAD) of the interviewees were gathered.*

*The theoretical and practical knowledge underlying the individual classifications was analyzed. The paper presents results on the different vocational functionalities attributed to the mathematically different types of representation of solids. A comparison of draughts(wo)men using traditional technical drawing facilities and those using CAD will finish the paper.*

### 1 Research Question and Methodology

The "utilitarian" view of mathematics strongly advocates for the teaching/learning of mathematics in general education by pointing to the importance of mathematics in vocational/professional situations. In order to strengthen this position, it is desirable to identify and describe the vocational/professional use of mathematics in order to shape the work-related teachning/learning of mathematics (Bromme&Sträßer 1991). As a consequence, research on the use of mathematics in the workplace often tries to directly identify mathematics used by (qualified) workers (cf. e.g. the "research studies" of the Cockcroft-Report in the UK). This approach proved to run into a dilemma: The "users" of mathematics in the workplace often deny to use any mathematics at all - while researchers from mathematics or mathematics education tend to classify too many practices of the workplace as involving mathematics (for a discussion cf. part 2 of Harris 1991 and Dowling 1991 for a major reason of this dilemma). For a psychological theory of professional knowledge, this dilemma is very interesting because professional work in qualified professions requires ´cognitive inter-disciplinarity' i.e. merging theoretical and practical knowledge of different origins. As this knowledge is partly implicit, appropriate psychological methodes of data collection as well as psychological models about human knowledge are neceessary (Bromme 1992).

To learn about the way mathematics plays a role in vocational/professional practices, we chose to analyse the way geometry (as mathematical knowledge) is used in technical drawing within metallwork (as the vocational/professional domain). We offered a prepared set of 16 technical drawings to experienced draughts(wo)men and asked them for a classification. When they had formed groups of drawings, they were asked to comment on their classification by giving a short description, at least a catchword to every group they had formed. The classification and the comments were protocolled by the interviewer. The draughts(wo)men were also asked to evaluate the respective importance of vocational training against workplace-experience and aspects of technical drawing which they came to learn at the workplace. Only at the very end of the interview, they were directly asked about topics from mathematics relevant to technical drawing. To detail their use of mathematics, they had to additionally describe the latest situation when they used mathematics for their drawing. Data on the educational and professional biography, the type of company they were working for as well as the experience in technical drawing and the use of advanced technology (esp.: CAD) were gathered. The draughts(wo)men interviewed should have finished a vocational training with a qualification, having at least 2 years of professional experience in technical drawing after their examination. To avoid too large a qualification spectre, we concentrated on draughts(wo)men, excluding technical designers.

A prior analysis of the role mathematics may play in technical drawings came up with the following dimensions: The most important issue of technical drawing (at least in metalworking) is the problem of representing solids (3D-objects) using only 2D-modelling, i.e. drawings. Technically spoken, this problem is solved by means of certain conventions developed for technical drawings, namely a certain set of representations as specific projections (like isometries etc.) or systems of orthogonal views as normalised for instance by the DIN- or ISO-norms ("DIN"=Deutsche Industrie-Norm, "ISO"=International Organization for Standardization). For the planning and control of production, information on the measures of the solids described is most important. The norm-systems give detailed conventions for this type of information. From the process of drawing and production, the appearance of symmetries in the drawing or object described is most important because this may facilitate or complicate the actual drawing or production of an object. The issue of symmetries becomes even more important when using Computer Aided Design (CAD) and advanced production technologies like Computer Numerically Controlled (CNC) techniques and integration of both.

With these issues in mind, we tried to have a set of 16 technical drawings which could be a-priori classified in three dimensions: 8 of them should have symmetries, the other 8 should be a-symmetrical, 8 of them have measures, 8 have not and 8 are drawn as projections while 8 give systems of 1,2 or 3 orthogonal views. A quick calculation shows, that every cell of the cube (cf. fig.1) can be filled by two drawings satisfying these descriptions. We decided to have at least one central perspective in the drawings (cf. fig. 2.4 on the following page, figures.2.1 to 2.3 are three other drawings of the 16 drawings presented, all four shown with a reduction of 50 %).

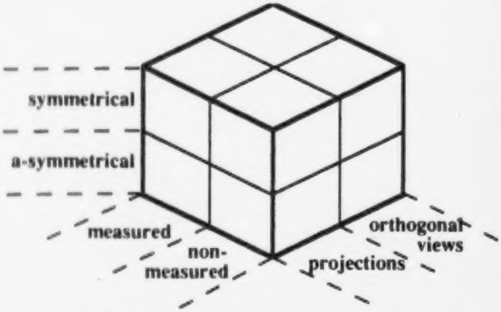
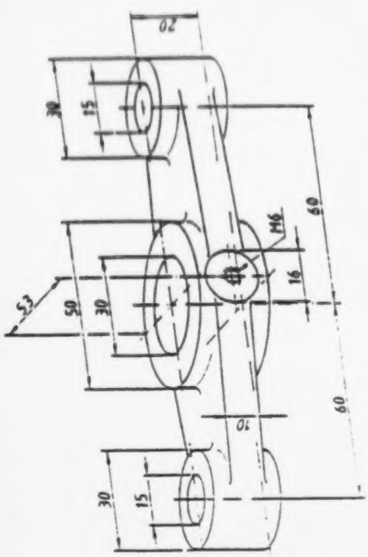


Fig.1:"a-priori"-structure of drawings

**Fig. 2.1**


Ø20
30
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Ø10
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**Fig. 2.2**

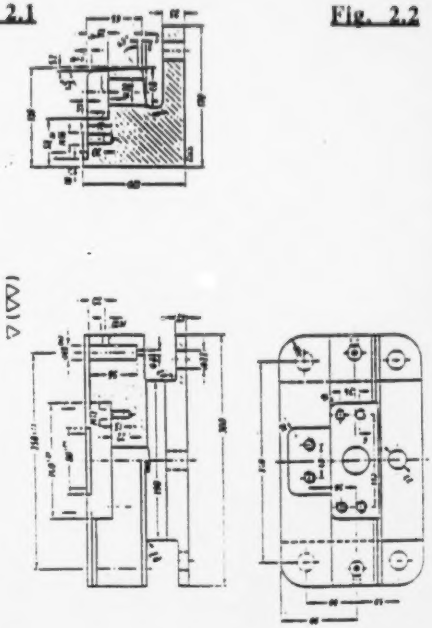


**Fig. 2.3**

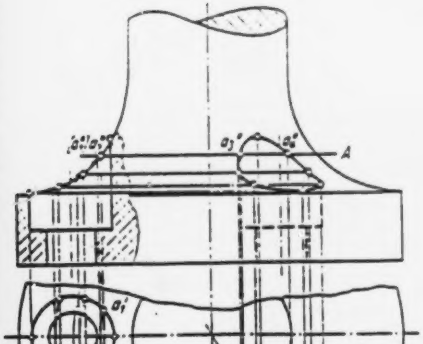
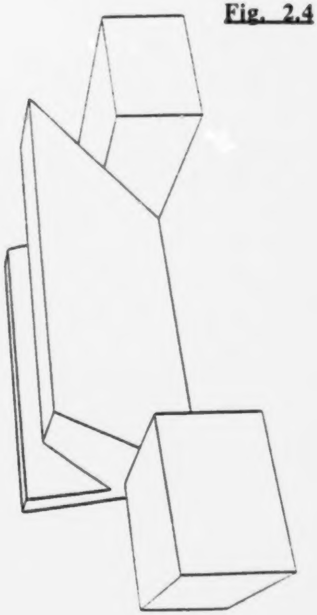


**Fig. 2.4**



## 2 Results

### 2.1 The Interviews

From December 1990 to August 1991, 20 draughts(wo)men were individually interviewed, in most cases near their workplace and during their usual working time. The 6 male and 14 female interviewees had an average age of 29,6 years (with a span from 22 to 40 years of age). The majority (13 of them) had a middle-school education ("Realschule") and all but one had passed an examination as a qualified draughts(wo)man. They had 2 to 18 years of professional experience as draughts(wo)men (with a median of 9,5 years). At the time of the interviews, only 6 of them did not use CAD-techniques at their workplace. Even this minority of traditional technical drawing is in the sample only because, at the end of the interviewing phase of the project, we rejected some potential draughts(wo)men using CAD.

### 2.2 The Classification

When asked to sort and classify the drawings, at an average, the draughts(wo)men formed 5,9 sets of drawings from the 16 drawings (cf. table 1).

Table 1: Number of sets formed

| number of sets | 2 | 4 | 5 | 6 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|---|
| interviews | 2 | 1 | 4 | 7 | 4 | 1 | 1 |

If we come back to the "ex-ante"-classification mentioned in part 1, we can look for symmetry, measurement and type of representation in the "ex-post"-classification of the draughts(wo)men. At a first glance, we find that symmetry is never mentioned in the classification of those interviewed. They never commented their classification relating it to symmetry. If we compare the ex-ante-classification on symmetry and count the ex-post-groups formed by the draughts(wo)men which contain both - symmetrical and a-symmetrical - drawings, nearly 50 % of the groups formed (average: 49,9 %) contain both types of drawings. Spontaneously, symmetry does not matter to the draughts(wo)men interviewed.

For measurements, the analysis of the classifications show a different pattern: 11 draughts(wo)men explicitly use the existence of measurements as a way to classify the drawings. 6 draughts(wo)men sort the drawings in in a way that they have no groups which mix measured and non-measured drawings. On an average, only 23,7 % of the individually established groups have measured and non-measured drawings mixed together. One draughtsman (no. 21) even replicates this dimension by identifying three groups of drawings which contain all drawings with measures (and no other drawings) and which he comments explicitly as being measured. To experienced draughts(wo)men, the existence of measurements seems to be a major aspect of a technical drawing.

Taking the average occurance of drawings individually mixed together in one group, the type of representation is nearly as important as the measurement aspect. On an average, 26,7 % of the individual groups contain perspective drawings as well as systems of orthogonal projections. 7 draughts(wo)men never mix these two types of representations in one group of their classification. One draughtswoman has a special group with 6 of 8 possible perspective drawings (which she characterises as "perspective diagrams"; German: "perspektivische Darstellungen"). As mentioned above, only one central perspective was offered with the 16 drawings (cf. fig. 2.4). 14 of the 20 draughts(wo)men sorted out this drawing as a special group containing only this drawing and two of them commented that drawings of this type are irrelevant to their work. At least for metalwork, the central perspective is an unusual type of representation of spatial configurations. A possible reason for this may be the problems in exactly reading off measures in central perspective drawings.

### 2.3 Signs and Functionalities

For a more detailed analysis, we classified the phrases used to describe the classes formed. In order to learn about the importance of mathematics and the way mathematics is related to the vocational field, the 118 descriptions of classes were classified according to the alternative "related to representations and processes of descriptive geometry" versus "related to production". Descriptions which indicate how detailed and/or complete an object is accounted of (e.g.: "partial drawing", "complete drawing") are classified "related to production". To give some examples, "assembly drawing", "milling" and "partial drawing" are classified as related to production, while "3D-drawing","front view with measures" and "perspective" are classified "related to descriptive geometry". "Partial drawing with measures in isometry" is an example of the category "mixed". Descriptions with no interpretation within the alternative "production" vs "descriptive geometry" (7 descriptions, i.e. 5,9 % of 118, in two cases the descriptions of the central perspective of fig. 2.4) are not taken into account in the following analysis.

Table 2: Classification of descriptions

| Interview | No.of Classes | no inter-pretation | Pro-duction | Mixed | Descr. Geom. |
|---|---|---|---|---|---|
| 4 | 2 | | 2 | | |
| 5 | 7 | | 2 | | 5 |
| 6 | 2 | 1 | 1 | | |
| 7 | 11 | | 3 | 4 | 4 |
| 8 | 7 | | 2 | 2 | 3 |
| 9 | 4 | | 4 | | |
| 10 | 5 | | 2 | 1 | 2 |
| 11 | 6 | 1 | 3 | 1 | 1 |
| 12 | 5 | | 3 | | 2 |
| 13 | 6 | 1 | 5 | | |
| 14 | 7 | | 3 | 1 | 3 |
| 15 | 5 | | 3 | | 2 |
| 16 | 5 | | 4 | 1 | |
| 17 | 6 | 1 | 2 | 2 | 1 |
| 18 | 6 | 1 | 2 | 2 | 1 |
| 19 | 6 | | 2 | 1 | 3 |
| 20 | 9 | | 4 | 2 | 3 |
| 21 | 6 | 1 | 2 | 2 | 1 |
| 22 | 6 | | 3 | 1 | 2 |
| 23 | 7 | 1 | 5 | 1 | |
| **Sum** | **118** | 7 | **57** | **21** | **33** |
| **% of 111** | | | **51,4 %** | **18,9 %** | **29,7 %** |

Table 2 gives our classification of all catchwords which the draughts(wo)men had choosen. If we only take into account the percentages of table 2, it is obvious that descriptions from production are of utmost importance for the draughts(wo)men: About half of the descriptions stem from production or indicate the completeness of the drawing compared to the object to be produced. In 10 interviews, most of the classes are characterized by means of catchwords from production. In all interviews, at least one description related to production is used, while 6 interviews lack any description related only to descriptive geometry. Four of them have only descriptions purely related to production. On the whole, only about 30 % of the descriptions are clearly related to descriptive geometry and only 3 of the interviews mostly use catchwords from descriptive geometry. Aspects of the production process obviously dominate the explicit classification of the drawings.

## 2.4 Mathematics in Technical Drawing

A closer look to the classifications related to mathematics reveals the following pattern: The number of views presented in the drawing is often used to classify when commenting on orthogonal views, while parallel projections with a spatial effect are normally classified as "perspective drawing". Within the perspective drawings, a certain subclassification plays a role - namely that of "isometry" and "dimetric projection" Fig. 2.3 (showing the construction of parts of a front view using information from the top view) was given special reference as the construction of a boring or a "penetration".

As was already stated, the question of measurement plays an important role in the classifications (and is responsible to sort a description under "mixed" in 12 of the 21 cases because "measured" or "non-measured" is part of the "mixed" description). Apart from measurement aspects, the completeness of a drawing is the second aspect to link production and geometry and appears in 11 descriptions classified "mixed". For a technical drawing, the completeness and detailedness of a drawing and the measurements seem to be the aspects tying production and mathematics together.

When asked explicitly about the role mathematics plays in technical drawing, most of the draughts(wo)men hesitate to mentioned any mathematics. Sometimes with a bit of pressing, trigonometric functions are the topic mentioned most frequently (15 interviews), followed by elementary arithmetic (11 interviews). The third most frequently mentioned topics are formulae and calculation of area and volume (both in 5 interviews). The other topics mentioned could already be termed "vocational" mathematics (for instance statics: 4 interviews, calculation of forces: 3 interviews). In the interviews, four of the draughts(wo)men explicitly state that no geometry is needed for technical drawing.

In order to know more about the actual workplace situation, we asked the interviewees to linearly rank qualifications according to their importance for an experienced draughts(wo)men. With nine qualifications offered, the draughts(wo)men interviewed came up with three qualifications rather clearly ranked most important. "Comprehension of the purpose and functioning of the object to be drawn" was ranked in the first place, "comprehension of a sketch or an order" came in second and "comprehension of geometrical relations of the drawing" was ranked in the third place. The result is some sort of confirmation of the utmost importance of production-related aspects of a technical drawing - while mathematics, esp. geometry seems less important.

### 2.5 Use of CAD

In the interviews, it is difficult to find results which stem from the different workplace situations analysed in the interviews. As far as we have analysed, we could not find any substantial difference between those drawing in a traditional situation and those using CAD. A sort of confirmation of this statement is the fact, that none of the draughts(wo)men interviewed came up with symmetry when sorting and classifying the drawings - whereas using this property can be of great help when drawing with CAD-facilities. Those qualifications directly linked with CAD and ranked according to necessity to learn at the workplace (characteristics of the CAD-program / of the computer and system-software) were ranked rather low by the CAD-users in the sample (overall ranks of 6 and 9; we did not ask the "traditional" draughts(wo)men to rank these qualifications). In the interviews, there were some explicit , but more or less isolated and contradictory hints to changes comparing the traditional and the CAD situation: In two interviews, draughts(wo)men stated that the ranking of qualifications important in the workplace should be different according to the drawing situation (CAD or traditional). Three of the draughts(wo)men thought training would be less important with CAD while four of them thought training is even more important when using CAD.

As for mathematics, 8 of the 14 CAD-users explicitly state that CAD facilitates the mathematics involved in technical drawing or even makes mathematics superfluous. Four of the CAD-users think that no geometrical knowledge is necessary for technical drawing - and only CAD-users state that geometry is useless. When analysing the mathematical topics the draughts(wo)men mentioned, we could not find any significant difference between the two sub-samples. In order to analyse possible differences between CAD-users and traditional draughts(wo)men in more detail, we will replicate the study with additional traditional draughts(wo)men this year and hope to get more data for this comparison.

## 3 Conclusion

For experienced draughts(wo)men in metalwork, geometry is part of their professional routine in a way that geometry does not come to their mind when speaking about mathematics. Predominantly, mathematis is arithmetic and calculations. These two aspects of mathematics are necessary to control a most important aspect of technical drawings, namely the size, the measures of the object drawn. Measurements are so important in technical drawing that they are spontaneously used to classify technical drawings. As for the (descriptive) geometry used in technical drawing, systems of orthogonal projections and parallel projections as "perspectives" seem to be as important as the measurement aspect. The central projection - as a second way to describe spatial configurations - is unimportant and classified "useless" by experienced draughts(wo)men. When drawing at the workplace, draughts(wo)men do not spontaneously use symmetry to analyse and classify drawings.

When asked to classify drawings, descriptions related to production processes or the completeness and detailedness of the drawing seem to be more important to the draughts(wo)men than characteristics related to descriptive geometry. Some of the draughts(wo)men classify all of the drawings according to criteria from the production process. Completeness and detailedness of a drawing as well as measurements are the aspects tying production and mathematics together.

Within the interviews, no substantial difference between the use of Computer Aided Design ("CAD") techniques and traditional technical drawing facilities can be shown. Judging from a minority of the interviews, CAD-use in technical drawing seems to integrate further geometrico-mathematical knowledge into computers/CAD-software in a way that geometrical knowledge is devalued. In the sample, we find contradictory statements of CAD-users about the consequences of CAD-use to training and everyday practice in technical drawing.

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# EVOKING PUPILS' INFORMAL KNOWLEDGE ON PERCENTS

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*Summary*

*The leading question behind this paper is how to teach percents. Depending on ones approach of mathematics education this question can be answered in different ways. This contribution reflects the realistic approach. One of its main features is starting from pupils' informal knowledge. A small classroom exploration (8 pupils) in grade 5 served as a framework to investigate whether and to what degree little daily-life stories are suitable to evoke pupils' informal knowledge or not and eventually what suchlike knowledge looks like. The exploration showed a variegated range of knowledge and abilities is present in pupils before teachings starts, which can used as a starting point for shaping further teaching.*

## 1. Introduction

The well-known National Assessment of Educational Achievement (NAEP) in the United States of America has had its counterpart in our country for some years now, namely the Dutch National Assessment Program in Education (the so-called PPON). The results of its first periodical assessment of instructional level were published at the end of 1988. In general the results were not too bad (cf. Wijnstra (ed.), 1988). Only one striking and at the same time discouraging result will be dealt with, that is the outcome of the test on percentages. The results fell considerably short of what could be expected. Even the easiest problems concerning concept and flexible computation were only answered correctly by above-average pupils. It may be obvious — among other things — that the factor instruction is of influence here, that is how percentage was treated in it.

## 2. Approaches in mathematics education and percents

There are different views on both mathematics and its education (Davis & Hersh, 1983; Treffers, 1987). Three approaches and their consequences with respect to percents will be considered here, namely: the traditional or mechanistic approach, the structuralistic and the realistic one.

*Mechanistic*

The key-features of the mechanistic approach are: building up step by step, steering instrumentally, rules-oriented, demonstration followed by imitation, applications, if at all, afterwards. For this approach percentage can be defined as follows:

- one percent (1%) is (means) one part of one hundredth (1/100), and;
- one hundred percent (100%) is (means) the full amount (everything, the whole) (100/100).

After this, 'definition' the pupils are expected to solve problems like: 1/2 = ../100 = ...%; which percentage is 1/20? 8/20? 3/4?; 75% = ../100 = ./4; draw 8 children, 75% of them are girls; 4% of 800 = ...; 300 eggs in a basket and 8% of them are cracked = ... eggs cracked. The given problem sequence is supposed to pave the way to problems like : 3 dollars is 1/4% of ....? In some cases at best a matrix like $\begin{array}{r|l} 3 & \frac{1}{4} \\ \hline & 100 \end{array}$

has been added in order to help the pupils to organise their computations.
The traditional approach proceeds from case to case with increasing complexity. Any connections with (decimal) fractions and ratio and proportion, if at all, are only made at too an advanced level of abstraction and algorithmisation. This approach probably has to do with the discouraging educational progress which is not very encouraging as shown by PPON. The expression 'one to ten pupils succeeds' does not even apply here, at least in The Netherlands.

*Structuralistic*
The key-features of the structuralistic approach are: emphasizing the structures within and of mathematics, insightful reproduction of the system, connecting related topics, making progress within mathematics. In this approach percents dealt with as particular cases of (decimal) fractions. Percentage is thus defined as a fraction having 100 as denominator: 'For p/100 one also says p percent, written as p %.' Also the aspect of function of percentage is considered.
So this approach rather corresponds to the traditional (mechanistic) one, albeit that the links with related subjects are established more carefully.

*Realistic*
The key-features of the realistic approach are: starting from reality as a source both to bring the mathematics forward and to apply it, using models, schemes, symbols and so on to bridge the gap between the concrete and the abstract, intertwining learning strands and giving the pupils room to contribute to the teaching-learning process by means of their constructions and free productions. Compared with the previous approaches the realistic one sheds a different light on the percents. Among other things this means: (Streefland, 1991).

- the firm intertwinement of percentage with ratio and proportion and (decimal) fractions from the very beginning of the teaching/learning process;
- the historical origin of interest ('one to ten') as an area for application of percentage after the decimalsation of money;
- the use of ratio and proportion in general and of percentage in particular as ways of relatively comparing situations having related magnitudes or quantities;
- the consideration of compound interest as an access to (exponential) growth, and last, but not least;
- the consideration of essential features of percents like their asymmetrical behavior depending on the point of view taken and related to this learning to take the correct stand with respect to whole amount and the percentage to be taken from that.

The previous description of the different, possible approaches makes clear in advance, that ones line of approach is decisive for ones ideas on teaching, learning or researching whatever the mathematical topic may be, percents included. In traditional teaching and research the imitation of prescribed procedures and rules will predominate. An important research question here is for instance: Do pupils recognize different types of problems related to the rules and procedures they have been taught? (see for instance Venezky & Bregar, 1988). The structuralistic approach will stress, among other things, the functional character of percents (see for instance Davis, 1988). An important research-question here might be: After the course, are the pupils able to make the structural switch from p % more or less to (1 ± p/100) times.....? Finally the realistic approach, due to the way it was characterised in brief will be interested in the answer to questions like: What informal knowledge do the pupils have and what informal strategies including models do they apply to shape their problem solving. Compared with the mechanistic researcher the aim the realistic one is not to reveal the ability of the pupils to start the fixed programs but to search for starting points for teaching. An extra complicating factor with respect to the informal knowledge of the pupils is that the older the pupils are, the less open-minded they will be. So this will have consequences for the way their informal knowledge will have to be evoked.

## 3. The present study

As said before the present study concerns the revealing of pupils' informal knowledge on percents. Informal knowledge means knowledge acquired outside school or knowledge that cannot be considered as the effect of a teaching-learning process aiming at suchlike knowledge.

The two main questions to be answered in the present study are:

1. Little, problem-like daily life stories on percents are they a suitable means to evoke pupils' informal knowledge on percents?
2. What does that informal knowledge look like?

The daily-life stories used in this study are part of a teaching-learning unit, designed by the authors (Van den Heuvel-Panhuizen & Streefland, 1992) for the 'Math in Context'-project in the USA (Madison-Wisconsin). For answering the questions a classroom exploration was organised in The Netherlands. In this the stories were tested. The group of pupils involved consisted of eight fifth graders. This complete group of fifth graders belonged to a school, which, generally spoken, is of below average level due to socio-economic circumstances (unclassified labourers and so on). This group was fresh with respect to teaching percents. The stories were presented on paper and treated in the following way. First discussing the story, its meaning and its problem in the group and then making a brief note on the question individually, a drawing — if preferred by the pupils — illustrating the question included. Ten stories were treated in this way. Four of them and the concerning results will be discussed in the next section.

## 4. The stories and the results

*a. The soccer training*

This problem concerns the use of percents in an expression on chance (see fig. 1).

> Geert's soccerteam has its training on Wednesday afternoon.
> Maybe the training will be removed to Tuesday.
> "My goodness, what about your practice with the brassband on Tuesday", his mother objects concerned.
> "Don't worry, it's a 95 percent chance the soccer training will stay on Wednesday", Geert replies.

>> Do you think this answer will set Geert's mother at ease?
When she will not have to worry?
In what case something will certainly have to be arranged?

figure 1: the soccer training story

*Results*

* Understanding:

- 5 out of 6 pupils showed the understanding of '95 percent chance'. This appeared from expressions like 'it is almost sure'.
- 2 out of 5 pupils showed an extended level of understanding. Examples of reflect expressions that this are 'it is almost certain; when it is 100 percent it is certain', or,'when it is 40 (percent) for example, it is uncertain'.

* Use of models: 3 out of 5 pupils were able to make a kind of drawing in order to support their way of reasoning. The worksheets showed drawings of different character. Some pupils used circles, other numberlines (see fig. 2).

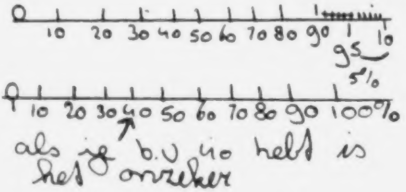


Figure 2: a pupils reaction to the soccer training story ('When it is 40 it is uncertain').

*b. The increasing rent*

This problem concerns a small computation with percents (see fig. 3).

"Dear mom,
I have some bad news! I can't believe it, but next month my house-rent will increase 25 percent! That means the end of my old 200 dollar rent. The invoice for the next month didn't arrive yet, but I think I have to pay something in between two and three hunderd dollar. Please send me - to be on the safe side - one hundred dollar extra this month.
Many kisses, Stan."

>> What will be the reaction of Stan's mother?

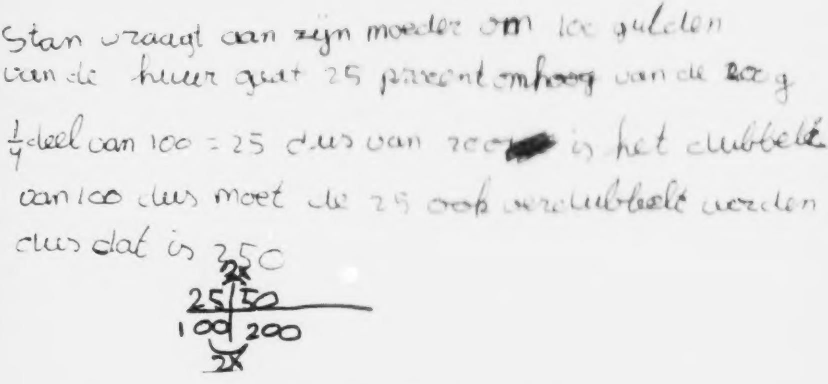
figure 3: the increasing rent story

*Results*

* Understanding:

- 2 out of 8 pupils did 'nothing' with the little story except, for instance, repeating it.
- 2 out of 8 pupils obviously made a calculation. e.g. '..., his rent is *f* 200,—now and he must pay 25% more, then it will be *f* 250,—'.
- 4 out of 8 pupils showed not only the outcome but were able to show their way of calculation.

* Use of models: 2 of the latter 4 pupils use a model or scheme (see fig. 4).



Stan vraagt aan zijn moeder om 100 gulden
van de huur gaat 25 procent omhoog van de 200 g
$\frac{1}{4}$ deel van 100 = 25 dus van 200 is het dubbele
van 100 dus moet de 25 ook verdubbeld worden
dus dat is 250

2x
25
50
100
200
2x

figure 4: a pupils' reaction to the increasing rent story ('1/4 part of 100 = 25, so of 200 it is double of 100, so 25 must also to be doubled, that is 250')

*c. The twin discount*

This problem focuses on the relative character of percents. This means that 'bare percents' cannot be compared in an absolute way (see fig. 5).

> Agnew and Andrew are twins.
> Agnew: "We both got 10 percent discount and yet we didn't get the same amount of discount. I got a ten dollar discount and you got only a five dollar discount. That isn't twins alike, is it?"
> Andrew: "Oh, I can live with it."

>> Is that possible, the same percent of discount and not the same amount of discount yet?

figure 5: the twin discount story

*Results*

* Understanding:

- 2 out of 6 pupils understood the relative character of ten percent. e.g.: 'It does not matter how much the discount is, but where you take it from, I mean from the total price it has cost.'
- 2 out of 6 pupils did not only understand the previous level but were also capable to elaborate its quantitative consequences, e.g.: 'Peter says they both got a ten percent discount. Peter got 10 guilders and John only 5 guilders. Peter has 100 guilders. John only 50, because 5 guilders = half of 10.'

* Use of models: Mathematical models were not used. Two pupils, however, illustrated their way of reasoning by means of contextual drawings of the twins.

### *d. The exam*

This problem also focuses the relative character of percents, connecting relative and absolute data included (see fig. 6).

> A fifty percent score is needed to pass the exam.
> John missed 14 problems.

>> What do you think, can we congratulate John or not?

figure 6: the exam story

*Results*

* Understanding:

- only one of 8 pupils was not able to understand the relative character of the question. 'No..., when you have 14 mistakes your mark is 1 and this is not good'.
- 2 out of 8 pupils understood the relative character of '50 percent of ...'. e.g. 'Actually we do not know, because we do not know how many items the test had'.
- 5 out of 8 pupils were able to deal with both the absolute and the relative data in the problem. e.g. '50% = one half. First you have to know how many items John made. At least 50% of the items must be answered right. If there are more than 28 items or exactly 28 then he succeeded, because you have to do 2 x 14'.

* Use of models: 1 pupil used a strip to illustrate the second level of understanding.

## 5. Conclusions

First some remarks will be made concerning the suitability of the daily-life stories as used to evoke informal knowledge. After that the quality of this knowledge will be considered and its implications for teaching.

*The stories*

In the foregoing only four out of ten stories and their results were dealt with. They showed to be very revealing although it must be admitted that not all the ten stories were equally unveiling.

The success of a problem story depended on its room for construction by the pupils, for instance:

- by means of manipulating the data in order to produce a desired outcome like;
  - * creating success for John doing his exam (d), or;
  - * improving Stan's global calculation of his new rent (b), or;
  - * making a decision on Geert's soccertraining (a).
- by means of applying different strategies of calculation, like for the 25 percent rent increase (b), or the 'equal' discount situation of Andrew and Agnew (c);
- by means of embodying different levels of understanding (see the description of the results).

*Informal knowledge*

As far as the informal knowledge is concerned the following can be said: half of the pupils (4 out of 8 resp. 3 out of 6) were rather successful with all the story-problems.

Aspects of their knowledge were:

- knowing key-percentages like '25 percent of ...' and their relations with simple fractions like '1/4 of ...';
- being able to make simple calculations, with percents and to make supporting drawings.

At a more general level the following aspects played an implicit role:

- being aware of different points of view in situations with percents, and;
- being able to relate relative and absolute numerical descriptions in situations with percents.

The remainder of the pupils only showed vague and implicit notions on percents, albeit that one of them was hardly able to take a relative stand, if not at all.

*Consequences for teaching*

The results show that a part of the pupils already know a lot of percents. So it will not be necessary to start the teaching for them at 'the very beginning' of percents. This advanced starting level, however, may not be used as an alibi for an even more premature algorithmization of percents than it used to be, because this will block the further development of the insights. The results also show the existence of a big difference among the pupils. Helping these weaker pupils is a big challenge for mathematics education. A possibility to meet this problem of differentiation could be achieved by exploiting the ideas, drawings, explanations etc. of the expert-pupils in the class. That will be the next goal of our developmental research.

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# THE "MULTIPLIER EFFECT" AND SIXTH-GRADE STUDENTS' PERFORMANCE ON MULTIPLICATION WORD PROBLEMS WITH UNIT-FRACTION FACTORS

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This study investigated the levels of success achieved by 194 sixth-grade students in solving multiplication word problems with unit-fraction factors. The strategies which students used were also catalogued and examined for patterns. The problems were generated from a model of multiplication problem-types which allowed for systematic manipulation of three problem features which have been shown to affect students' thinking about in similar contexts. It was found that students' performance was affected by the semantic features of the problems and that students used different strategies to solve problems with different semantic features.

## Background

Evidence that the semantic structure of word problems affects the levels of students' performance on the problems and the strategies which they utilize in an attempt to solve them has been accumulated in several domains. These include young children's approaches to addition and subtraction problems (Carpenter and Moser, 1982), older students' solutions to multiplication problems containing decimal factors (Bell, Greer, Grimison, and Mangan, 1989), and some college students' solutions to multiplication problems containing common fractions (Hardiman and Mestre, 1989). Based on previous research it was hypothesized that three problem features might affect students' levels of successful performance and the strategies they would use to solve multiplication word problems. These features are the action performed in the problem (Carpenter and Moser, 1982), the kind of multiplier--whole number or rational number (Bell, et al., 1989) and type of quantity (Hiebert and Tonnesen, 1978; Pothier and Sawada, 1983).

A model of problem types was designed which systematically varied these three problem features (Taber, 1991). The four problem actions were Combining equal parts, Comparing two quantities, finding a Part of a quantity, and Changing the size of a quantity through expansion or contraction. Multipliers were either whole numbers or unit-fractions (1/n), and the quantities in the problems were either discrete objects, such as cupcakes or baseball bats, or measured continuous quantities such as length or volume.

The levels of performance on these problems exhibited by 4th-grade students who had not had instruction in either finding 1/n of a quantity or in multiplying fractions were significantly affected both by the kind of multiplier and by the action performed in the problem, and these students also used different strategies on problems with different features (Taber, 1992). This suggested that students, prior to instruction in these kinds of problems, attend to the semantic structure of the problems. The study reported here investigated whether students who had been instructed in finding 1/n of a quantity and in multiplying by fractions would also attend to the semantic structure of word problems.

## Method

Students from four school districts, each using a different textbook, participated in the study. In order to sample from as varied a population as possible, two urban/suburban districts and two rural/suburban districts were selected. School administrators selected two classes of sixth-graders who were representative of the students in the school for participation in the study, giving 194 subjects. Standardized achievement test scores on mathematics and reading comprehension were available for 172 of the 194 students; the

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mathematics scores ranged from the 7th to the 99th percentiles and the reading comprehension from the 1st to the 99th percentiles.

Students were asked to solve three examples of each of the ten problem-types generated by the model. The 30 problems were presented in a booklet, 5 1/2 inches by 8 1/2 inches, with two problems per page. Examples of the problems are:

> Robert has 32 pieces of candy. Hillary has some cookies. She has 1/4 as many cookies as Robert has pieces of candy. How many cookies does Hillary have? (Compare action, Fraction multiplier, Discrete quantity)
>
> Josh's job is to fill water balloons with water. Each balloon holds 1/6 of a quart of water. How many quarts of water will it take to fill 18 balloons? (Combine action, Whole multiplier, Measured Continuous quantity)
>
> It is 30 miles from Marie's house to the shopping center. If Marie rides her bike 1/6 of the way to the shopping center, how many miles has she ridden? (Part action, Fraction multiplier, Measured Continuous quantity)
>
> A rubber band is stretched to 30 times its original length. If it was 1/3 inch long before, how long is it now? (Change action, Whole multiplier, Measured Continuous quantity)

Students' responses yielded two kinds of information. First, students were given a performance score of correct or incorrect for their answer to each of the 30 problems on the written test. The strategies which students used in solving the problems were coded separately, so that students' patterns of strategy use could be compared with the problem features.

About five students from each of the eight classes, 43 in all, were also interviewed individually at a later time. They were asked to solve four problems from the test, and then asked to explain what they had done and why. These interviews verified the strategy categories (agreement was achieved on more than 99 % of the items) and provided additional information about students' thinking.

## Results

### Performance Levels

Analyses of Variance were performed to check for differences in levels of performance on the problems (see Taber, 1991, for details). The results for the effect of type of quantity in the problem were inconclusive. There was a significant effect of quantity in just one of the three analyses which examined the effect of quantity; students performed better on problems with discrete quantities when their performance on the Combine and Compare problems was examined.

The performance of 4th-graders on all problems had been shown to be affected by the type of multiplier in the problem; they performed significantly better on problems with fraction multipliers (Taber, 1992) than on problems with whole-number multipliers. Although the 6th-graders also had higher scores on problems with fraction multipliers, the differences in performance were not significant.

Differences in 6th-graders' performance were significantly different for each of the four problem actions, $F$ (579, 3) = 37.08, $p < .001$. As shown in Table 1, students performed best on Part problems, followed by Combine, Compare, and Change problems.

This suggests that the students' performance was affected by the action of the problems; that is, that they attended to the semantic structure of the problem.

Table 1

Sixth-Graders' Mean Scores and Standard Deviations on Problems with Different Actions. $F$ (579, 3) = 37.08, $p < .001$)

| Action | Score | S.D. |
|---|---|---|
| Part | 4.8 | 2.0 |
| Combine | 4.6 | 1.9 |
| Compare | 4.3[a] | 2.0 |
| Change | 4.0 | 2.0 |

[a] There were 12 Compare problems, but 6 Part, 6 Combine, and 6 Change problems; the number of correct responses to Compare problems was divided by 2 for comparison purposes.

## Students' Strategies

Further evidence that students attend to the semantic structure of the problem was supplied by the strategies which the students used in order to solve the problems. All 30 of the problems could be solved by simply multiplying the whole number and the unit-fraction factors, or by dividing the whole number by the denominator of the unit-fraction. There were 4907 instances of codable strategies which were coded into 12 strategy categories. The standard algorithm for multiplication of fractions **1/n X m/1** accounted for 45 % of the strategies used on all problems. Division, dividing the whole number by the denominator of the unit fraction accounted for 29 % of the coded strategies. The remaining 26% of the instances were spread over the remaining ten strategies.

The kind of quantity in the problem did not seem to have an effect on the kinds of strategies used by the students. The picture is different, however, for the type of multiplier and the problem action.

Half the 30 problems had whole number multipliers and half had fraction multipliers. Problems with whole number multipliers attracted 62 % of all instances of the standard algorithm, while problems with fraction multipliers attracted just 38 % of this strategy. On the other hand, problems with whole number multipliers attracted just 29 % of the division strategies, while problems with fraction multipliers accounted for 71 % of the instances of this strategy choice. This suggests that although there were not significant differences in the students' performance on problems with whole number and fraction multipliers, some students may have thought that the problems required different solution strategies. Some of the students who were interviewed claimed that a problem with a whole number multiplier had to be solved by multiplication while a problem with a fraction multiplier had to be solved by division because they needed to find "a part of" the starting quantity.

The distribution of students' strategies across the problem actions was also examined. When the multiplier was a fraction, the instances of strategy use for both the algorithm and division strategies on the three kinds of problems, Compare, Part, and Change, were close to the expected proportion, 40-40-20. When the multiplier was a whole number, a different picture emerges. If problem action made no difference to students' choice of strategy, we would expect Combine and Compare problems each to account for 40 % of the instances of the strategy, while Change would account for 20 %. In fact, Combine problems accounted for 33 % of the instances of the algorithm while Compare problems accounted for 46 % and Change for 21 %. The departure from the expected distribution was greater for the division strategy with 67 % of the instances of this strategy occurring on Combine problems, 21 % on Compare problems and 13 % on the Change problems. These results suggest that students were less sensitive to differences in the problem action when the multiplier was a fraction than they were when the multiplier was a whole number.

Interviewed students who used division on Combine problems explained that they had done so in order to find out how many whole units would result from combining the m parts. This is how Chris explained his solution to the pizza problem, 3) 27 :

> Pizza from Dumbo's pizza comes cut in 1/3s (thirds). How many pizzas will be needed to give 27 children one slice each of pizza?
>
> C: If you cut it into thirds and there are 27 children, then I tried to figure out how many times 3 goes into 27. One third means a third of a whole.
>
> Interviewer: Could you do it another way?
> C: Would it be 3 times 3?

## Discussion

The results of this study suggest that the semantic features of multiplication problems with unit-fraction factors affect both students' performance and the strategies they use. Although significant differences in levels of performance occurred only on problems with different problem actions, the patterns of strategy choice suggest that there were differences in how students thought about problems with whole number multipliers and fraction multipliers. The patterns of strategy use across problem actions suggest that students tended to use the same kinds of strategies on all problems with fraction multipliers, regardless of the problem action, but this was not true of the three problem actions with whole number multipliers.

Although these students had been instructed in multiplication of fractions during two or more years of school, many of them attended to the semantic differences in the problems, particularly problem action and type of multiplier, even though all the problems could be represented by the same kind of mathematical expression. That is, instruction in the algorithm for multiplication of fractions does not seem to level out the semantic differences in multiplication problems of this kind.

This study suggests that several problem features may combine with instruction to affect students' thinking about multiplicative problems. At this point we do not have a theoretical model which predicts how students will perform on multiplicative word problems with rational numbers.

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# TELLING STORIES ABOUT PLANT GROWTH: FOURTH GRADE STUDENTS INTERPRET GRAPHS

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*Through the development and evaluation of curricular activities for fourth grade, we analyzed how children produce and interpret graphs describing changes over time. The distinction between two approaches to understanding graphs, that we call pointwise and variational, became salient and relevant for the teaching and learning of the graphical representation of change. In this paper we describe these approaches and how they were used by children.*

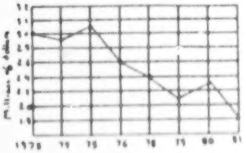
As part of the TERC's Number, Data, and Space project, we developed a unit for fourth grade about change over time. The first section of the unit poses situations of discrete change (people in and out of the children's homes); the second section introduces situations of continuous change (height of plants). In a previous paper ( Tierney & Nemirovsky, 1991), we discussed the graphs children spontaneously developed of the population of their homes. In this paper, we discuss the cartesian graphs children developed representing plant height vs time.

The curriculum addresses two main areas:

1. The interplay between a sequence of accumulated values and a sequence of changes (e.g. given a list of daily heights of a plant, what can you tell about its pattern of growth?)
2. The contrast between graphing measured data point by point (attention to the plant height each day) and graphing variation of qualitative data without specifying magnitudes (telling an overall story)

Creation of cartesian graphs of continuous variation has usually been postponed in the United States until secondary school, with some interpretation of graphs beginning in sixth grade. Typically, the graphs students interpret provide a scale and ask for identification of specific data points. The Boston schools mathematics curriculum (1987) recommends that, in eighth grade, children be able to read points on line graphs that have axes with only some points labeled (figure 1a). The focus is primarily on scale and on particular data points, but doesn't ask children to look at the graph as a whole.

6. This is a graph of a city's expenditures over a period of years.

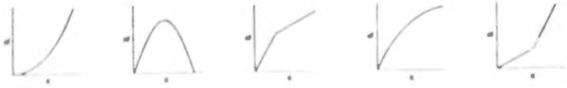


In what year did the city spend $7,100,000? ____

What was the drop from the highest year's expenditure to the lowest?

**1a. Boston**

**Mary and her brother John leave home together to walk to school. Mary thinks they are going to be late, so she starts out running, then tires and walks the rest of the way. John starts out walking and starts to run as he nears the school building. They arrive at the same time. The graphs in figure 8.4 show the distance from their home on the vertical axis and time on the horizontal axis. Which graph best represents Mary's trip? John's trip?**



**1b. NCTM**

In recently developed graphing curriculums, the focus has shifted away from reading and making graphs pointwise towards a more holistic approach in which the shape of the graph as a whole is considered. Curriculums have been developed in the Netherlands and Great Britain for middle and secondary school (Krabbendam, H. (1982); Swan, 1982, 1985) and adapted in Australia for primary school ( Lovitt & Clarke, 1988). In the United States, the NCTM Standards (1989) suggest that middle school students be able to associate stories of changing situations with graph shapes. (figure 1b).

The children's work we discuss in this paper suggests some advantages of this trend toward making graph shapes central to the curriculum together with pointwise graphing.

## Overview

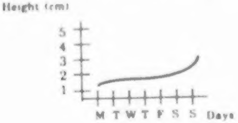
This study took place in a fourth grade class of 20 students in a city public school that draws mostly from lower to middle class families. The work we will report on took place over a two week period in 10 class sessions of approximately one hour each. Our data include a video tape of one session (mystery graphs), detailed notes of each class by one or two observers, and the children's work on two written assessments.

## The activities

Each pair of students planted bean seeds and kept track of one plant's growth. On a Thursday, Friday, and Monday, they measured and recorded plant heights on a chart. On Monday, they recorded these measurements on a graph and, in a different color, made a line to show their prediction of the plant's growth over the next week. They continued to measure the heights each school day for two weeks, and record them on their chart and graph. Activities in connection with the plant growth work included: telling plant growth stories for their own graphs and for curves we provided on a worksheet; using their graphs to compare patterns of growth; describing the pattern of growth of plants from looking at lists of daily changes in height; making "mystery graphs"(Tierney & Nemirovsky, 1992) of the population of people doing familiar things over a 24 hour period.

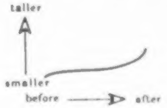
## Issues

Through our analysis of this experience we recognized the centrality of the distinction illustrated above in the Boston and NCTM examples (figure 1) between the two common approaches to understanding cartesian graphs: interpreting the graph as displaying correspondences between coordinates versus interpreting the graph as telling a story of "ups" and "downs" over time[1]. We will call the first (figure 2a) "pointwise" (after Monk 1989) and the second (figure 2b) "variational/qualitative". It became apparent to us that these involve different types of knowledge, expectations, and cues. We will describe how each approach was manifested by students and the difficulties, as well as the learning opportunities that students experienced as they worked with them.



Graph a

**2a.pointwise**



Graph b

**2b. variational/qualitative**

### 1. Pointwise approach toward graph construction

In the pointwise framework, a graph is a display of correspondences between coordinates. Issues of scale and units are critical in order to locate points in the graph.

In general, the students tended to include and to allow space for only the data they had measured rather than by a systematic plan. We saw this tendency earlier in the work on graphing population of their homes, and we refer to it as "information-driven", as opposed to "system-driven (Tierney and Nemirovsky,

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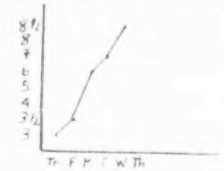
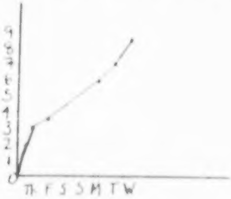
[1] We did not include a third approach to graph interpretation and construction commonly reported in the literature (e.g. Hart, 1981), called graph as a picture. It did not happen in our experience. The graph as a picture approach is typical in representing situations of motion, such that the curve can be seen as a trajectory. This is not the case with plant growth, or with any situation of amount-change, like people in a room or variation of temperature .

1991). In graphing plants, some manifestations were (figure 3a): a) all pairs of students in our study began the vertical scale with the first measurement of their plant ; b) since some of their data was in whole centimeters and some in half centimeters, students tended to mix their scales along the vertical axis, sometimes one square to a whole centimeter and sometimes one square to a half centimeter, creating a non-uniform height scale; c) since the children had no measurements for the weekend days, most simply wrote Monday following Friday as closely along the horizontal axis as Friday followed Thursday, creating a non-uniform time scale. When the teacher suggested making the vertical axis more systematic by starting at zero and going by 1cm.intervals, several students drew a segment linking the zero point with their first measured point and took it to represent a day's growth (figure 3b).



**3a initial graphs**



**3b corrected vertical scale**

Although students ignored consistency of scale when they made their own graphs, their way of interpreting their graphs took a consistent scale for granted. They interpreted differences in gradient as meaningful without being sure that the scale was the same (that two regions of the same width represent equal intervals of time). When the teacher asked on what days did their plant grow fastest, students mentioned segments with the highest gradient to represent fastest plant growth--the weekend and the initial segment tying the first measurement to zero--disregarding lack of regularity along the horizontal axis.

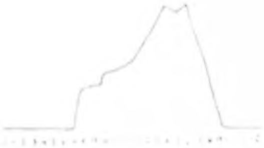
**2. Variational/qualitative approach** In the variational/qualitative approach the graph is perceived as telling the story of the way in which something varies over time, the story being an account of increases and decreases at different rates. The emphasis is on the relative height and steepness (or gradient) of a graphical shape rather than on issues of range, units, or point-values. This approach is variational because what stands out is how high and steep the shape is over a certain region, relative to how high and steep it is before or after that region; it is

**qualitative because the variables are identified categorically, that is, by type and direction of increase, rather than by a unit-based quantification.**

## 2.1 Mystery graphs

Students generated graphs which were qualitative/variational in nature to show the number of people in different places in their town over a 24 hour period. We provided for this a sheet on which we wrote hours along the horizontal axis and left the vertical axis empty.

Students attempted to include information that would hold true for all people, but needed to use their own limited experience as a guide. Thus, their graphs were a combination of both generalizations as well as very specific data that pertained to their own personal knowledge. Figure 4 is an example of a mystery graph made by two boys and the story they tell about it. We observed that students made these graphs by approximating heights on the graph to represent number of people present at key hours. When they reported, they accounted for the key points and for the slope of the segments that join them.



It's people at a playground. Not a lot of people at a playground at night and they start coming at 7 when they're going to school and it keeps going up because little kids like two year olds and three year olds come, and older kids come after school and then at night it goes down so at 9:00 there is no one.

4. mystery graph

## 2.2 Plant growth

Students had generated graphs at the beginning of this work to show their prediction of plant growth. The predictions were mostly linear. One pair made a step-wise graph, showing alternation of growth one day, none the next after they decided that a continuous line would make the plant "too tall".

After doing the mystery graph activity, they wrote stories associated with curves showing plant growth. For the graph in figure 5, one child wrote "The plant started slow then went fast then all of a sudden it dropped (it most likely fell off) then it started up slow then it went very fast." Another wrote: "It started slow,

**faster, fast, dive, stop, slow, and faster." These stories describe a succession of discrete intervals on the graph.** The second description distinguishes between accelerated speed as"faster" and steady speed as"fast".

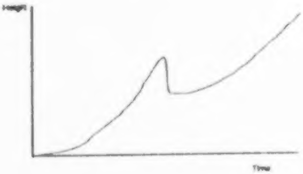
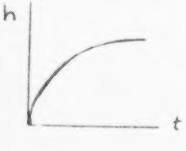


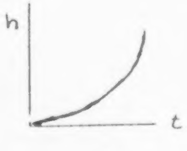
**figure 5**

**For the construction of these verbal descriptions, the recognition of the basic types of variation (Nemirovsky, 1992) and their associated graphical shapes play a critical role. On a worksheet of types of variation, the fourth grade students picked out places showing the plant growing fast or slowly or speeding up or slowing down.**

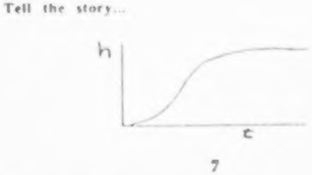
There were only two types of curves on the worksheet that proved difficult for students to interpret. The first described an event that happened very fast at first and then slowed down [figure 6a]. Seven out of 18 students either described this curve as slow, then fast or as all fast. On the other hand, all 18 of them were able to interpret a curve that described an event that began slowly and then increased in speed as "slow" then "fast"or "faster" [figure 6b ]



**6a**



**6b**

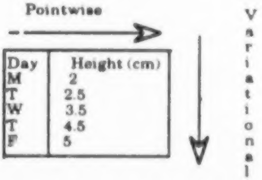


**7**

The other type of curve that students did not describe accurately was one that showed no change in status for a period of time (figure 7). The children had not yet experienced the leveling off of their graphs as all their plants were still growing. Instead of interpreting this as stopping, 13 out of 18 students described it as "steady". "Steady" was used to mean either slow growth ( e.g. "It grew steady. Then it grew fast. Then it grew steady again.") or any linear growth including a period of no growth ("It started slow. Then it grew fast. Then it grew really slow, then it grew steady").

### 4. Graphs and number tables

There is a parallel between the pointwise and the qualitative approaches to production and interpretation of graphs on the one hand and to number tables on the other. This may be illustrated with the following table:



| Day | Height (cm) |
|---|---|
| M | 2 |
| T | 2.5 |
| W | 3.5 |
| T | 4.5 |
| F | 5 |

From a pointwise approach the table is a set of horizontal correspondences. Each correspondence can be translated into a graphical point. From a variational approach the table is read as a sequence of heights whose successive differences show whether the plant grows more or less as time passes. The first column (Days) becomes just an indicator of order showing what comes after what.

The children told stories from lists of numbers that were similar to the stories they told from graph shapes. Examples of this were the descriptions students provided for these list of changes:
"slow, fast, faster" : 1/2, 2, 2, 1, 2 1/2, 4
"steady, very steady": 1/2, 1, 1, 1, 1, 1/2
They overlooked the messiness of these lists to see the general picture as they did in graph interpretation.

### Conclusions

From a pedagogical point of view we are interested in fostering children's abilities to move flexibly and consistently between variational/qualitative and pointwise approaches. This is because a meaningful use of graphs involves a dialectic between both approaches. The variational perspective enables the identification of remarkable points and the pointwise framework allows one to establish the landmarks for a global variation.

It is clear from our experience that fourth grade students can interpret and construct graphs using both approaches. It is natural to them to interpret gradient and height qualitatively by comparison of one portion of the graph with another. However, in both cases their inclination is to be faithful to the information that they have without preserving systematicity. Thus, when they make their own graphs, they tend to ignore consistency of scale; it is not a relevant element of the graph for them. When they interpret graphs, on the other hand, they take a consistent scale for granted; again, because it is not a relevant element for them, they do not notice whether it is present or absent.

They take their personal knowledge to be the general case when both producing and interpreting graphs. Thus, they were less likely to recognize shapes of plant growth graphs that they had not experienced, and their construction and interpretations of the mystery graphs were based largely on their own families' patterns of behavior.

In accordance with Krabbendam (1982) , we think that children's concern about systematicity has to emerge from the need to tell more accurate and consistent stories. We expect that shifts between pointwise graphing and qualitative graphing and between making and interpreting graphs will lead to the perception of the importance of systematicity and the refinement of the story-construction process to describe change over time.

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# OVERGENERALIZATION BETWEEN SCIENCE AND MATHEMATICS: THE CASE OF SUCCESSIVE DIVISION PROBLEMS

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*Recent studies have shown that students tend to produce similar responses to subdivision problems which are essentially different but are figurally similar. This study presents 7th to 12th grade students with three subdivision problems.. The first problem concerned an ideal, geometrical line segment, while the other two dealt with material substances (copper wire and water). All three problems involved the same process: successive division. Two of the problems (line segment and copper wire) were also figurally similar. Our data indicate that the similarity in the process had a profound effect on students' responses.*

## Introduction

Previous studies in science and mathematics education have shown that when asked to consider two problems, one of which concerning the successive division of a line segment, and the other dealing with the successive division of a copper wire, a substantial number of seventh and eighth grade level students state that these two processes will come to an end while the tenth and twelfth grade students, majors in mathematics, state both processes were endless (Fischbein, Tirosh, Stavy, & Oster, 1990: Stavy & Tirosh, 1991). One possible conclusion that may be drawn from these findings is that many students in these grade levels tend to perceive mathematical and physical objects as behaving similarly in regard to subdivision processes. This conclusion is partly supported by well known scholars: Piaget & Inhelder (1967/1948) and Hilbert (1964/1925). They argued that humans tend to perceive mathematical and physical objects as identical in respect to continuity. However, while Piaget and Inhelder suggest that our intuitive perception of external objects is discontinuous, Hilbert declares that our naive conception of matter is that of continuity. Piaget and Inhelder described the development of a conception of division processes as endless only with regard to mathematical objects. They argued that the perception of subdivision processes as endless presupposes a discrimination between physical and mathematical objects.

The two problems we presented in our previous studies resembled each other in two aspects: They related to the same process (successive division) and were figurally resembled (line segment and copper wire). One could attribute the identity in students' responses to the figural resemblance of the two problems. This interpretation is in line with findings of recent research which reported that students tend to rely on external features of problems (Carey, 1985; Chi, Feltovich & Glazer, 1981; Chi, Glaser & Rees, 1982; Larkin, 1983; Larkin, & Rainhard, 1984; Hardiman, Dufresne & Mestre, 1989). In the present study an additional problem (a successive division of water) was added to the two previous ones. This will allow us to find out whether students conceive successive division processes of mathematical and physical objects as identical regardless of their figural similarity and content.

## Method

### Subjects

Two-hundred students from upper middle-class students participated in this study. Fifty students, each from the seventh, eighth, tenth and twelfth grade levels in the same school were tested. The tenth and twelfth grade students were studying in classes that majored in mathematics.

At the time of the study, the seventh grade students had finished a unit on the particulate nature of matter. They received no instruction concerning geometry or infinite processes. The eighth grade students had started studying Euclidean geometry. In that year their science instruction had related to elements, compounds, and the periodic table. The tenth grade students had studied Euclidean geometry (i.e., undefined and defined terms, axioms, postulates, definitions, theorems and proofs), but had received no additional instruction in science concerning the structure of matter. The twelfth grade students took an introductory course in calculus in which they dealt with infinite series, limits and integrals. They also studied chemistry as a minor subject (stoichiometry, the structure of the atom.

### The Problems

1. The line segment problem: "Consider a line segment AB. Divide it into two equal parts. Divide one half into two equal parts. Continue dividing in the same way. Will this process come to an end? Explain your answer."

2. The copper wire problem: "Consider a piece of copper wire. Divide it into two equal parts. Divide one half into two equal parts. Continue dividing in the same way. Will this process come to an end? Explain your answer."

3. The water problem: "Consider a cup full of water. Pour out half of it. Again, pour out half of the water left in the cup. Continue pouring out half of the remaining water in the same way. Will this process come to an end? Explain your answer."

The correct responses are that in the case of the line segment the halving process is endless, whereas in the case of the copper wire and water the halving process stops when the atomic (molecular) level is reached.

### Procedure

The three problems were administered to all students during one class period (about 45 minutes). In order to counterbalance the effect of the order of presentation of the problems, half of the students in each grade level received the mathematics problem first while the other half received the science problems first. The mathematics problem was given on one sheet of paper along with other, irrelevant, mathematics problems. The science problems were given on another sheet of paper along with other, irrelevant, science problems. Each sheet was collected after the students had responded. After collecting the data, ten students from each grade level were interviewed to better understand their responses to the three questions.

## Results

### Responses to the three problems

Two types of responses were given to each of the problems (see Table 1). The first response was that the process is endless. This response is regarded as an infinite response. The second response was that the process ends. This response will be referred to as a finite response. In each of these problems, the frequency of the infinite responses significantly increased with grade level while that of the finite responses significantly decreased ($x^2 = 27.82$, df=3, $p < .001$ in the line segment problem; $x^2=7.91$, df=3, $p<.047$ in the copper wire problem; $x^2=11.46$, df=3, $p<.009$ in the water problem).

Table 1: Responses to the line segment, copper wire and water problems

| | Line segment | | | | Copper wire | | | | Water | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Grade | | | | Grade | | | | Grade | | | |
| | 7 | 8 | 10 | 12 | 7 | 8 | 10 | 12 | 7 | 8 | 10 | 12 |
| **The process is endless - Total** | 30* | 50* | 76* | 72* | 26 | 30 | 48 | 46 | 22 | 12 | 38 | 36 |
| **The process will end- Total** | 70 | 50 | 24 | 28 | 74* | 70* | 52* | 54* | 78* | 88* | 62* | 64* |

*correct responses

Three types of justifications were given for the "endless" response. The most dominant in all grade levels was: "One can always divide by two." The percentage of students who used this justification, which refers to the dynamic aspect of the process, increased with grade level. The second most common justification to the judgment that the process never ends was: "There is an infinite number of points/particles. And therefore the process will never end." This type of justification was offered only for the line segment problem, mainly by the upper grade students. The third, least frequent, justification was: "We shall reach a point/particle but it too, can also be divided." This justification was offered only to the line segment and the copper wire problems. For instance, "At a certain point we will reach an unobservable, very small particle, but it will still be possible to break it down."

Students used three types of justifications for their response that the successive division processes would come to an end. A substantial number of seventh and eighth grade students provided the following concrete, finite response: "We will not be able to divide anymore because it will become extremely small." The second common justification was that "there is a finite number of points/particles." Few students in this category added that the segment is bounded or similarly, that there is a finite quantity of matter in the copper wire/water. The third type of justification was: "We shall not be able to divide anymore as we shall reach a point/particle."

The data we have presented thus far show that the same answers and even the same justifications were given by many students to all three problems. However, these data do not provide systematic information about the consistency of responses at the individual level. Such insight was obtained by examining students' response patterns.

## Patterns of responses to the three problems

The eight possible response patterns are presented in Table 2. Students who gave the same response to all problems (either finite or infinite) are included in the concordant patterns. Those who gave different answers to the problems are included in the discordant patterns.

Table 2: Response patterns to the line segment, copper wire and water problems

| | Grade | | | |
|---|---|---|---|---|
| Segment Wire Water | 7 | 8 | 10 | 12 |
| **I Concordant patterns - Total** | **74** | **40** | **44** | **48** |
| a. finite finite finite | 60 | 38 | 18 | 18 |
| b. infinite infinite infinite | 14 | 2 | 26 | 30 |
| **II Discordant patterns - Total** | **26** | **60** | **56** | **52** |
| a. Correct pattern | | | | |
| infinite finite finite | 8 | 24 | 24 | 32 |
| b. Reversed pattern | | | | |
| finite infinite infinite | 2 | 2 | 2 | 2 |
| c. Figural patterns | | | | |
| finite finite infinite | 4 | 4 | 2 | 2 |
| infinite infinite finite | 6 | 20 | 18 | 8 |
| d. Non-figural patterns | | | | |
| finite infinite finite | 4 | 6 | 2 | 6 |
| infinite finite infinite | 2 | 4 | 8 | 2 |

Concordant Patterns. The concordant patterns emerge when the same response is provided to all problems. A common aspect to all three problems is the process of successive division. Thus, the percentage of students included under the two concordant patterns gives an indication of the effect of the similarity in the process on students' responses to the problems. Table 2 shows that a substantial number of students at all grade levels responded identically to all three problems: either that the process of successive division would come to an end (Ia on Table 2) or that the process was endless (Ib on Table 2).

The frequency of the finite-concordant pattern significantly decreased from grade 7 to 10 ($x^2 = 18.6$, df=2, $p<.001$) while that of the infinite-concordant pattern significantly increased from grades 8 to 12 ($x^2 = 11.96$, df=2, $p<.002$). The decrease in the finite-concordant response pattern indicates that students in the upper grade levels gave up their initial, adequate, finite response to the copper wire and to the water problems. The increase in the infinite concordant pattern indicates that the older students who were majoring in mathematics overgeneralized the infinite response also to the material tasks.

It is interesting to note that almost all the students who showed a concordant response pattern also offered the same justification for all problems. Most students who evinced the finite-concordant pattern either said that "we will not be able to divide anymore because it will become extremely small", or that "we will not be able to divide anymore because we will reach a point/particle." Students who showed the infinite-concordant pattern used the "you can always divide by two" justification. Some students explicitly referred in their answers to the apparent concordance of these problems. For instance, Oded (grade 11) explained that: "The process (of dividing a copper wire and water) is endless, and this is exactly as with the segment".

Discordant patterns. The discordant patterns can be divided into four groups.

The correct pattern (IIa). This response pattern was the most frequent among the discordant patterns. Although the frequency of this correct response significantly increased with grade level ($x^2 = 12.24$, df=3, $p<.001$), even in the highest grade no more than 32% of the students showed it. Students included here were able to discriminate between the problems according to their theoretical frameworks. They were capable of ignoring the similar surface features of the problems.

The reversed pattern (IIb). A finite response to the line segment problem and an infinite response to the copper wire and the water problems. This pattern was produced by very few students at each grade level.

"Figural" patterns (IIc). Students who gave the same response to the line segment and the copper wire problems but a different response to the water problem are included in the two figural patterns. The percentage of these students gives an indication of the effect of the figural similarity (segment and wire) on students' responses.

"Non-figural" patterns (IId). Students who gave the same response to the line segment and the water problems and a different response to the copper wire problem are included in the two nonfigural patterns. The frequency of these patterns at each grade level did not exceed 10% .

## Discussion

This study is part of a project which is concerned with factors affecting secondary school students' choice of models for solving problems in mathematics and science. In a previous study, which dealt with the successive division of a line segment and that of a copper wire, it was suggested that the figural similarity between the two problems was the dominant factor triggering similar responses (Fischbein, 1987; Fischbein, Tirosh, Stavy , & Oster, 1991; Stavy & Tirosh, 1991). However, the present study indicates that many students gave the same response to the water problem, although this problem is figurally different from the line segment and the copper wire ones. Our results indicate that a substantial number of students (at least 40%) in all grade levels showed a concordant pattern of responses, i.e., gave the same responses to all three problems.

A closer examination of our data reveals that while the majority of the concordant patterns in the seventh and eighth grades were finite, the majority of the concordant patterns in the tenth and twelfth graders were infinite. The responses of the younger students are in line with Piaget's observation related to the first stages of the development of the idea of continuity, namely, They regard mathematical and physical successive division processes as finite. Piaget argued that in the fourth stage of the development of continuity, children can differentiate between successive division of mathematical objects and that of physical objects. Our findings show that indeed, about a quarter of the upper grades students could differentiate between the mathematical and the physical successive division tasks and correctly answered all of them. Another pattern of responses was observed in the upper grades. About a quarter of the tenth and twelfth grade students exhibited an infinite concordant pattern of responses to all three problems. This may suggest that children who have become aware that subdivision may continue endlessly, view all these processes as endless regardless of the constraints imposed by the nature of the objects (mathematical objects vs. physical objects). This perception that mathematical as well as physical objects are "ultimately divisible" was suggested by Hilbert in 1925. Hilbert claimed that this view is the first naive impression of natural events. In contrast with Hilbert, our data, as well as others' relating to the development of the concept of infinity (Fischbein, Tirosh, & Melamed 1981; Martin, & Wheeler, 1987; Sierpinska, 1987) suggest that this view is not a naive one. Such view develops with age and/or instruction.

Many studies have shown that students tend to generalize knowledge they acquire within the boundaries of the same content area (Clement, 1987; Stavy, 1991). The present study shows that students may also overgeneralize knowledge between content areas. It seems that students create a scheme in which processes of successive division are grouped together. Such a scheme may prevent students from discriminating between successive division problems according to different theoretical frameworks. Thus, although the process of generalization is essential to learning, it is of crucial importance to help students distinguish between problems according to the theoretical frameworks in which they are embedded. This demanding task can be done by exposing students to examples and non-examples of subdivision processes in mathematics and science and by encouraging them to examine the validity of the schemes they use when solving problems in each framework.

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# EXPLORING UNDERSTANDING AND ITS RELATIONSHIP WITH TEACHING: VARIATION AND MOVEMENT

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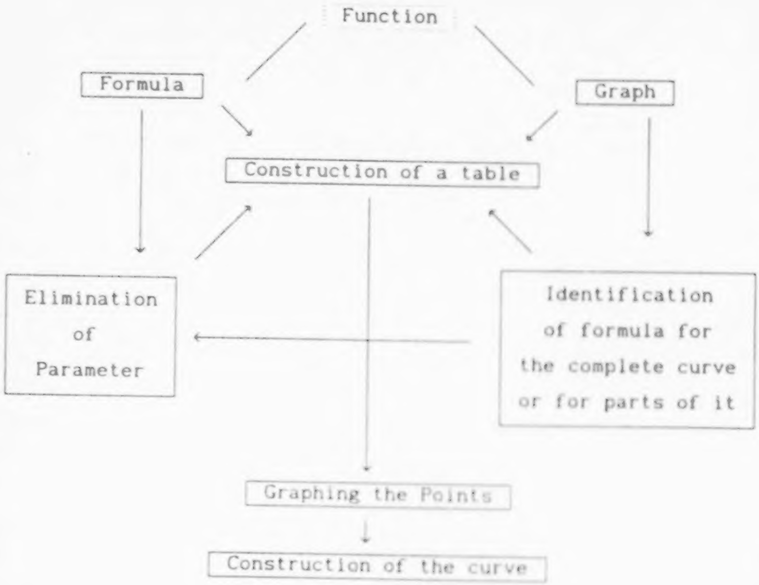
*The concepts of variable and variation are explored by analyzing the answers of students when they deal with curves expressed in parametric equations. Some teaching strategies that result from this analysis are outlined stressing their relationship with concepts in Calculus.*

## 1.- Antecedents and Problem Presentation

In [Cantoral & Trigueros, 1991] we reported about the students difficulties with the understanding and graphing of curves expressed in terms of parametric equations. We said then that those difficulties could come either from situations related to representation in different contexts: numerical, algebraic or geometrical, or with the transference of information from one of these forms of representation to one of the others or with the understanding of the concepts of variable and variation.

When we observed the behavior of the students in the classroom and when we later explored it by means of clinical interviews with some of them, we recognized and analyzed the strategies they follow when dealing with problems related with parametric representation. Those strategies can be schematized, grosso modo, as shown in the next figure.

The numeric strategy is used independently of the context. The algebraic strategy is used as a resource to support the numerical strategy and as an independent tool, the geometric strategy, by contrast, is not used in a spontaneous way and it seems to be closely related to the idea of movement.


Function
Formula
Graph
Construction of a table
Elimination of Parameter
Identification of formula for the complete curve or for parts of it
Graphing the Points
Construction of the curve

That research pointed out that the understanding of the concept of variable, the interrelation between different variables and the graphing of curves expressed in parametric form are related with the idea of movement, in particular with the concept of variation.

In the present research study we started from the observations recognized during that first stage of exploration and focus our attention to the way the students analyzed the different situations in order to be able to propose some teaching strategies based on the introduction of problems and concepts related to motion which can be explored later as they are used in the classroom.

## 2.- About the Experience

We worked during five semesters in a classroom context with first year students majoring in Applied Mathematics. We started the analysis of strategies by means of questionnaires that were solved during the lessons. Afterwards we chose three students that had successfully completed the course to conduct the interviews.

The clinic interviews were realized during four two hours sessions. During the interviews we asked the students some very general questions about variation to explore their strategies. At the beginning we worked problems with one variable but after a while we discussed problems that required a parametric representation.

The concept of parameter was not clear, and we found out that it came about more naturally when the students chose problens that dealt with objects in motion.

The students used always numeric or algebraic strategies, so we had to lead them, through the introduction of more complex problems, to use geometric and qualitative strategies. It was within this context that the idea of variation emerged more clearly as the role of the parameter and of the curve acquired a more dynamical nature.

## 3.-Learning Episodes

The first difficulty encountered by the student when dealing with curves in the plane or in space, is the meaning of the parameter itself. This is why a teaching strategy must start precisely at this point, looking for ways to introduce and give meaning to the parameter [Cantoral & Trigueros, 1991].

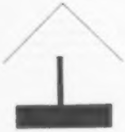
### *Learning Episode 1*

### *A Dynamic Variable*

From the analysis of some of the questionnaires and from the students' answers during the interviews, we could detect that when the parametric equations were introduced in the context of the description of the path of a moving object, the confusion with the meaning of the parameter: as a constant that can take en arbitrary value or as a variable in itself,diminished:

> S- (Movement)...*in nature...is something that is not static, for example an animal that changes its position...*
>
> I- How could you represent an animal that is moving?
>
> S- *With its velocity, how much it takes it to move from one point to another...*(for example)...*or measuring each year (the height of) a tree that is growing.*

**REPRESENTATION**

| *Pictoric* | *Numeric Height* | *Verbal* |
|---|---|---|
|  | 1<br>3<br>6<br>10 | The variable year is only expressed verbally. It is not explicitly represented in a symbolic form. |

The natural variable is time. That is why it doesn't need a symbolic representation. This idea stays with their explanations, so we will conserve it as an axis of our didactic approach.

### *Learning Episode 2*

### *Time stays on the representation*

After a debate period in which the convenience of the representation system, in order to have an adequate interpretation of the phenomena, was discussed, the evolution of the students' representation system ended up within a graphic context. It was precisely at this point that the explicit representation of time as a variable was necessary:

> S- *I could draw something in two dimensions and this is how I would have a representation that indicates that the velocity is varying* (with time).

### *Learning Episode 3*

*Breaking the movement in its components*

The decomposition of the movement along a curve in its components is not easy; even when it is taught explicitly. This is also the case when it is taught in Physics courses, for example when studying projectile motion or motion on an incline; the students memorize the formulas but they don't understand why the decomposition can be done.

During the interviews when we talked about the description of the trajectory of an object moving on the plane, the students used a numeric strategy (pointing out some of the points over which the object passed) or to an algebraic strategy (trying to remember a formula from their Physics lessons), but didn't break the problem in its components.

To induce the decomposition we asked some questions as:

> I- You have an object moving on a plane. It follows a specific trajectory. Can you say something about the motion of the components?
>
> S- *I would chose time as a parameter*...(I observe) *the position of that point along the trajectory as time* (goes by)...*time changes constantly and both* (x and y) *change with respect to time.*

Our analysis of concrete situations together with the students, drove us to the introduction of the idea of the shadow the trajectory projected on each of the axis and of a variable (time in this case) that could be used to relate one of the shadows with the other.

This qualitative analysis made it possible to introduce ourselves in the geometric context very easily: beginning with the path of a moving object and analyzing qualitatively the shadows made by the trajectory over the axis while the time passes, the students could build, in a qualitative or numeric way, the graphs of the components of the motion and they understood why the parameter is not shown explicitly on the graph of the path.

### *Learning Episode 4*

### *Representation of curves in the Plane*

The reconstruction of the trajectory when only the shadows projected over the axis are known was more easily and more straightforwardly accomplished during the interviews than the decomposition. Once again the geometric context was the natural one, allowing us to take aside, at least for a while, the need to know algebraic equations to solve the problem:

> I- Now we only have data for the shadows and we want the path.
>
> S- ...*in both graphs I have to take the same moment* (time)...*otherwise it won't be the same point...*

At this moment the parameter was already assumed as a variable, so, always in the geometrical context, we posed some non conventional problems to explore how clear this idea was in the mind of the students. Even though there were some difficulties with the interpretation of the questions asked, their handling of the parameter was the expected one.

### *Learning Episode 5*

### *Toward the Ideas of Calculus*

The strategies related with the notion of local variation, the analysis of the part to understand the whole, appeared in the geometric context we have already talked about. Small variations along the curves imply small variation on its components and the other way round. These strategies were not based on a numerical or algebraic aspects, so it showed a significant step forward in dealing with the initial problem.

## **4.- Towards Possible Teaching Strategies.**

The strategies used by the students and their answers to the different situations presented to them during the clinical interviews allow us to point out the advantages of working in the context of parametric curves, signaling out some of the aspects that can be taken as a starting position to deal with the problem of variation.

The introduction of the curves as a way to represent the path of a moving object, in which the natural parameter is time, provides a propitious environment to make clear that when dealing with problems related to variation the variable plays a dynamical role in contrast with the more static concept of function which is more tied up to algebraic strategies.

This dynamic conception of variable allows the student to conceive the curve as something that is constructed as the parameter varies. In this manner the curve may be viewed not as a static picture of the path but as a dynamical object. Also, when studying the curve behavior, the parameter plays a central role as opposed to its being only a mathematical tool needed to find the curve and some results about its global behavior, a tool one can eliminate after it has fulfilled its purpose.

When the question about the possibility of describing the path followed by the object by looking only at the shadows it made over the axis of the reference system is addressed explicitly, it is possible to induce in the students the idea of breaking the problem in its components, and in this atmosphere, that requires working the problem using a geometrical strategy that is at the same time dynamic in nature, one can handle the concept of local variation which plays an important role in the construction of the concepts of Calculus.

As the students tend to start up using numeric or algebraic strategies it is important to lead them through the idea of motion to use a geometrical and dynamical strategy where the relationship between variable and variation can be handled qualitatively and where one can explore and analyze local variation (as local linearity) without having to struggle exclusively with algebraic representations.

Once the role of time as a dynamic variable has been understood, the notion of variation emerges naturally in a contextualized form. This makes it possible to handle other problems where time is not necessarily the parameter, without losing the central idea of the role of the parameter as the generator of the curve, and as a dynamical variable.

All this seems important to us, not only because of the relevance of the problem of parametric representation in itself but because of the close relationship of these strategies with concepts as continuity and differentiability in Calculus. We think this aspect has not been dealt with in the literature [Dreyfus, 1990]. It can possibly open new lines of research in the psychology of advanced mathematical thinking.

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# STUDENTS' AWARENESS OF INCONSISTENT IDEAS ABOUT ACTUAL INFINITY

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*Recent research has indicated that students at different ages and at different levels of mathematics learning hold ideas that are incompatible with each other. So far, little has been done to find out whether students are aware of their own conflicting notions, and how students suggest to resolve the contradiction in their own answers when confronted with them. Hence these are the main objectives of our study.*

*The findings of our research, performed by interviewing 32 students aged 13-18 years in advanced mathematics classes, show that about two-thirds of these students responded in an inconsistent manner to problems dealing with infinite magnitudes. Yet most of them recognized their conflicting answers as incompatible with each other. Attempts to solve the identified inconsistency included: (1) perceiving such a situation as legitimate in mathematics, (2) resolving the contradiction in a way that was incompatible with conventional mathematical theories, and (3) attempting to formulate rules and theorems which defined equality between infinite sets.*

## Introduction

Mathematical doctrines are, in a way, games of logic and abstraction founded on several independent ground rules (axions), from which the players (mathematicians) are allowed to derive new laws (postulates). One restriction of these games is that of avoiding contradictions between the basic rules: The name of the game is consistency.

Recent research reveals that students often hold inconsistent ideas, occasionally caused by their primary intuitions (Fischbein, 1987: Kahneman, Slovic & Tversky, 1982). Each of these incompatible ideas is often grasped by the student as self-evident and certain. It is not likely that these inconsistencies will be resolved by the learners without special didactical intervention (Brown, 1992, Fischbein, Nello, & Marino, 1991). Therefore, it is most important to raise students' awareness that their beliefs might sometimes mislead them and direct them to contradictory conclusions.

How can this be done?

In the last decade several educational approaches toward resolving inconsistencies have been suggested, including "conflict teaching" (i.e., Swan, 1983), "teaching by analogy" (i.e., Strauss & Bichler, 1988), and "the generic environment approach" (i.e., Tall, 1990). It has been shown that the conflict teaching method, which is based on raising the student's awareness of contradictions in his/her own thinking and of the illegitimacy of this in mathematics, can be effective in removing and correcting various misconceptions (Tirosh & Graeber, 1990). However, not enough attention has been paid to students' reactions when confronted with negations of their own statements.

The exploration of students' reactions to inconsistences in their own thinking has led us to construct an "infinite-sets game", created to facilitate cognitive conflict. Through several activities with infinite sets, chosen because the concept of infinity raises conflicting intuitions (Martin & Wheeler, 1987; Sierpinska, 1987, Moreno & Waldegg, 1991), the experimenter: (1) encourages a variety of different conflicting ideas, (2) follows the student's contradictory answers, (3) promotes awareness of resulting inconsistencies, and (4) guides the student toward recognizing the discrepancies in his/her considerations. The present paper describes the infinite-sets game and a first attempt to explore its efficiency in (a) evoking students' awareness of inconsistencies in their own thinking about infinity, and (b) examining students' reactions when they discover inconsistencies in their answers.

## Method

### Subjects

Thirty-two students between the ages of 13 and 18 from advanced mathematics classes of the Israeli public school system participated in this study. Six to seven students from each of 8-12 grade levels, who were identified by their teachers as most likely to specialize in mathematics or to use mathematics in their advanced academic studies, took part in the activity described ahead.

### Materials & Procedures

The infinite-sets game is a three-phased didactic tool which uses "card-activities" to present each student with two different representations of the same problem, and encourages him/her to reflect on his/her answers.

The stages of the game ran as follows:

Stage 1: Encouraging "part-whole" considerations

Instruction 1: "Take card A. This card refers to the set of natural numbers. (a) Explain the meaning of the sign '....' at the right end of the card. (b) Are the numbers: 987853, 1947, (-7), 22.5 also included in this card?"

Card A: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21...}

Instruction 2: "Take card L. (a) Circle the numbers divisible by 4. (b) Explain once again what the sign '....' at the right end of the card stands for. (c) Which of the following numbers belong to this card: 1246000, 1113, 16567, 4400080, 1/4, 842.

Card L: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21...}

Instruction 3: "Copy the numbers encircles previously on a new empty card marked B."

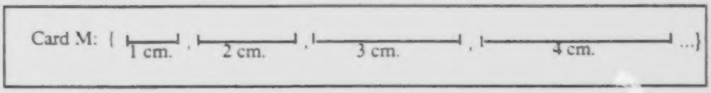
Instruction 4: "Card A contains the natural numbers. Card B contains the the numbers divisible by 4. In your opinion, are the numbers of elements in cards A and B equal? Please explain your answer."

Card B: {4, 8, 12, 16, 20...}

Stage II: Encouraging one-to-one correspondence considerations

Instruction 1: "The following card refers to an infinite set of segments. The first segment is 1 cm long, the second segment is 2 cm long, etc. Each segment is one cm longer than the previous one. Please write down, on card 1, the set of numbers which represent the lengths of the segments described in this card (in cm.)."

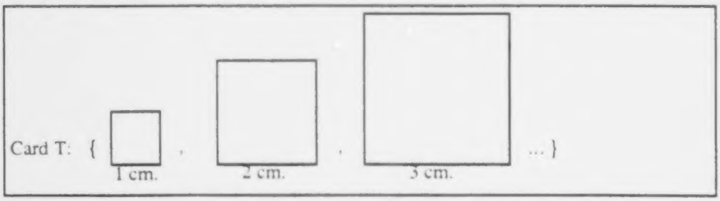
Card M:



Card 1: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21...}

Instruction 2: "Imagine yourself trying to construct squares in such a way that the previous segments are their sides. In your opinion: (a) Is it possible to build more than one square for each segment? (b) If the side of a square is X cm, how long is its circumference? Is this the only possible solution? (c) Given the circumference of a square is Y cm, how long are its sides? Is this the only possible solution?

Instruction 3: "Consider a set of squares, each square using one of the previous segments as its side (see cards M and T). How many squares are there? Is the number of squares equal to the number of segments?"



Instruction 4: "Now, write down, on card 2, the set of numbers that represent the lengths of the circumference of these squares (in cm). In your opinion, are the numbers of elements in cards 1 and 2 equal? Explain your answer."

Card 2: {4, 8, 12, 16, 20...}

Stage III: Confronting the student with his/her inconsistent ideas

At this stage, cards A, B, 1 and 2 were laid out in front of the student to encourage him to notice the identity of (i) cards A and 1, and (ii) cards B and 2. He/she was then asked to explain his/her conceptions about the consistency in his/her answers to the problems presented in the previous stages. The conversation aimed to help students who gave conflicting responses to recognize the inconsistencies and suggest possible ways to cope with the situation.

The game was played, individually, by each of the students, under the guidance of the researchers. A typical session lasted 35-50 minutes. All sessions were audiotaped and transcribed.

## Results

### Consistent Answers

Twelve of the 32 participants maintained the same answers in stages 1 and 2 of the infinite-sets game. All of them argued, consistently, that the number of elements must be equal in all cases. In their explanations, nine students exhibited a concept of **singularity of infinity** (i.e., "All infinities are always equal", "There is only one magnitude of infinity", or "Infinity can only be reached through the limit of a mathematical process, and as such, it must be unique"); two students explained that infinite quantities are incomparable (i.e., "There are no correct or wrong answers to these problems because it is illegitimate to compare infinite magnitudes"), and one student used a lexical argument, stating that, "Individual names were given to numbers representing different quantities. For instance, '7' indicates the same quantity of elements no matter what kind. Likewise, if there was more than one infinity, then there should have been matching names to describe each 'quality' of infinity. But as the number of elements in all those sets is called "infinity", it must indicate a single magnitude".

Students belonging to the "consistent answers" group did not face refuting statements, and thus the issue of awareness of contradictions did not arise in their interviews.

### Inconsistent Answers

Twenty students gave contradictory responses in the two stages of the game. They argued, in the first stage, that the set of natural numbers includes all the numbers divided by 4 and other elements, and thus the numbers of elements in cards A and B are not equal. In the second stage of the game, these students claimed that every square relates precisely one element from card 1 (segment lengths) to one element in card 2 (circumferences) and thus the number of elements in cards 1 and 2 are equal.

#### Recognition of Inconsistences

By the end of stage II of the game, seven of the twenty students noticed themselves that their answers to the two stages were incompatible. The other 13 students were asked to re-examine cards A, B, 1 and 2 carefully. After some prompting questions (i.e., What do you see in these cards? What does this tell you about your initial answers?), all these students noticed the inconsistencies in their answers.

#### Students' Reactions to Inconsistencies in Their Own Answers

Each of the 20 students who gave incompatible responses in the two stages of the infinite sets games was encouraged to suggest possible ways for resolving the contradiction. Their responses fell into two categories:

a) Six students recognized and identified the two conflicting statements as incompatible with each other but perceived such a situation as legitimate in mathematics. Three explanations were given to support this view:

i) Referring to the relativity of mathematics. Two students stated that a situation of contradictory answers is acceptable in mathematics if each answer is embedded in a separate, independent mathematical system. These students claimed that while the first activity dealt with numbers, the second activity was geometric, and this might affect the results.

ii) Drawing inappropriate analogies. Two students provided inappropriate examples to account for the existence of situations like $B<A$ and $B=A$ simultaneously. Both students mistakenly claimed that: "$\sqrt{4}=2$ and $\sqrt{4}=(-2)$". On the basis of this inappropriate statement, they continued arguing that "In this case [the square root] I know for sure that both solutions are acceptable, so I presume it will work here [in the sets] too."

iii) Misunderstanding the independent nature of mathematics. Two participants expressed their perception of mathematics merely as a tool for solving various problems in physics, chemistry or other sciences. These students asserted: "The correctness of a mathematical statement is established through checking its usefulness in facilitating a certain physical or chemical procedure.... it has to fit reality... but if the mathematical law is not applicable right now, we mustn't discard it as it may be functional in a way not yet known."

b) Fourteen students identified the contradicting components of an inconsistent situation and grasped the situation as problematic. They used four different ways to resolve the contradiction.

i) Grasping infinite magnitudes as incomparable. Three students claimed that it is impossible to maintain simultaneously that $A>B$ and $A=B$. In line with Galileo, they suggested that the attributes "equal", "greater", and "less" are applicable only to finite sets and not to infinite ones.

ii) Viewing infinity as singular. Three students declared that their previous arguments were mistaken and concluded that all infinite sets have the same magnitudes.

iii) Preferring one-to-one correspondence for comparing infinite sets. Two students rejected their claim (in the first stage of the game) that the number of elements in the sets is not equal, viewing the second stage of the game as a mathematical proof.

iv) Preferring "part-whole" considerations when comparing infinite sets. Two students said that the idea that the whole is always bigger than its part should be used to compare infinite sets. They felt that it is natural to think in terms of inclusion.

It is noteworthy that four students demonstrated sound mathematical thinking by clarifying that they lacked the information needed to resolve the contradiction. A typical response was: "Either one of the conclusions (A>B or A=B) is wrong or they are both wrong. I lack definitions and rules to guide me in the subject of infinity." Along with their awareness of contradiction as a cause of 'trouble' in mathematics, these students showed an understanding that performing within a mathematical theory requires a solid knowledge of its ground rules - definitions, axioms and postulates. In particular, they required a systematic law pattern in order to compare infinite sets accordingly.

## Discussion

This study suggests that when solving mathematical problems, students are not necessarily concerned with the consistency of their solutions. Even when only two problems are presented within one rather short session, many students refer to each problem separately without regarding the compatibility between their answers. Much like in other studies, our data indicate that the representation of the problem has a sound impact on students' solutions (Arcavi, Tirosh, & Nachmias, 1989; Fischbein, Tirosh, & Hess, 1979; Silver, 1986). Thus, in planning a sequence of instruction of the Cantorian set theory, the comparison of the segments and squares suggested in Stage II of the game, can be used as an anchoring example to encourage preference for one-to-one correspondence as a means for comparing infinite magnitudes.

A major aim of this study was to evaluate the efficiency of the infinite-sets game in evoking students' awareness of inconsistencies in their own thinking about infinity. Our data indicate that most students included in our study perceived the situation in which two contradictory statements are provided to the same mathematical problem as inconsistent and therefore problematic. Yet their attempts to reconcile the discrepancy between their own statements uncovered misconceptions related to issues such as the nature of consistency, the relativity of mathematics, and the relationship between mathematics and the physical world. These data clearly indicate that the road from awareness of an inconsistency to its resolution in a way that is compatible with conventional mathematical principles is a rocky one. It is only to be expected that this should be so in the case of infinite sets. However, we would like to suggest that even in less extreme instances, this reconstruction is by no means easily achieved.

In sum, our primary results suggest that the infinite-sets game may serve not only as a means of exploring students' ways of coping with conflicting notions in their own thinking, but also as a springboard for discussing a number of central issues in mathematics such as (a) the nature of consistency and its role in mathematics, and , (b) the causes of inconsistencies in our own mathematical thinking (i.e., discrepancies within our intuitive knowledge, disparity between the everyday meaning of a notion and its use in mathematics). Currently our efforts are directed toward further explorations of possible ways of using the infinite-sets game as a means of convincing students that consistency should be a standard for checking the validity of their own mathematical thinking.

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# MATHEMATICS AND GENDER: AN INTERACTIONAL ANALYSIS

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*ABSTRACT: The purpose of this study is to investigate whether there were any gender-related differences in the patterns of interaction during mathematics instruction in a second grade mathematics classroom with a problem solving, mathematics tradition. The findings indicate that when students' answers were not questioned by the class, there were no gender-related differences in the interaction patterns. However, when students' answers were questioned by their classmates, differences in the pattern of interaction which may be related to the gender of the students emerged. During these periods, boys appear to be more active verbal participants during whole class discussion than girls.*

## RATIONALE

There is a long-standing "school mathematics" tradition of teaching and learning mathematics (Richards, 1991) into which teachers and students have long been acculturated. In order for students to successfully learn and understand mathematics, all students are taught to think about and do mathematics in specific ways. To this end, students work on activities intended to make them more proficient in using standard procedures to solve problems. The smooth flow of the classroom discourse in these mathematics classes has traditionally depended upon a specific answer to a teacher question as teachers unknowingly "funnel" their students' responses by steering them to a predetermined answer or solution method (cf. Voigt, 1985).

In a recent study by Jungwirth (1991) conducted in Austrian middle and high school classrooms, the interactions of teachers and their students during mathematics classes were reconstructed in order to determine if there were any differences in this typical pattern of interaction related to the gender of the participants. The findings indicated that even though all students were funnelled to the correct answer, often the funnelling of girls' solutions was much more directive than the teacher's actions of steering the boys' thinking, causing girls to appear less competent in mathematics.

Because the classes that Jungwirth studied were all traditional math classes, I questioned whether there would be any gender-related differences in the interaction patterns in a classroom where the traditional question response-evaluation pattern (Mehan 1979) found in most mathematics classes was not the classroom norm.

## PURPOSE

The purpose of this exploratory study was to determine if there were any indications of gender related differences in the patterns of interaction during whole class, teacher-led activities in a second-grade mathematics class in which problem solving is the focus of children's mathematics activities (Cobb, Wood, & Yackel, 1991; Cobb, Yackel, and Wood, 1989).

## RESEARCH QUESTIONS

1. What is a typical pattern of interaction that occurs in this classroom?
2. Are there any gender related differences in the typical patterns that could influence children's learning opportunities in mathematics?

## THEORETICAL FOCUS

The theoretical frameworks that guide this research are symbolic interactionism (Blumer, 1969) ethnomethodology (Mehan & Wood, 1975). Under these assumptions, the meaning that people give to events arise through their social interactions. Individuals in the group construct their knowledge by interpreting and adjusting their meanings based on the reactions of others in the situations where the interactions have occurred.

By looking through the lens of ethnomethodology, the patterns of interaction that people engage in certain contexts can be used as a window into understanding the culture of the group. Because the purpose of this study is concerned with gender-specific patterns of interactions, and that gender differences are products of the culture in which they were formed, it seems appropriate to combine these two theoretical frameworks for this study

## DATA COLLECTION STRATEGY AND ANALYSIS

Four consecutive days of videotaped data were collected in the observed second grade classroom during math period during the month of March. As a secondary data source which was used for purpose of triangulation, an informal interview with Mary, the classroom teacher, was conducted after a preliminary analysis was made

The videotaped data was analyzed using and analytic induction approach (Goetz and

LeCompte, 1984). During the initial phase of analysis, transcription notes of the four days of whole class discussions were compiled. The notes were studied and then coded according to similar incidents or discrepant cases of those events that appeared to be related to the gender of the student. The flow of the classroom discourse was documented by a chart of the interactions illustrating the activities of the participants.

## FINDINGS AND DISCUSSIONS

As the following sections will illustrate, the pattern of interaction differed greatly from the traditional tripartite pattern of interaction. Following the description of the typical pattern of interaction in this classroom, are several descriptions of the modifications to this pattern which appear to be related to gender of the student.

### TYPICAL PATTERN OF CLASSROOM INTERACTION DURING WHOLE CLASS DISCUSSIONS

During the whole-class teacher-led discussion in the observed classroom in which the focus was on arithmetic activities, the pattern of interaction could be described as Inquiry, Response, Evaluation, and Solution.

When Mary asked her students a question during math time (Inquiry) her purpose was not to elicit a predetermined response from the students based on explicit precursory instruction as is the norm in traditional mathematics classes, because she never instructed her students on any specific procedures they must follow to derive correct answers. In response to Mary's question, the student called upon would offer a numerical answer (Response). Whereas in traditional instruction this response would be followed by a teacher evaluation, Mary would never comment on whether the answer was correct or incorrect. Instead, she asked her students if they agreed with their classmate's answer (Evaluation).

Following an evaluation by their peers, the student was then expected to give an explanation or justification for their answers (Solution). Mary would then solicit several other students' solutions before moving on to another problem. Because Mary did not explicitly evaluate her students' solutions, there were not any gender modifications of this type in the interactions. Other research studies on gender differences in mathematics instruction have shown that teachers initiate more contact with boys than girls (Pecker, 1981; Fennema & Reyes, 1981). Mary averted this problem in her instruction by alternating who she chose to call on based on gender.

## PATTERN OF INTERACTION --(WHEN ANSWERS ARE QUESTIONED)

When students were evaluated positively by their peers, there was no evidence of gender related difference in the pattern of interaction in this class. However, when the answer was questioned, instead of simply agreeing or disagreeing with the answer given, the class talk "breaks out" becoming somewhat chaotic as students called out their disagreement, answers, and/or solutions to this problem. During these break outs, Mary reacted by remaining relatively silent until the time when the discussions reached a point where many children were speaking simultaneously. She then regained control over the talk by silencing the class and calling on a student to give his or her solution.

In situations such as the one above, when the classroom talk broke out and the teacher relinquished her position as turn-taking monitor, the "fair" boy girl turn taking practice which had been orchestrated by the teacher disappeared. As the students take over the group discussion, modifications in the pattern of interaction which may be related to the gender of the students begin to emerge. During these debates, the students in this class that are the more active verbal participants in whole class discussions are boys.

## GAINING THE FLOOR- The Boy's Talk

One of the possible reasons that many of the boys in this study appear to be more active verbal participants, and therefore are more likely to dominate whole class debates, is that they are more likely to call out their solutions and enter into the discussions at moments when their opportunities to be heard and "gain the floor" are maximized.

When entering into the dialogue, many boys would begin their turn by first making an evaluative comment such as "Yeah," "No," "I know that," etc., before voicing their opinions. By linking their statements to one given by the student before them, they are very successful at entering into the dialogue and maintain a smooth flow in the discussion. An example of this talk is illustrated

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in the following segment.

Context: After working with their partner's on the following problem in pairs, Mary places this problem on the overhead to be discussed.
"Stewart has this much money (pictures of 1 quarter, 1 dime, 2 nickels, and 3 pennies are given). He has 30 cents less than Scott." After David gave their group's solution and answer of 18 cents, the class talk break out and the following student-directed discussion ensues.

1. Brian: Scott doesn't have any of that. All of that is Stewart's money.

2. Albert: Yeah. And Scott has, and so he has, (pause).

3. David: But it says this: It's Stewart that has 30 cents up there.

4. Amy: Yeah, it does.

5. Albert: He- No, it says Stewart has 30 cents less than Scott. It didn't say 30. Stewart has 30 cents.

6. Albert: It says- it didn't say that Stewart had 30 cents of that money up there. It says that he has 30 cents less than Scott. And it doesn't show Scott's money because, that would give you the answer right away.

7. Lee: I know. But they didn't write the problem, problem right.

8. Amy: How do you know?

9. T: (to Lee) Oh, that's an excellent statement. How do you think it should, what do you think it should say?

10. Lee: It said "Steward had this much money." Stewart has 48 cents in all. And then it says "He has 30 less than Scott." It should say, "Scott has 30 cents less than Stewart."

In this segment, Albert's evaluative statements (lines 2, 5) of "yeah" and "no," Lee's statement (line 7) of "I know," and David's statement (line 3) of "But it says," illustrates the use of evaluative introductory phrases. In each case these statements are a link between a prior students comment and their own justifications. Each boy, then, was able to maintain their position as discussant by immediately supporting their own introductory statements with a follow-up statement that would justify their position.

This segment also illustrates a significant teacher move that can be seen throughout this analysis. When Mary breaks into the talk with her reaction to Lee's statement, she is encouraging him to further elaborate on what she views is an important aspect of the problem. By highlighting a student comment which had been given during a student-directed discussion, she was not only sanctioning this mode of turn taking but also fostering this form of open discussions.

## GAINING THE FLOOR The Girl's Talk

Even though girls in this class also made evaluations of their peer's solution throughout class discussions, they were less likely to follow an evaluation with a justification. By this action the girls limited their own opportunities to speak. In the class dialogue above, Amy's statement (line 11) illustrates that she agrees with David's statement. Because she doesn't give any elaboration of it, her turn ends.

On the other hand, when girls did call our their arguments in whole class discussions, they tended to speak out during periods when many students were calling out their solutions, making it more difficult for them to gain their teacher's attention and to have their solutions highlighted. When a girl did take advantage of a pause in the discourse and follow up on her peers' statements, it was common for another child to jump in with his/her comments making it difficult for the girl's justification to be heard.

## REGAINING ORDER

Because Mary accepted answers that students called out, there were instances when several students called out their solutions at the same time. Since Mary had to regain some order before anybody could be heard explaining or justifying his/her answer, she had to silence the class talk. To do this, Mary would call on students with their hands raised by making a statement such as, "Stacy has her hand up," signaling to her students that their hands are supposed to be raised if they want to give their solutions.

At these times, Mary chose to call on children who were more reticent about openly calling out their solutions. Because many more girls than boys fell into this category, they were usually the ones chosen. Since hand raising proved to be an effective way for them to gain the floor during whole class discussions, it may be possible that the girls choose to participate in this way, avoiding the open and often heated discussions in which boys were more inclined to participate.

## MARY'S VIEW

By allowing her student to call out their solutions, boys had more opportunities to speak during whole class discussions. It might be argued that Mary should have curtailed the children's tendency to call out their solutions instead of acting upon them. However, because Mary allowed open student-directed discussions to occur in her classroom, her students had many opportunities to engage in a rich mathematical dialogues which might not have evolved if the children's spontaneous interactions had been restrained.

When I asked Mary about the break outs in whole class discussion, she stated:

> (T)hat age-old problem teachers have is spontaneity--you know you want to have order in your classroom and yet you really love spontaneity, and it's so hard to keep that balance--(D)o you just let the ones who blurt out the answers, do you accept it, or do you say "I'm looking for somebody who's raising their hand?" That kind of parity (is what) we have to deal with all the time. I like to try and keep things as spontaneous as I can. that is a disadvantage to the quieter ones and so it's really a tough thing.

## CONCLUSIONS AND IMPLICATIONS

A study of videotaped data collected over such a short period of time in a single classroom and one teacher interview cannot and should not be interpreted as yielding conclusive evidence which illustrates gender related modifications to the pattern of interaction during math time in the observed classroom. All of the indicators of gender differences in this class might not be contributed to "the girls actions" or "the boys actions" during whole class discussions. It is possible that the actions of the students in this class during their math time have very little to the students' gender. It might be that their actions are solely related to the unique qualities of each individual in this class.

However, the findings in this study illustrate the need for further study in classrooms where the patterns of classroom interaction are similar to the one described in this study. Even if there is further evidence of gender differences in classes with similar mathematics traditions, the question remains: Does the girls' reticence in open discussions, or their inability to gain the floor when they attempt to participate in those debates influence their learning of mathematics?

In conclusion, the purpose of this study was to determine if there were any indications of gender related differences in the patterns of interaction in classrooms that did not follow the typical question-response-evaluation pattern. Even though the findings illustrate that there are many more opportunities for all students in this class to openly participate in mathematical communication than in traditional instruction, the findings indicate that boys in this class are better able to take advantage of those opportunities than girls, raising questions that could only be answered from further investigation into classrooms with similar mathematics traditions.

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# THE LEARNING OF TRIGONOMETRIC FUNCTIONS IN A GRAPHICAL COMPUTER ENVIRONMENT

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*A study is reported which compares the performance in mathematics of students who work with a function plotter on a computer with that of students without access to the graphical computer environment.*

*The results illustrate that, under the given conditions, access to graphical computer software can have a positive effect on the mathematical attainment as measured by a posttest. This effect of was even more marked on a retention test given three months after the experience. The relative attainment of female students in the computer group was superior to that of male students on the posttest, but not so on the retention test.*

## Background to the study:

In the first study of the ICMI Study Series (ICMI, 1986) the importance of exploration and discovery in mathematics is emphasized and also the way computer technology may be used in this endeavour. It is pointed out that through visualization via computer graphics, students may explore questions and discover results by themselves. Dreyfus (1991) presents more important arguments in favour of the potential of visual approaches for learning, specially in a computerized learning environment, because graphic computer screen representations of mathematical objects allow for direct visual action on these objects. Recent research on visualization is concerned with the effects of a visual versus a symbolic approach and how students relate both (Eisenberg, Dreyfus, 1989). There are studies which show the positive effects of visualizing in mathematical concept formation (Bishop, 1989), but there are dangers in doing this carelessly because visual presentations have their own ambiguities.

Some teaching strategies propose a combination of analytic/algebraic and visual/geometric settings. It is not clear, however, if students can establisch correct connections between different representations. There is evidence that they cannot unless they are tought explicitely to relate both (Schwarz, Burkheimer, 1988). Dreyfus and Eisenberg (1987) hipothesize that the comprehension of the relationship between multiple representations (algebraic and graphical) can be facilitated by computer software in a structured learning environment, but apparently, free exploration with the software was as successful as the structured activities. Several other studies show that computer environments seem to be an ideal tool to build a curriculum from a constructivist point of view which allows students to make transitions between algebraic and geometric representations (Gardiner, 1982, Dreyfus, 1990, Artigue, 1987, Schwarz, 1989 and Tall, 1984, 1987).

This study is based on the development of graphical environments with computers which enable students to discover and acquire function concepts in trigonometry at the highschool lelvel. The main objective of the research reported is to study the feasability and efficiency of a graphical environment for the construction of trigonometric function concepts. The graphical environment we describe in the next section assumes a constructivist viewpoint of both mathematics and its teaching. Such a cognitive approach is intended to induce meaningful learning by means of a "generic organisational system" (Tall, 1985). Generic means that a learner manipulates examples of a concept and his attention is directed at certain aspects of such examples. By means of concrete examples, the learner constructs higher order concepts. The organisational system consists of a generic organiser (software) and organising agents (study guide and teacher). The agents are necessary, since the organiser alone does not guarantee that the learner will get the best out of the learning situation.

## Method:

The teaching strategy as reflected by the study guide was developed and tried out in a series of three previous studies. (Wenzelburger 1989, 1990, 1991).

In March 1991, the fourth study, reported here, was done with a group of 31 highschool students enrolled in an eleventh grade geometry course. From the 31 students, 8 were chosen randomly to form the experimental group (computergroup). The reduced number of students in this group was due to limitations in the computer equipment. The rest of the students (classroomgroup) attended normal lectures of the classroom teacher on the same mathematical contents, trigonometric functions $y=a \sin bx$ and $y=a \cos bx$ which normally are not taught in the geometry course (nor any other course in Mexican highschools).

The Computergroup worked during 10 50-minute sessions in the computer center of the highschool with the function plotter and the study guide. Every student had his own computer. The investigator was present at all times.

A pretest, a posttest and a retention test was given to all students. The retention test was given three month after the experience.

## Teaching Strategy:

The teaching strategy as reflected by the study guide is inductive and emphasises exploration and guided discovery.

The didactical progression in the study guide was a follows:

Topic 1: Amplitude of functions of type $y = a \sin (x)$

Topic 2: Amplitude of functions of type $y = a \cos (x)$

Topic 3: Frequency of functions of type $y = a \sin (x), y = a \cos (x)$
Topic 4: Frequency of functions of type $y = a \sin (bx)$
Topic 5: Frequency of functions of type $y = a \cos (bx)$

It is supposed that by means of this directed experimentation with CACTUSPLOT (Losse, 1986), students get actively involved in the learning process and construct and interiorize concepts and do not learn by memorization.

Learning objetives and testing instrument:

Students had to learn about the role of a and b in the function equation and their effect upon graphs of these functions, their amplitude and their period.

The three main objectives were:

1. Students relate graphical and algebraic representations of trigonometric functions.
2. Students generalize from concrete examples to the general equation of trigonometric functions.
3. Students inteprete graphical or algebraic representations of trigonometric functions.

The posttest and retention test (and pretest) consisted of a closed part (16 multiple choice items) and an open part (14 open questions). The main tasks students had to perform on the tests were as follows:

Task 1: Given an equation $y = a \sin bx$, $y = a \cos bx$, where b are real numbers, identify or draw the graph (Objetive 1).

Task 2: Given a graph of the type $y = a \sin bx$, $y = a \cos bx$, identify or write the equation (Objective 1).

Task 3: Given a graph of the type $y = a \sin bx$, $y = a \cos bx$, determine the amplitude, period or frequency, (Objective 2).

Task 4: Given a graph, identify amplitude, period, maxima and minima (Objective 3).

Task 5: Given an equation, identify the amplitude, period, maxima and minima (Objetive 3).

Hypotheses:

1. The graphical computer environment directly stimulates the construction of concepts of the trigonometric functions under study and students in the computer group will do better than students in the classroom group on the inmediate posttest.
2. Students of the computer group do better on a delayed retention test (three months later) than students in the classroom group.

Results:

The pretest showed that the students participating in the experience had no previous knowledge of trigonometric functions $y = a \sin bx$, $y = a \cos bx$.

Table 1 shows the average percentages of correct answers by group on the posttest. According to the experimental design, a student's t was calculated on the gainscores. The t-value was 1.76 which with 29 degrees of freedom for a one-tailed test is significant.

TABLE 1

Average percent correct on the posttest (max. score 100 points)

| | Number of Students | Average | Standard deviation |
|---|---|---|---|
| Computer Group | 8 | 73.6* | σ = 18.4 |
| Classroom Group | 23 | 67.9* | σ = 19.3 |

* Significant difference at 0.05 level

Results of the retention test three months later are shown in Table 2.

TABLE 2

Average Percent Correct on the retention test (max. score = 100 points).

| | Number of Students | Average | Standard deviation |
|---|---|---|---|
| Computer Group | 8 | 67%* | σ = 0.88 |
| Classroom Group | 18 | 48%* | σ = 2.25 |

*Difference significant at 0.001 level

It was also observed, that girls in the computer group did much better on the posttest than boys, while on the retention test no difference could be found (Table 3).

TABLE 3

Average Scores for the Computer Group (100 = max. score).

| | Posttest | Retention Test |
|---|---|---|
| Girls<br>n = 4 | 85% | 64% |
| Boys<br>n = 4 | 63% | 66% |

### Discussion and conclusions:

The finding of superior performance on the average by the computer group on the posttest and retention test, suggests that the graphical computer environment had a positive effect on the formation of concept images and construction of concepts for students in the computer group.

These results are in agreement with similar studies reported by Artique (1987), Bishop (1989), Eisenberg, Dreyfus (1989), Goldenberg (1988), Tall (1985, 1987).

The reported results also suggest, that students who get more actively involved in the learning process, as they were expected to do in the computer group, retain the acquired concepts over a longer period of time. This is certainly not surprising but never the less worth emphasizing. The use of an interactive software together with a studyguide based on a discovery learning approach and visual reasoning may be a good teaching strategy for functions.

There was also a strong gender difference on the posttest in the computer group which has also been previously observed by Hoyles (1986), Ruthven (1990) and Tall (1991). Inspite of the apparent positive treatment effect observed, it is important to be aware of the limitations of the study. The most common mistakes observed on the posttest were the confusion of amplitude and period, sine and cosine functions, the parameters a and b, inability to associate an equation with a graph or to interprete a graph correctly. Both groups had difficulties with the concept of period. On the retention test the most common mistakes also consisted in the confusion of cycle, period or frequency and in writing the formula $2\pi/b$ upside down. This error showed up consistently in the computer group in which students remembered a formula, but not the correct one. The classroom group in general did not even remember that there was a formula for period at all. Again this may be due to the lack of active student involvement in the learning process in a traditional lecture classroom setting.

It must be said that the positive effect of the graphical computer treatment can be confounded with the effect of active, individualized learning which supposedly occurred in the computer group. This however does not rest validity of the research done, since both factors can be seen as additive rather than exclusive.

## Conclusion:

Although the study reported is limited in the sense that it concerns a very specific aspect of functions and makes use of a very specific graphic software, it provides some more evidence that an interactive, visual approach in a graphical computer environment, structured by means of a guided discovery strategy, influences positively the mathematical attainment of students, specially female ones, and helps them to remember acquired concepts over a longer period of time.

This study also suggests that information technology can be useful in mediating mathematics in the classroom, but that much thought must go into the way it can be used most effectively.

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# EUCLIDIAN CONSTRAINTS IN MATHEMATICS EDUCATION

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## 1 Introduction

*More girls than boys drop out of mathematics-education, or choose mathematics in further education, often with poorer results (Sherman, 1981; Pedro et al 1981; Leder, 1985). This pattern still persists today, in spite of various efforts to change this situation, be it politically or scientifically inspired (for instance Sangster, 1988). In this paper*[1] *I analyze this situation in terms of the (scientific) intervention methods employed. Such methods tend to emphasize individual choice and the factors that determine this choice, e.g. biological or socially induced differences between girls and boys. An alternative is to see mathematics-education as a sub-system of society determined and maintained by it. It itself determines both the content of the mathematical curriculum as well as the approach to teaching. It thereby constructs the gender pattern in the attitude towards mathematics. Changing this pattern implies changing the way the various sub-systems are maintained. It is assumed to be based on a 'language' which has to be replaced or adapted.*

## 2 Three intervention strategies

Whether or not one wants to change the gender-related choices of and behaviors in mathematics may be decided differently on different levels. On the individual level it can be a question of ideology, of just simply of taste. On the societal level it is a problem that should be made as independent of ideology or (cultural) taste as possible, the solution to which should be as coercion-free as possible. This latter type of motivation concerns me here.

The most frequently employed intervention strategy at the moment is not coercion-free at all. In fact, it is based on imposing norms and models (or on replacing them by others). The idea is to look for factors that determine high gender-related drop out rates and negative attitudes towards mathematics education, and then to formulate and implement intervention strategies in terms of such factors. An example of this approach in present stimulation programs is 'Kies exact', a government sponsored program in the Netherlands explaining mathematical career opportunities. Here the gap in career-awareness between boys and girls is taken as the major determining factor for gender-related choices of and behaviors in mathematics. Full career-awareness is taken as the norm.

This type of intervention strategy in general has turned out to be much less than effective. The norms often are felt to be 'alien', that is externally imposed. Also the concept of determining factors appears as (too) ambiguous in these areas of intervention: to base ones interventions on them hence at least appears premature, if not a lottery.

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[1] This paper is based on research described in my thesis (Witte, 1992).

Alternative approaches have been formulated and explored therefore. More generally we may distinguish three different types of change-strategies.

The first strategy is the intervention strategy mentioned above. It is based on what may be called a 'model of girls', a hypothesis about what the average girl wants or can do. Such a model may emphasize biological differences, such as lower spatial visualization levels for girls. Or it may emphasize the social construction of feminity as ways of behaving, choosing and acting. Such constructions eventually come to be imposed on individuals, 'modelling' them to show model behavior (Meece, 1982; Fennema, 1985).

The second strategy is based on what may be called a 'model of content', that is on the assumption that science and mathematics is a social construction made by men and therefore only suits for men(Keller, 1985). This strategy sees mathematics as culturally bound, as dependent on changes in points of view.

The third strategy is based on what may be called a 'model of organization'. It is assumed that people relate to each other and their environment in many ways. Certain forms tend to become preferred, however - and if preferred, relate to each other in complex ways: preferences in mathematical curicula relate to distinctions in the job market, distinctions in the job market relate to behavioral differences, behavioral differences relate to linguistic differences, etc. These relations are important. They tend to increase societal stability: changes in one preferred form will be repaired via their analogical relation to other preferred forms.

Interventions will differ according to these various strategies. One might try for example to prevent girls from learning mathematics, when biological evidence would predict failure and a waste of educational effort. Or one might try to change the imposed models of femininity and open up vistas of 'equal' careers. Or one might try to find ways of adding choices to education such that they stretch organizational preferences to the point of becoming more coercion-free.

The third strategy has received relatively little attention up till now. It is not based on a 'model of girls', neither biological or social, nor on the idea that there is something intrinsically male to mathematics. It starts from the idea that society is stabilized through the implementation of constraints in various sub-systems. Such constraints generally can be represented by a regulating language.

One of the most pervasive of such languages, for example, is the control language which orders many hierarchically regulated subsystems (and also the first type of intervention strategy above). It emphasizes rule-governed behavior, and de-emphasizes all that may generate deviations, such as emergent interactions.

Given the variety in how one can react to influences from preferred organizational forms, one may expect that some people will mostly fit the rules, and others will be prone to marginality and to being a deviant. Each language therefore will create its own correlations between various characteristics. Thus, in some societies high social position is strongly correlated to age, and not mainly to social-economic variables (as is the case in many Western societies). Intelligence similarly may be associated to the ability to see and express similarities; or conversely to the ability to see and express differences. Given various regulating languages, creativity may be connected to marginality or to being central, in the thick of things; it sometimes even gets visualized by 'artistic' clothes. In Western society only a few remnants are left of the very high Medieval correlation between social status and clothing.

In this paper I will focus on the third strategy, as a way of changing the correlation implied by gender-related preferences for and behaviors in mathematics-education. I take it that improvement will mean that this correlation gets weakened. An inverse example would be to dissuade girls from choosing more mathematics oriented classes, which would increase the correlation - whatever the reasons for doing so. To all such changes there might be negative side-effects too, of course, as happens sometimes when the correlation is weakened by separating boys and girls in special classes.

## 3 Regulating systems

The strategy which sees both the selection of what material is taught as well as how it is taught, as regulated by a language which itself again is an integral part of society, is part of a wider approach. Each sub-system is seen as a language-community. Its language determines which words or concepts are accepted and not accepted, and even regulates the interaction between individuals (See for instance Lacan, Foucault, Luhmann; Witte, 1992).

An example of one such sub-system is a scientific community, in which the interaction between members is structured by a paradigm (Kuhn, 1962). The paradigm determines what the members 'see' and what they presume to be of relevance. In other words, the paradigm differentiates between the accepted and non-accepted words or concepts, and even between who will be 'boss' or not. A change in paradigm can de-regulate all distinctions, and lead to a whole set of new 'bosses', with a whole new set of correlations (e.g. in terms of images such as the distracted professor, the white-maned Einsteinian genius). As is well-known, with scientists there is no correlation between standard interpretations of intelligence and 'bossness' (above a certain level of intelligence). Given the diversity of paradigms this phenomenon is easy to explain, and their relative high rate of change in some cases.

Another example of regulation is presented by the development of our ability to speak a natural language. Children are able to produce all sounds, of all languages, but are regulated by the language of their parents, in which just a few sounds are used. When they grow older, they do not find it easy to produce the more unusual sounds.

Mathematics education clearly also represents a regulating system with a language which regulates its entire organization, the curriculum, the presentation, the interaction in the classroom etcetera. It also regulates the possible actions of the students. The constraints of this language define what will be accepted ways of thinking or acting. Persons or groups which prefer other ways of thinking or acting have to choose: adapt, quit or do poorly.

## 4 The language of mathematics education

Dutch secondary mathematics education underwent many changes in the curriculum and in methods of teaching. The main aim of a recent proposal was to develop a form of mathematics education more open to all learners, including girls. This development went under the label of 'realistic mathematics'. It was deemed necessary to build upon what pupils already know: their experiences in real life, the various concepts and procedures they already developed. Thus many individual 'mathematics' were assumed possible.

Unfortunately, the results of these changes, as far as experimented with, did not have the expected effects. This may be due to a wrong interpretation of what was necessary (a wrong 'model of boys and girls', a wrong 'model of content'). It may also be due to the effects of a pre-existing dominant language, embedded in Dutch society with a strong immune type of resistance to changes. In my study I have found many indications to prefer the latter possibility.

Basing myself on a discussion by Lerman (1986), I distinguish between mathematics organized according to the Euclidian regulating language and according to a constructive form of regulation. In these terms the changes in the Dutch curriculum were aimed at replacing the former by the latter, but eventually came to be subsumed again under the Euclidian form. As the Euclidian form of language is a special case of the constructive form, we might have expected a weakening of the gender-related individual behaviors in and choices of mathematics. No such change could be detected, which supports the idea of a strong immunity level in the case of the Euclidian type of language.

### 4.1 The Euclidian language

The main characteristic of the Euclidian language is the presupposition that we can distinguish objects in our world and represent them as abstractions. The relations between these abstractions should be described, following the relations between the objects. Abstract relations can be expressed as sentences in a formal language, the aim of which is to support people's activities in their environment: e.g. to measure, to compute and to compare.

Unfortunately, given the variety in activities, the relevant sets of sentences usually will be underdetermined as a language. To remove this defect, and to make the languages unique as a tool, a sufficient number of constraints has to be added, usually in the form of axioms. Sometimes axioms can be turned into variables, as in the case of non-Euclidian geometry.

The ability of the Euclidian language to combine sentences in one set is represented by criteria for quality such as coherence, consistency and completeness, and the possibility of proof. Applied to mathematics these properties guarantee "a steadily accumulated body of knowledge, linear, hierarchical, dependable, reliable and value-free. Concepts do not develop, they are discovered" (Lerman, 1986, p. 71), similar to more empirical forms of knowledge accumulation. Or as Lakatos (1977) says: "in deductivist style, ... [m]athematics is presented as an ever-increasing set of eternal, immutable truths." As part of mathematical lore no "Counter examples, refutations, criticism can[...] possibly enter" (p. 142).

Given these assumptions it is only logical to take it as not wholly accidental that so much of mathematics eventually can be used in scientific work (which is based on the same assumption), and to wonder that mathematics sometimes does not seem entirely domain specific (Wigner, 1960). A similar consequence is the pervasiveness of the Euclidian language in nearly all sub-systems of Western society, in which abstraction dominates.

As Lerman (1986, p. 76) warns, the danger of the Euclidian language and of the mathematics based on it, is that one will tend to avoid methods of use that still harbor inconsistent, contradictory elements.

### 4.2 The restrictions on pupils actions

Confrey (1985) argues that if rules, procedures and mathematics are seen as steady, certain and objective, children will perceive their own ideas, procedures, constructions and reflections as inferior and irrelevant. The pupils are under the control of an external authority (the teacher or school-book) instead of under their own internal control, with results that are visible to themselves. Their attitude towards the subject is negative; it will generate apathy and aversion, according to Confrey. The educational process therefore will waste a lot of time and effort, and presumably, much talent that would have had a better chance when accessed by a different language.

Confrey stresses that research shows that, necessarily, in this kind of education "students mathematical knowledge is limited and rigid. They focus on answers; they expect whole number solutions, they lack multiple representations; they rely on memorization and imitation of examples; and their powers of generalization, abstraction, curtailment and flexibility are weak. They believe mathematics is objective, external and absolute" (Confrey, 1985, 477).

Mathematics education based on these ideas only stimulates deductive ways of thinking, while other ways of thinking are not accepted. Pupils who deviate from the use of the Euclidian language way are seen as not mathematically talented. "It has not yet been sufficiently realized that present mathematical and scientific education is a hot bed of authoritarianism and the worst enemy of independent and critical thought" (Lakatos, 1977, p. 142). In our society correlations apparently associate such types of behavior to characteristics of gender.

## 5 The possibilities of constructive mathematics education

We can say that the Euclidian language implies an emphasis on rules, but also that it needs deviations to keep its immune system on the alert. The opposite of course is a language that emphasizes variations, but will allow the development of invariants when the environment allows to do so. It does not emphasize the environment and how one sees it, but the way individuals deal with their environment (including other individuals), using its properties as constraints.

What may be called a constructivist language does not, therefore, help distinguish objects, but is taken as a generator of variable behavior - though some of the invariants of such behavior can be called objects (and some should be called strategies, procedures or methods). It does not need, therefore, to stress a criterion of quality such as completeness. It can only emphasize informational openness, with some possibility of temporarily maintaining organizational stability. There is no a priori need, therefore, for any correspondence with relations between objects in the environment, that is in reality. In fact, we may only expect a posteriori correspondences, to be created: as when designs correspond to their blue-prints.

The constructivist language is used in mathematics to generate what is called, since the beginning of this century, constructivist mathematics. The main initiator has been the Dutch mathematician Brouwer (1907), but many different forms have been developed on the basis of his work too. They all stress the need not only to deduce truths, but also to demonstrate the possibility of actually creating the conditions under which statements become true. Mathematics, in Brouwer's view, does not deal with the representation of objects in reality, but must be seen as the product of human activity, as a mental construction, possibly shared by more than one person.

Lakatos (1977) expresses the same view when he states that mathematical activity produces mathematics. Mathematics grows as a living and growing organism with a certain autonomy from the initial activity. Lakatos argues for using the Greek method of analysis and synthesis, the method of proof and refutation. To legitimize the existence of a mathematical system is not proved by logical consistency, which is usual in Euclidian style. "Logic may *explain* mathematics but cannot *prove* it" (Lakatos, 1978, p. 19). Mathematics therefore includes mathematical statements, but also the "counter-examples which led to their discovery." (Lakatos, 1978, p. 144).

To Brouwer a proof is a mental construction, starting with the concepts which are shown to be constructed. "I do not recognize as true, hence as mathematics, everything that can be written down in symbols according to certain rules, and conversely I can conceive mathematical truth which can never be fixed down in any system of formulas." (Brouwer, 1912, p.452) According to Brouwer mathematics is built up by producing sequences of things in time, on which people can base their actions. Each person can do so in a different way. Brouwer does not exclude the possibility, of course, that people can create mathematics as a shared construct.

### 5.1 A constructivist form of mathematics education

The constructivist language is not yet socially embedded in the sense that there are many different sub-systems of society that share this language, though there are some improvements in this respect. It may be possible however that by emphasizing variation as the main driving force of mathematical understanding, in stead of the lack of it (leading to rules and regularities), the construction of similarly based sub-systems can be stimulated further.

This approach leads to a radically different model of teaching mathematics than is common now, according to Confrey (1985). Mathematics education which adopts this view has to accept that pupils create their own mathematical constructions and activities, in different contexts. An important aspect of this view is the possibility of communicating and sharing certain language elements.

Children will be "constructing their own knowledge, by comparing a new problem, idea, object, hypothesis against their existing experience, and conceptual system" (Lerman, 1986, p. 73). General concepts such as these can be turned into better learning tools. Lerman thus stresses the problem solving approach: "the adoption of this view implies the tendency to see the teaching of mathematics through a problem-solving perspective." (Lerman, 1986, p. 73).

Unfortunately, sometimes the concept of problem solving appears to be so natural that it comes to dominate other approaches. This problem is visible in American education, according to Lerman, but can also be found in the Dutch 'realistic mathematics' curriculum. Other methods, however, are useful as well, such as the possibility of making decisions, or of negotiating concepts (Pask, 1987).

It should be obvious, though it can also be argued in a more closed form, that the use of the constructivist language (and its implementation in mathematics) will lead to a weakening of specific correlations. It can also reduce the amount of unused time and of unused resources of individual pupils, and hence increase their performance. The argument may be facilitated here by imagining the extreme case: each individual being helped to construct his or her own mathematics (possibly action-specific), though exhorted to communicate about these constructions.

In fact, Sutherland and Hoyles (1988) and Turkle (1986) report of two projects in which students were able to choose their own ways of problem-solving. Both projects used a computer environment which is based on LOGO as developed by Papert (1980). Both concluded that girls motivations and performances improved. This result supports the idea that pupils, when provided with a customized environment, will develop a wider variety of competent behaviors, thereby decreasing the presently existing gender-related individual choices for and behaviors in mathematics.

## 6 Conclusion

In this paper I am opting for a 'third' strategy to devise intervention strategies that can bring about a change in the relationship between girls and mathematics. The reason is that there are many drawbacks to, for example, excepting properties of girls (more oriented towards interaction with others than with nature; more caring, etc.), and basing teaching strategies on those properties. The same holds for seeing mathematics as part of a culture, and as prejudiced according to that culture. These approaches restrict pupils' ways of thinking, acting and learning, and induce correlations between various properties that people show as users of societal sub-systems.

This 'third' strategy deserves attention partly due to its potential to avoid such drawbacks, but also due to its vulnerability: whenever implemented it usually quickly is reduced to the 'first' and 'second' strategy. It therefore often could not be explored appropriately.

As has been argued, the use of the constructivist language may help weaken the presently negative correlations between gender and proficiency in and preference for mathematics. Such a weakening will only be permanent, however, when the language is used for a relatively large number of societal sub-systems: not only to select constructivist ideas in mathematics, but also to teach mathematics as an attitude; and also to create societal sub-systems that can enhance immunity to a reduction of the constructivist language to the Euclidian language.

Introduction of the constructivist language may create new 'bosses' in mathematics, even one which may pre-select women. The latter is not what is argued here for, however. Improvement should mean additional ways of teaching and of thinking about mathematics, such that correlations between individual choices for and behaviors in mathematics and other properties disappear.

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# Short Oral Presentations

# The Concept of Speed
## Two case studies in the primary school

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Abstract

In every case there exists a connection between the methodical procedure employed by the teacher and the learning behaviour of the pupils. In particular we are interested to discover in what way the work of the pupils in small groups is influenced by the teacher's behaviour. Special question complexes are aimed at the dependence of single remarks or actions by the pupils

- on directly previous remarks or actions by the teacher or by fellow pupils,
- on the problem or task and the way presenting or representing it,
- on the level of abstraction and the framework situation of the conceptual contexts presently being dealt with.

We observed problem solving processes of pupils of 3rd and 4th grade using the concept of speed. The analysis shows that the pupils prefer a special method in solving the problem, taking a constant distance and looking for the shorter time when they are comparing the speed of two bicyclists. The pupils oppose with this concept the teacher's desire to use an other idea for solving the problem.

## ACHIEVEMENT AND THINKING STRATEGIES ON "REVERSED ITEMS"

Alex Friedlander and Jeanne Albert

The Weizmann Institute of Science

*"Reversed tasks" are considered an important means to promote higher-order thinking. Student performance on a written test interviews and classroom observations were analyzed and led to the conclusion that children were capable of reversed thinking at a reasonable level.*

*The main source of conceptual errors was a mechanical use of the inverse operation and a lack of awareness for the need to estimate and check answers.*

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## THE PROCESSING LOADS AND RELATIONS BETWEEN COUNTING AND PLACE VALUE

GILLIAN. M. BOULTON-LEWIS. SCHOOL OF LEARNING AND DEVELOPMENT.

FACULTY OF EDUCATION. QUEENSLAND UNIVERSITY OF TECHNOLOGY. KELVIN GROVE [1]

***A sample of 55 children in years 1, 2 and 3 in three suburban schools in low to high socioeconomic areas of Brisbane was tested for knowledge of the structure of the counting sequence and for place value. Analysis of the processing load of levels of knowledge of the counting structure and place value was undertaken using Halford's structure mapping theory of cognitive development. Children's responses for counting and place value were analysed and categorized. The relation between counting and place value knowledge and implications for teaching are discussed.***

# FROM RESEARCH TO TEACHING:
## An analysis of students' performance on calculus

*Ricardo Cantoral-Uriza*
MEXICO

Departamento de Matemáticas
Instituto Tecnológico Autónomo de México. ITAM
Sección de Matemática Educativa
Centro de Investigación y de Estudios Avanzados del IPN. CINVESTAV

### I. Abstract

*Within the framework of a study to explore the possibilities of modifying the curricula, taking into account results in the research fields of psychology of mathematics education and epistemology we have analyzed some paths to make the transit form research situations to teaching situations possible. This paper reports on the result obtained when teaching infinite numeric sequences during a first course in college calculus, with 34 applied mathematics students (18 and 19 years old) at ITAM.*

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# METAPHORS FOR THE MIND AND THE WORLD IN THE PSYCHOLOGY OF MATHEMATICS EDUCATION

Paul Ernest
University of Exeter

In the psychology of mathematics education the range of competing research paradigms includes Traditional Empiricism and Information Processing, Radical and Social varieties of Constructivism. Beyond the well known epistemological differences between them, these paradigms differ in their underlying metaphors for the mind and the world. Recognition of these metaphors and their strengths and limitations allows for a fuller evaluation of the research paradigms, and may be a significant factor in further progress in the field.

# THE PSYCHOLOGY OF PUPIL'S INTELLECT DEVELOPMENT IN THE PROCESS OF TEACHING MATHEMATICS

E.G. Gelfman, L.N. Demidova, M.A. Kholodnaja[*], N.B. Lobanenko,
J.J.Wolfengaut
Tomsk State Pedagogical Institute, Russia
[*]Kiev State University, Ukraine

**Abstract**. *New technology of teaching school mathematics aimed at pupils intellect development and realized in a series of special training aids is discussed. In this technology the following principles are the basis of teaching influence:*

*1) considering psychological peculiarities of the process of the notions mastering ;*
*2) forming of the basic intellectual qualities ;*
*3) providing the psychologically comfortable regime of intellectual activity;*
*4) developing pupils' metacognitive information.*

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# IMPLEMENTING THE NCTM STANDARDS: RECONCILING THE PLANNED IMPACT WITH THE EXPERIENCED REALITY IN AN URBAN SCHOOL DISTRICT

Rochelle G. Kaplan and Mahendra Jani, William Paterson College
Amy Schmidt, City University of New York

This paper describes the effects of an inservice mathematics education program designed to improve the content knowlege and pedagogical skills of elementary-certified mathematics teachers in grades 6 - 8 of an urban school district. Workshop topics and delivery formats were chosen to be consistent with the NCTM driven changes in the district's curriculum. The broad purpose of the analyses here is to compare initial expectations for meeting our goals with participants' realities and expectations and to decribe how discrepancies between them can affect program outcomes.

## CHILDREN USING THE TURTLE METAPHOR TO CONSTRUCT A COMPUTATIONAL TOOL IN A GEOMETRICAL LOGO MICROWORLD.

**Chronis Kynigos**
University of Athens, School of Philosophy,
Dept. of Philosophy, Education and Psychology

**ABSTRACT:** *A case - study is reported, investigating how two children, working in a Logo microworld environment, constructed a generalised computational tool for an isosceles triangle by initially using computational measuring instruments to construct the figure, based on very little knowledge of its properties, and then exploring these in the process of generalising from their measurements. The children, aged 11 - 12, were Logo - experienced and worked for a total of 15 hours. The evidence from the study supports the argument that powerful computational tools in a Logo geometric environment may be based on the turtle metaphor, enabling children to employ the intuitions which seem to accompany its use.*

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## UNDERSTANDING MATHEMATICS CLASS DISCUSSIONS

Jane-Jane Lo, Arizona State University West

Grayson H. Wheatley, Florida State University

### Abstract

In this paper, we present our current understanding of mathematics class discussion based on a year-long interpretive study. First, we identify the disagreement in mathematics class discussion as one possible source for creating potential learning opportunity. Then we try to trace the origin of the disagreement and spell out the important elements so a disagreement can become constructive discussion. In conclusion, we re-emphasize the importance of negotiation and provide our views of the goals for mathematics class discussion, in particular, and for mathematics instruction, in general.

# Cognitive Origins of Students' Errors in Writing Equations

Mollie MacGregor and Kaye Stacey
Institute of Education, University of Melbourne, Australia

*Information about the errors that students are known to make when writing equations was obtained from the literature. A set of test items was designed to eliminate as far as possible the factors believed to cause these errors. The test items were given to 281 year 9 students in seven schools. The high incidence of errors in the students' responses could not be attributed to documented causes. A theory of cognitive models was proposed to account for the errors as well as for the various forms of correct equations. According to the theory, cognitive models of relations between two variables represent contrast between large and small entities and do not conform with the structure of equations. Errors are made when students try to represent these cognitive models directly.*

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# TOWARDS APPLIED PROBLEM SOLVING *

Nicolina A. Malara, Department of Mathematics, University of Modena, Italy
Consolato Pellegrino, Department of Mathematics, University of Modena, Italy
Rosa Iaderosa, "Leonardo da Vinci" Middle School, Cesano Boscone (Milano), Italy

SUMMARY: *In this paper we present the guidelines and the first results of a research, carried out with 11-12 year olds, aimed at improving their ability to mathematicize problem situations and to improve their attitude and performance when faced with "the problem". The research is based on a certain number of key problems which depict realistic situations and in which the pupils are required to understand which are the questions to be asked, which are the relevant data and the difficulty or cost involved in obtaining them. Furthermore, the problems are presented in such a way as to induce the pupils to "optimise" their solution methods. The first results show that the pupils acquire a broader understanding of the concept of "the problem" and acquire the ability to approach it hypothetically, thereby enhancing their metaknowledge and improving their ability to solve problems.*

# Visual Estimation

Zvia Markovits and Rina Hershkowitz

The Weizmann Institute of Science, Israel

*Visual Estimation occurs when one is presented with a large group of objects for a short period of time and is asked to evaluate their number. Four strategies were expressed by three grade children in visual estimation situations: counting, grouping, comparison and global perception. After going through some visual estimation activities, some changes in children strategies were found.*

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# WRITTEN MATHEMATICAL COMMUNICATION: THE CHILD'S PERSPECTIVE

Candia Morgan, South Bank Polytechnic, London

*Written work in mathematics is a major basis for teachers' evaluations of children's mathematical activity and understanding. There is, however, a possibility of a mismatch between teachers' and children's understandings of the purposes and the conventions of such written work which may lead to lack of communication and hence to inappropriate assessments. In this paper, some examples are offered of children's use of particular forms of written communication in mathematics and their beliefs about the functions of these forms. These are discussed and the implications for effective communication considered.*

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## ROTATING CANDY BARS AND REARRANGING ORANGES: A STUDY OF CHILDREN'S UNDERSTANDING OF COMMUTATIVITY

Terezinha Nunes & Peter Bryant
Institute of Education, London, & University of Oxford

*Two studies investigated the hypothesis that different aspects of multiplication may be more easily understood in different contexts. Children correctly solve isomorphism of measures problems earlier than product of measures but the latter problems offer a better context for understanding commutativity. Although 9/10 year olds in our studies showed some difficulty in using commutativity appropriately, in both studies they performed better in product of measures than in isomorphism of measures problems.*

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## INTERPRETING CHILDREN'S UNDERSTANDING OF GEOMETRIC CONCEPTS: A COMPARISON OF THE VAN HIELE THEORY AND THE SOLO TAXONOMY.

John Pegg, Department of Science, Technology and Mathematics Education, University of New England, Armidale, NSW, AUSTRALIA
Geoff. Davey, Division of Education, Griffith University, Brisbane, QLD, AUSTRALIA.

*The van Hiele Theory has become a popular framework, used by researchers, to help explore and identify student growth in geometry. Less well known, but also used to investigate geometric understanding, is the SOLO Taxonomy of Biggs and Collis. This paper compares the categorisations provided by both theories in the light of the findings of a series of studies undertaken by the authors. The findings support the view that : (i) both theories share many common traits; and, (ii) the SOLO Taxonomy, with its modes of functioning and levels of attainment associated with each mode, provides a broader base to describe student behaviour than the van Hiele Theory.*

# ARE THERE ANY DIFFERENCES IN PUPILS' CONCEPTIONS ABOUT MATHEMATICS TEACHING IN DIFFERENT COUNTRIES?

## The case of Finland and Hungary

Dr. Erkki Pehkonen, Department of Teacher Education, University of Helsinki
Dr. Klára Tompa, National Centre for Educational Technology, Budapest

*Summary:*

*This paper reports on an international comparison of seventh graders' conceptions of mathematics teaching. A questionnaire survey was made of the conceptions of a total of 200 pupils in Hungary and Finland. The main results were, as follows: The Finnish pupils were more in favour of calculation-centered working, where the teacher is always in control of the learning process than their Hungarian counterparts. The Hungarian, in contrast to the Finns, placed greater emphasis on exact teaching methods which strive for understanding according to pupils' capabilities, they also stressed computational aspects of mathematics, such as rapid performance, correct answers, the memorisation of rules and a belief in the existence of proper procedures.*

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# Introducing Calculators to Six-Year-Olds: Views on Support for Teachers.

**Pat Perks,**

**School of Education, University of Birmingham, U.K.**

*This paper describes one aspect, that of support for teachers, of an evaluation of a research project in a large City, where funding had been received to distribute calculators to 100 schools with classes of six-year-olds, a third of the schools in a Local Education Authority. Data used comes from two strands of an evaluation of the project, interviews with those staff in what were designated as pilot schools in the project and questionnaires to other teachers of middle infant (Y1)[1] pupils. The data used refers to teachers' views on the support they would have liked to have received as part of the project. The findings show differences in the perceived needs for support between the different groups.*

# HIERARCHIES OF COGNITIVE DIFFICULTY IN EARLY ALGEBRA

Br.Cyril Quinlan, fms, Australian Catholic University - N.S.W.

*Hierarchies of cognitive difficulty for several tasks in early algebra were proposed following logical task analyses based on established psychological principles. Specific statistical criteria were set in order to examine whether or not written test responses could provide empirical support for the propositions. Data were collected in 1990 from 517 students spread across Years 7 to 12 secondary classes. Their responses did support several propositions, three of which are discussed in this paper.*

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# Pupil Strategies for Solving Algebra Word Problems with a Spreadsheet

Teresa Rojano, Research Centre for Advanced Studies, Mexico/PNFAPM
Rosamund Sutherland, Institute of Education University of London

*In this paper we report the results of a two year collaborative Mexican/British project which aims to investigate the ways in which pupils (aged 10-11) use a spreadsheet to symbolise and solve algebra word problems. The study has shown that although most pupils of this age do not spontaneously think in terms of a general rule when first presented with a spreadsheet environment, they can learn to do so through working on a range of activities which focus on symbolising and generalising. Pupils communicated these general rules either by pointing to the spreadsheet cells or by incorporating the symbolic spreadsheet language into their talk. When using the spreadsheet to solve the word problems pupils were taught to use an approach which involved: representing the unknown by a spreadsheet cell; expressing the relationships within the problem in terms of this unknown; varying the unknown to find a solution. In the paper we describe the ways in which the pupils linked their own informal strategies with this more formal algebraic approach.*

# Children's Developing Concepts of Landmarks in the Number System[1]

Andee Rubin and Susan Jo Russell
TERC, Cambridge, MA

### Abstract

This paper describes what we believe is an important aspect of the notion of "number sense": landmarks in the number system. We believe that people who are adept with number operations - computating, comparing, and estimating - have a non-uniform view of the whole number system. In our recent work with third and fourth graders in an urban public school, we have begun to investigate aspects of this critical landmark knowledge. In particular, we have been looking at multiples of 10 and especially at 100 as key landmarks in the development of number sense at this age. Four aspects of these landmarks are described: additive structure, multiplicative structure, the generation and analysis of mathematical patterns, and the construction of mathematical definitions.

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# EQUATIONS AND INEQUALITIES — PROCESSES WITHOUT OBJECTS?

Anna Sfard and Liora Linchevsky
The Hebrew University of Jerusalem

This paper is a continuation of [2], which dealt with the introductory part of our present study. In our investigation, we tried to test certain hypotheses regarding students' understanding of equations and inequalities. In [4] and [5], a distinction has been made between operational and structural conceptions on one hand, and pseudostructural conceptions, on the other hand. In the present study the main focus is on the latter. In the case of equations and inequalities, we say that the student thinks in a pseudostructural way if the propositional formulae are conceived just as strings of semantically void symbols, for which the formal transformations used to find the solution are the only source of meaning. This approach to algebraic symbolism seems deceitfully close to the views on algebra endorsed by such mathematicians as Peacock, deMorgan and Hilbert. The difference between this and our students' positions is thus carefully studied and explained. Our investigation, carried out among secondary school pupils, shows that the pseudostructural conceptions may be more widely spread than suspected.

# Assumptions and intentions in distance learning materials for mathematics

Christine Shiu, Open University, UK

*The curriculum materials used as the basis for Open University distance taught courses are planned and prepared by course teams. In this paper a description is given of the use of reflective debriefing in the final session of a three-day meeting held to draft the structure for a new mathematics foundation course. Written responses from the participants are examined for the individual assumptions and intentions they reveal.*

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# WRITING IN MATHEMATICS:
# IS IT ALWAYS BENEFICIAL?

Kevan V. Swinson & Barry D. Partridge

School of Mathematics, Science & Technology Education

Queensland University of Technology

### *Abstract*

*The positive effects of using writing as a means of reflection in mathematics have been well documented. The research reported in this paper examines the effect on students' comprehension of a unit of statistics, when writing was used as a vehicle for reflection. In this project the writing activities were conducted over a limited period of time. The results of the investigation suggest that when writing is used for a limited period it may be detrimental to student learning.*

# THE IMPACT OF UNDERSTANDING AND EXPECTATIONS OF PERFORMANCE ON COLLEGE STUDENTS' SELF-CONFIDENCE

Anne R. Teppo
Montana State University

*This article examines the impact of understanding and expectations of success on students' self-perceptions of a mathematical learning experience. Two groups of students in a unique college general-education mathematics class were identified on the basis of self-descriptions of their mathematics ability. As a result of acquiring new levels of mathematical understanding, it was found that some students improved their self-perceptions of ability and exhibited more positive affective responses to the course than did other students. It is postulated that these differences are related to whether or not the students' initial expectations of success and perceptions of mathematical ability were contradicted by their actual classroom performances. These findings suggest that more attention be paid to providing significant learning experiences to the "less mathematically able" student.*

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# Experiences and effects of realistic mathematics education: The case of exponential growth

Anno van Streun Department of Mathematics
P.O. Box 800 9700 AV Fax 31 50633976
Groningen The Netherlands

### *Abstract*

*This report deals with a part of ongoing classroom research about the transition from a rather traditional curriculum into a new national curriculum, based on the principles of realistic mathematics education. A radical change in the curriculum, as we're involved in the Netherlands, requires and provokes a change in the attitudes and beliefs of students and a change in the attitudes and beliefs of their teachers. The predominant resource used by teachers is the basic textbook. So we're developing new textbooks brought into line with the new goals and classroom research findings. We will report about the learning processes of students, the teachers attitudes and working methods and the influence of the structure of the new teaching materials. The examples come from the domain of exponential growth or exponential functions (grade 10).*

# IKONIC AND EARLY CONCRETE SYMBOLIC RESPONSES TO TWO FRACTION PROBLEMS

Jane M. Watson, Kevin F. Collis and K. Jennifer Campbell
University of Tasmania (Australia)

*Summary*

*This paper will present responses from young children to illustrate the ikonic and concrete symbolic modes of functioning in the SOLO Taxonomy. The responses were part of a larger study of common and decimal fraction understanding in children from Prep to 10. They are used here to illustrate the transition from ikonic to concrete symbolic functioning and the continued ikonic support available in the first cycle of the concrete symbolic mode.*

# Conceptual Splatter and Metaphorical Noise: The Case of Graph Continuity

Steven R. Williams and Sharon B. Walen
Washington State University

## Introduction

The investigation reported here grew out of a curiosity about the nature of students' views of graphs, particularly as they relate to questions of continuity and limit in calculus. Although considerable attention has recently been given to the way students relate graphs to functions, (Romberg, Carpenter, & Fennema, in press; Leinhardt, Zaslavsky, & Stein, 1990), this research deals more with the question of what students believe a graph *is*, or stated more informally, what a graph is made of. Previous research (Williams 1991) suggested that students had an implicit faith in the continuity and regularity of graphs and that they relied on this faith to answer questions about limits and the behavior of functions near points. In an effort to determine the nature of this faith, 81 students from four precalculus classes were given a questionnaire about graphs and graphing, and 10 of these students were selected for more in-depth interviews. Two classes were chosen from a school which used a graphing-calculator-based curriculum, and two were chosen from a school using a more conventional curriculum, in order to take into account possible effects of graphing technology. Although some interesting similarities were noted across the ten students interviewed (Williams, Walen, & Cockburn, in preparation), this report deals only with data from the interview with one such student, called Anne. Anne was considered unexceptional in most respects, and representative of the other nine students interviewed.

# TRYING THE THEORY ON THE DETERMINATION TO STUDY

## - Applying mathematical activities based on varied problem solving -

Minoru Yoshida
The Center of School Education
University of Tsukuba

Making mathematics funs is an essential theme for organizing mathematical activities.

To attain this theme, students in Japan are taught to solve a single problem with various methods.

This is exceptionally effective when one considers the mental structure behind the dislike of mathematics in Japanese children and students.

This paper argues this effectiveness in terms of the interpreting process for literary texts while relating the occurrence of fun with metacognition and its mechanical control.

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# Inverse of a Product - a Theorem out of Action.

**Rina Zazkis**
Faculty of Education, Simon Fraser University,
Burnaby, B.C. Canada V5A 1S6

*This study investigated the mathematical behavior of college students in a problem situation that requires them to find the inverse of a given compound element. The problem was posed in an environment of computer drawings, Logo Turtle graphics. Clinical interviews conducted with 30 students focused on their process of problem solving. The results demonstrate that in a search for the compound inverse, even "mathematically literate" students tend to invert the elements, but not their order. The discussion suggests possible explanations for this phenomenon.*

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# Poster Presentations

# TEACHING MATHEMATICS IN THE FIRST YEARS OF ELEMENTARY EDUCATION: KAMII'S PROPOSAL IN ACTION.

ROBERTO BAROCIO QUIJANO Y JAVIER PREÑA SANCHEZ.
SCHOOL OF PSYCHOLOGY,NATIONAL UNIVERSITY OF MEXICO

Accepting the fact that teaching mathematics at the elementary level has been a matter imbedded in an inappropiate vision about the manner in which children approach their own learning,and about the manner in which teachers can support such learning,Kamii (1985 & 1989) has been able to work a proposal which takes into consideration the pedagogical implications that Piagetian theory has in teaching mathematics.Nevertheless,actual implementation of the proposal involves solving a previous problem:that of training.

This paper describes an experience in training a group of teachers who work in the first and second grade of elementary education.The main purpose of the experience was to inform the teachers about the genetic approach in its relation to teaching mathematics (Kamii,1985).

The refered experience involved training activities (seminars and miniseminars) and training support activities (classroom planning,observation and feedback,meetings with the teachers,etc.) The effect of the program on the behavior of children and teachers was assessed through a control-experimental group design and pre-test post-test measures.

Results indicate in general,that the training experience was succesful in terms of its main purposes.In comparison to traditional approaches,specific effects of Kamii's proposed approach are shown as related to children's construction of numerical concepts.

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# ANALOGICAL REASONING: BASIC COMPONENT IN PROBLEM SOLVING ACTIVITIES

Gerhard Becker
University of Bremen

## *Summary*

*Students' impressions towards problem solving is very often determined by anxiety from being overcharged, having its roots in a feeling of not being familiar with the applicable means. Extending or modifying a starting problem by using analogy transfer successively amplifies both the area of problems and the repertory of means to tackle them. Additionally, single steps in problem solutions can be obtained by analogical reasoning.*

*Analogy, as a correspondance between relations, can be specified by reducing or extending complexity of a topic, the number of parts determining the topic, exponents in equations, function terms, inequalities or dimensions of parts of a geometrical figure, changing number or kinds of parameters determining the topic, forming new combinations of partial premises or partial conclusions. These categories are illustrated by examples.*

# PUPILS' EVALUATIONS OF LEARNING ACTIVITIES

Alan Bell, Rita Crust, Ann Shannon and Malcolm Swan
Shell Centre for Mathematical Education
University of Nottingham, UK

ABSTRACT: As part of the evaluation of a 2-year project aimed at investigating and enhancing pupils' awareness of their learning in mathematics, pupils in nine secondary school mathematics classes responded to a questionnaire concerning the evaluation of different classroom mathematics activities. These covered whole-class learning, reflective activities, pair and group discussion, problem solving and mathematical investigation.

The responses required covered the frequency and the perceived value of each activity. Results showed, amongst other points, that listening to teacher's explanations was the most frequent and one of the most highly valued activities, while discussing ones own mistakes was the most valued though it occurred with only middling frequency.

# RACE AND GENDER INTERACTIONS AND CONSTRUCTIVIST TEACHING

SARAH B. BERENSON
NORTH CAROLINA STATE UNIVERSITY
Raleigh, NC 27696-7801

## Introduction

Student performance on the Scholastic Aptitude Test (SAT) is a high priority for high school mathematics teachers. In the midst of calls to reform the teaching and learning of mathematics (National Research Council, 1989; National Council of Teachers of Mathematics, 1989) teachers are reluctant to abandon traditional instructional methods that have produced "satisfactory" SAT-Math scores. This study enlisted 13 teachers as partners in action research to compare active learning strategies and traditional methods and their effects on SAT-Math scores. The research questions addressed in this poster concern the relationships between active learning strategies with respect to gender and race on SAT-Math items.

A number of studies have indicated that African American students may not benefit from traditional instruction based on lecture methods (Stiff, 1990; Dossey et al., 1988). Stiff (1990) concluded that small group activities, laboratory work, and other innovative forms of instruction may benefit African-American students. The individual competitiveness found in traditional instruction may place females at a disadvantage (Linn & Peterson, 1985), and cooperative learning and other methods that draw on students' verbal strengths may increase mathematics achievement for females.

## Method

The investigation involved 30 intact college preparatory mathematics classes with 448 white students and 145 African American students. Teachers used active learning strategies including cooperative learning, journal writing, problem solving, problem-solving skills, and SAT coaching with their experimental classes and taught the comparison classes with their traditional methods. Pre and posttests were given before and after the eight-week treatment period and post-posttests were administered three months after the posttest. Three previously administered SAT-Math tests, taken from *10 SATs* (College Board, 1990) were used in combination as measures.

## Results and Discussion

Teachers reported that they spent between 1.6 and 16.8 times as many minutes on the combined treatment methods in the experimental classes as in their control classes. Correlations between gain scores and treatment minutes were computed and one-sided hypothesis tests were conducted to test if treatments were positively associated with gains. For all students, gains were associated with more time spent in cooperative learning, journal writing, problem solving problem-solving skills, and total treatment. These relationships were strongest over the five month period (pretest - post-posttest). In a second analysis of all students, male gains after the 8 week treatment were associated with more time spent in journal writing, problem solving and SAT coaching. Female gains were associated with problem solving and SAT coaching in the first eight week treatment period. Experimental white students' gains were associated with more time spent on problem solving and problem-solving skills, while experimental African American gains were associated with more time spent on cooperative learning and journal writing.

In viewing the combined results of the amount of time spent in class on experimental treatments, I concluded that more time spent on the experimental treatments was associated with higher gains. Females tended to benefit during the first 8 weeks of the treatment, while male gains were delayed. Further, it was noted that African American students responded differently to the treatments; their gains were associated with active verbal strategies. Summarizing these findings, I concluded that there is no "best" active learning method of teaching high school students, but white and African American, male and female students would benefit from a variety of active learning activities.

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# Schematic structures of mathematical form

Christer Bergsten
Linköping University, Sweden

*Abstract: In this theoretical paper figurative aspects of mathematical symbolism are discussed and related to the notion of image schemata. The analysis shows that the importance for the understanding of mathematics of understanding mathematical form (referring to spatial characteristics of mathematical formulas), and form operations, should not be underestimated.*

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# TRANSFORMATIONS OF FUNCTIONS USING MULTI-REPRESENTATIONAL SOFTWARE: VISUALIZATION AND DISCRETE POINTS

Marcelo Borba Jere Confrey
Dept. of Education Cornell University

## *Abstract*

*This paper will report preliminary findings of a research project which focuses on how students understand transformations of functions using multi-representational software. Transformations were taught starting with the visual and graphical forms and extending to table data and algebra equations. Specifically this paper will present how one student, Doug, interacted with a multi-representational software. Doug used points sampled from a continuous graph and the linkage of these points with the table as a strategy for understanding of the stretch of a function.*

# Relational Thinking and Rational Numbers

David William Carraher

Universidade Federal de Pernambuco
Recife, Brazil

*Tasks for learning about rational numbers unfortunately often sustain additive thinking even though the relations inherent in the problems are multiplicative. Special tasks are described that call for relational thinking about number and quantity and can not be solved through additive means nor rote computation. In the model underlying the tasks, rational numbers acquire meaning as operators and measures through specifying actions upon quantities. As they come to describe relations of quantities and variables, they become imbued with algebraic meaning.*

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# F(X) = G(X)?: AN APPROACH TO MODELLING WITH ALGEBRA

Daniel Chazan
Michigan State University

*The paper argues that equations can usefully be conceptualized as the comparison of two functions and that such a view helps make algebraic modeling more accessible. This assertion is then examined in the light of three sample algebra word problems.*

# THE OTHER CONSEQUENCES OF A PROBLEM-BASED MATHEMATICS CURRICULUM

David Clarke and Margarita Wallbridge
*Mathematics Teaching and Learning Centre,*
*Australian Catholic University (Victoria)*

Sherry Fraser
*EQUALS, Lawrence Hall of Science, Berkeley*

## Summary

*Implementation of a problem-based mathematics curriculum (IMP) at three high schools in California has been associated with more than just differences in student achievement. Among the outcomes which distinguished those students who participated in the IMP program from those who followed a conventional algebra/geometry syllabus were the students' perceptions of the discipline of mathematics, of mathematical activity and the origins of mathematical ideas, of the mathematical nature of everyday activities, and of school mathematics and themselves as mathematicians. A coherent and consistent picture has emerged of the set of beliefs, perceptions and performances arising from such a program. Students who have participated in the IMP program appear to be more confident than their peers in conventional classes; to subscribe to a view of mathematics as having arisen to meet the needs of society, rather than as a set of arbitrary rules; to value communication in mathematics learning more highly than students in conventional classes; and to be more likely than their conventionally-taught peers to see a mathematical element in everyday activity. These outcomes occurred while the IMP students maintained mathematics achievement levels at or above those of their peers in conventional classes. Since the achievement outcomes are comparable, the mathematics education community must decide whether it values these other consequences of a problem-based curriculum.*

# Design of a Logo Environment for Elementary Geometry

Douglas H. Clements Julie S. Meredith Michael T. Battista
State University of New York at Buffalo Kent State University

We are engaged in a 5-year curriculum development project, funded by NSF, that emphasizes meaningful mathematical problems and depth rather than exposure.[1] Our responsibility is developing the geometry and spatial sense strands. Students will engage in investigations involving a wide range of approaches and materials; we are designing a modified Logo environment specifically for the project. The design is based on curricular considerations and a number of implications for the learning and teaching of geometric concepts with turtle graphics (Clements, 1991; Clements & Battista, 1990, 1991).

1. Allow students to build on their visual strengths, but concomitantly demand increasingly complete and precise specifications and analytic thinking. Computer geometry tools need to constrain students' actions. They must serve as a transition device, connecting the intuitive and the visual to the more abstract.

2. Realize the potential of ostensibly simple tasks (e.g., drawing a "tilted" rectangle in Logo); overly complex tools may not be necessary.

3. Establish close ties between representations—Logo code, the action of the turtle, and the resultant figure.

4. Provide tools for easy editing, repeating, "undoing," "stepping," etc.

5. Encourage students to create procedures, alter them, and reflect on them. Highlight procedural-conceptual connections.

6. Allow students and teachers to pose and solve their own problems, encouraging exploration and conjecture.

7. Explicate the theoretical model underlying the design.

These results effectuated the following features of our Logo environment. Students enter commands in "immediate mode" in a command center, or as procedures in a "teach" window. Any change to commands in either location, once accepted, are reflected automatically in the drawing. Because the sequence of commands in the command center reflects what is displayed graphically, they represent a potential procedure. A palette tool copies these into the teach window, applies a child-supplied name, and thus defines the procedure.

A step function allows children to "walk through" any procedure, or the commands in the command center. As they do so, they may edit any command in the sequence. The change is immediately reflected on the screen.

Tools are provided on a palette for defining procedures, stepping, measurement, and other sundry details.

Of course, no environment operates in a vacuum; the larger classroom environment must encourage the necessary constructions and connections. Finally, development of computer environments for learning geometry should be synergistically integrated with research efforts. Researchers must discover how we can systematically build on the geometric knowledge students learn each year.

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[1] "Investigations in Number, Data, and Space: An Elementary Mathematics Curriculum" is a cooperative project among the University of Buffalo, Kent State University, Technical Education Research Center, and Southeastern Massachusetts University. National Science Foundation Research Grant NSF MDR-9050210.

# MULTIMODAL FUNCTIONING IN MATHEMATICAL PROBLEM SOLVING

Kevin F. Collis, Jane M. Watson and K. Jennifer Campbell
University of Tasmania (Australia)

## *Summary*

*Problem-solving in school mathematics has traditionally been considered as belonging only to the mode of thinking concerned with making logical connections between data and the mathematical model and then teasing out the relationship between the variable in the model and the concrete symbolic mode. Little, if any, attention has been given to the place of the intuitive processes of the ikonic mode at this level. This project has set out to explore the interface between logical and intuitive processes in the context of mathematical problem-solving. The paper will present some of the results obtained in the early stages of the study.*

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# THE IDEA OF VARIATION AND THE CONCEPT OF THE INTEGRAL IN ENGINEERING STUDENTS: SITUATIONS AND STRATEGIES.

**Francisco Cordero-Osorio**
**Centro de Investigación y Estudios Avanzados del IPN.**
**Instituto Tecnológico Autónomo de México.**
**México.**

*We present results of a research and after having identified some significances of the integral in teachers of Engineering schools, we constructed an experience of controlled teaching with Engineering students, analyzing situations that relate notions, as well as strategies deriving from them. This experience was made under conditions of continuous variation related to the integral.*

# The Role of Metacognition in Mathematical Problem Solving Among Ph.D. Mathematicians

Thomas C. DeFranco, University of Connecticut

*This descriptive study involved two groups of Ph.D. mathematicians. Group A consisted of 8 mathematicians who have achieved national or international status in the field of mathematics. Group B consisted of 8 mathematicians who have not achieved such a reputation. Its objectives were: 1) to describe the problem-solving strategies used to solve four mathematics problems and 2)to examine and contrast these individuals' metacognitive knowledge in relation to mathematics and the problems they attempted to solve. Thinking aloud, introspective and retrospective techniques were used to gather information using a Person-Strategy-Task (PST) Questionnaire and a Problem-Solving Booklet. The results indicated that subjects in group A solved the problems more accurately and possessed a dissimilar metacognitive knowledge base than their counterparts in group B. The problem-solving performance of subjects in both groups seemed to be influenced by their metacognitive knowledge.*

# CHILDREN'S ACCEPTANCE OF THEOREMS IN GEOMETRY

Michael de Villiers, Faculty of Education, University of Durban-Westville, Durban, 4000, South Africa

As part of a previous study (see De Villiers, 1991), the author asked high school children to judge 42 geometric theorems from the formally prescribed syllabus according to the following statements: Code 1: Believe it is true from own conviction; Code 2: Believe it is true because it appears in the textbook or because the teacher said so; Code 3: Do not know whether it is true or not; code 4: Do not think it is true. It was then found that the certainty or conviction of the majority of pupils (50% - 70%) seemed to be based on Code 2, i.e. authoritarian grounds, rather than on personal conviction.

In an attempt to verify the above findings in township schools in the Durban area, and also to gain further information about why children made certain choices, each of the four categories above were subdivided into three or more subcategories to give a total of 19 subcategories, and in August 1991 139 Grade 9 - 10 high school children from 7 different schools were asked to evaluate 15 geometric statements according to them. In contrast to the previous study, four false statements were also included to see whether, and why, they would choose category 4. A preliminary analysis of the data confirmed the previous finding, and also indicated that children do not easily identify false statements, uncritically accepting them as true.

**Reference**

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# EXPLORING MATHEMATICAL BELIEFS

Brian A. Doig

The Australian Council for educational Research

There have been many mathematics attitude inventories produced over the last two decades or so all of which have added to our knowledge of the röle of affective factors in mathematics learning. However they are all flawed. This flaw is in the interpretation phase. While the developers of each instrument have gained considerable knowledge about the group with whom the instrument was developed, users are faced with the means and standard deviations of their sample of subjects. This is not helpful! Consider a five-point Likert scale, (1 ~ 5). What does it mean for two groups to have means of 3.5 and 3.8? Is there any significance in the difference? What useful information can be gleaned from this?

The fault lies not with the various instruments of course but with the statistical procedures used to encapsulate the data. Traditional analyses do not give sensible results in the case of these rating scales. What can be done? The answer lies in the use of item response techniques especially designed for the analysis of rating scales. With these procedures the probability of any given response can be judged against the individual's total score. That is, knowing the over all score suggests the likely response to any given question on the instrument. Graphical layouts of the analysed data are then used to make clear the situation contained in the data.

The benefits of this approach are enormous as such a visual assessment can be performed quickly giving an immediate guide to a student's belief structure on all questions, thus helping to identify individuals and groups with particular problems or attitudes.

One such graphical analysis is given, with sample interpretations, on this poster.

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# AN ASSESSMENT OF STUDENTS' GRAPHING STRATEGIES IN A TECHNOLOGY-RICH ENVIRONMENT

Christine L. Ebert
University of Delaware

This study deals with an assessment of the graphing strategies of college students enrolled in a one-semester pre-calculus course which utilizes several forms of technology to enhance the learning environment. Students were assessed on a wide variety of graphing tasks which are characteristic of the translations between the algebraic and graphical representations. The primary objective of this study was to document evidence of the strategies students use to translate from a graphical representation to an algebraic representation. The findings suggest that these strategies fall within particularly well-defined categories which may be exploited in instruction.

# TEACHING MATHEMATICS EDUCATION AT A DISTANCE: THE DEAKIN UNIVERSITY EXPERIENCE

N.F. Ellerton and M.A. Clements
Deakin University, Geelong, Victoria, Australia

Over the past five years, Deakin University (in the state of Victoria) has developed the most comprehensive and popular set of professional development programs for practising teachers of mathematics in Australia. Altogether, about 500 practising teachers from all states and territories of Australia are enroled in mathematics education units that form part of the Deakin postgraduate Bachelor of Education, Graduate Diploma in Mathematics Education, Master of Education, and PhD or EdD programs.

In this poster session, a summary of the following four distinctive features of the Deakin mathematics education off-campus programs will be presented.

1. All of the programs, including the doctoral programs, are offered in the off-campus (distance) mode. There are no compulsory attendance requirements.
2. There are no formal written examinations.
3. For each unit they undertake, students receive a Unit Guide, and associated monographs, software and audio- and/or video-tapes.
4. Assessment is based on a range of investigative reports made by students on tasks that relate to their everyday professional lives. The aim is to involve students in collaborative investigations in which they (a) reflect on their current practices, (b) design and carry out programs in which they modify or change their existing practices, and (c) then evaluate the effects of the changes. These evaluations give rise to further cycles of change. Students are expected to read widely in relevant mathematics education research literatures, and to apply theory to practice.

Deakin University mathematics education Unit Guides and Unit monographs will be available for inspection during the poster session. Over the past three years, 25 mathematics education research monographs have been published by Deakin University Press, and these provide the basic reading for the mathematics education professional development units. The monographs have been favourably reviewed in international mathematics education research journals, and it could be argued that they represent the most scholarly and comprehensive set of mathematics education research monographs ever published ... anywhere. Writers with international reputations, from both within and outside Deakin University, have been commissioned to write on topics in which they have special expertise. While the Unit Guides and the Unit monographs are linked, in fact the monographs can stand alone; this is evidenced by the fact that many of them are prescribed reading for students taking mathematics education research courses offered by other universities.

In 1990 Christopher Ormell, the English mathematics educator, was commissioned to carry out an independent evaluation of the Deakin University off-campus mathematics education programs. In his report, which he wrote after a large-scale investigation in which he interviewed many teachers taking the Deakin programs, he commented on the enthusiasm, commitment, changed teaching approaches, and high quality written submissions of the teachers taking the courses. He pointed out that the case of Deakin University casts serious doubt on the validity of the commonly held belief that the quality of learning by distance is usually inferior to the quality of learning in the on-campus, face-to-face mode.

# Communication process in learning mathematics

Hideyo EMORI. Graduate School of Education. UT. Japan
Nobuhiko NOHDA. Institute of Education. University of Tsukuba.

## summary

Small groups activities and whole-class discussions provide opportunities for students to raise new questions. discuss and share their ideas. and summarize their discoveries in writing when solving problems by themselves. We do not doubt that these kinds of activities are very important for all students who are learning mathematics.

To introduce small group activities and whole-class discussions into mathematics classes is one of the major topics in Japan. But some serious problems confront Japanese teachers. One of the problems is that Japanese students are reluctant to express their opinions in small groups activities and whole-class discussions. Teachers always make effort to forster communication in mathematics classes by asking questions or quoting some students' opinions. The other problem is that Japanese teachers worry that small group activities or whole-class discussions provide only a few students with good opportunities.

We think these kinds of worries are originated from our misconception. that is. the more communication occurs. the more fruitful activities have done. But to communicate well is not a natural ability. We have to know how to communicate each other in mathematics classes. Our study "Communication process in learning mathematics" is to focus on the process of communication itself what consists of sending a message and receiving the message.

When analyzing communication process. we can find many communication gaps between a sender and a receiver. The communication gaps are treated as bad aspects for communication. Therefore. many researchers have studied how to diminish the communication gaps. And they recommended teachers teach communication skill more.

But we found that the communication gaps sometimes bring a new idea to solve a problem. For example. one 5th grade student solved a problem in a better way as a result of communication was not a directed result of what another student intended to communicate.

The purpose of this study is to analyze communication process and to know its dynamism.

# PROBLEM FAMILIARITY AND EXPERTS THE CASE OF TRANSITIVITY

By Fairouz Farah-Sarkis
American University of Beirut

*Subject matter knowledge of teachers is related to the ability of applying their knowledge of a concept to a new situation. In other words, the ability to demonstrate transfer behavior. The present study investigated the ability of inservice math teachers transfer behavior in the concept of transitivity. Twenty inservice math teachers participated in the study. A test of 10 items including two different instances of transitivity with different content and context dimensions was used. Results showed that most teachers were able to identify one instance of a transitive relation and failed to identify a second instance on all content and context dimensions.*

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# CHILDREN'S USAGE OF STATISTICAL TERMS

Iddo Gal, Patricia Mahoney and Susan Moore

University of Pennsylvania

*Examined children's interpretations of statistical terms ('average', 'sample', 'random') which often come up in statistics education. Intuitive interpretations children may informally develop for such terms may support, but also impede, acquisition of statistical knowledge. Results suggest that some children have useful statistical intuitions regarding the meaning of such terms, but that many children associate with such terms meanings which are distant from, or in conflict with, formal notions. Implications are discussed for teaching practices, e.g., regarding the use of learners' preexisting knowledge as a basis for analogical transfer, and the need to help children to explicate and differentiate their interpretations of terms used in mathematical and everyday contexts, to facilitate effective communication and learning.*

# THE STATUS OF NEGATIVE NUMBERS IN THE SOLVING PROCESS OF ALGEBRAIC EQUATIONS

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Centro de Investigación y Estudios Avanzados del IPN. México

Programa Nacional de Formación y Actualización
de Profesores de Matemáticas

*Upon the basis of a previous historico-epistemological analysis on the status of negative numbers in the context of linear equations, three hypotheses concerning the avoidance and acknowledgement of negative solutions in students are formulated. Some evidence provided by observations made so far at the individual level makes plausible the hypotheses already mentioned and gives rise to a second stage of this research, which is described in this paper.*

# Study of identities in the school course of algebra

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M.A. Kholodnaja++, J.J. Wolfengaut***

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+Tomsk State University, Tomsk, Russia
++Kiev State University, Kiev, Ukraine

The problem which every teacher faces is how to acquaint a pupil with such an element of culture as algebra, to develop pupil's ability to deal with algebraic material freely, how to make a pupil more active in the process of studying algebra? It is obvious that algebra leaves its mark on characteristics of thinking, in our opinion, basic peculiarities of thinking as a cognitive psychic activity become apparent.

That is why mastering algebraic notions supposes special work on organizing pupil's cognitive experience. This work should be organized so that it could promote creating a subjective image of an algebraic notion.

There arise questions: what stages a pupil should go through in the process of forming this or that algebraic notion; what set of tasks it should be to stimulate the development of conceptual thinking, playing the guiding role in psychological development of a child.

It seemed possible to us to single out five stages in the process of forming a subjective image of an algebraic notion.

1. **Motivation** is realization by pupils of the necessity to find a new way of describing their experience. At this stage the tasks applying to arithmetical, geometrical, physical and past algebraic experience are widely used.
2. **Generalization** is an increase of generality degree of notion properties. The main consideration here should be given to organizing different sign-image transference.
3. **Enrichment** is accumulation and differentiation of a pupil's experience to deal with an introduced notion. Pupils are given a possibility to consider different properties of an algebraic notion being studied and appreciate a degree of their generality and importance.
4. **Transference** is realization of a possibility to apply notions in new situations. Pupils are offered the tasks, which allow them to include new knowledge into the system of associations with other notions.
5. **Compression** is an urgent reorganization of the whole complex of a pupil's knowledge, associated with the given notion and its conversion into a generalized informational structure. This stage, in our opinion, is a great problem of methodology. At present it seems to us that among the techniques of methodology, which may promote compression, we can point out such as using paradoxes, focus-examples, complex tasks and so on.

During studying algebraic material it's important to take into account not only psychological prerequisites for mastering a notion, but also to keep to a number of requirements. So, for example, in the process of teaching a comfortable regime of intellectual work should be created. It's desirable that a pupil, considering his own peculiarities, could choose strategies and the pace of learning. With regard to this it's useful to state the same algebraic material in different kinds of languages: figurative, logical, practical and also to offer pupils different levels of training and control.

Let us note that it's advisable to include into the system of tasks those, which would form intellectual qualities, such as an ability to plan their activity, to control it and to see a perspective and so on.

We tried to realize the state principles in a series of textbooks on mathematics, in particular, in the books "Acquaintance with Algebra" and "Identities".

# SOME WRONG STRATEGIES TO DETERMINE PROBABILITIES IN 8th GRADERS. REPORT OF A PRELIMINARY STUDY.

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Science Teaching Centre , Hebrew University of Jerusalem *

## ABSTRACT

*Some known strategies are involved in probabilistic comparison situations. The stability of their use is [illegible]amined. We explore the inconsistence of several strategies, and the use of intuitional knowle[illegible] in very closely approaches. Perceptual cues and inverse results are also observed.*

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# From Concept to Proof: A First Step

Merilee D. Goldberg and Rina Hershkowitz
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## Abstract

*Our goal in this paper was to investigate certain problems in the procceses students use when they write proofs. To achieve this we designed a self-contained questionare in two parts. In the first the students developed a definition for a "new geometric concept" and in the second they were given problem situations in which they had "to prove" some statements concerning this concept. We found that students relied very heavily upon the definition and that in some cases students relied in different ways on examples as justifications. We administered our questionnaire to senior high school students in the Greater Boston area. Some responses to selected problems are given.*

# METACOGNITIVE STRATEGIES IN THE CLASSROOM: POSSIBILITIES AND LIMITATIONS[1]

Zahra Gooya
University of British Columbia-Vancouver

This paper reports on a study of the effects of the "metacognitive-based instruction" on students' beliefs about mathematics and mathematical problem solving, and about themselves as doers of mathematics. The 60 hours of instruction occurred in the context of a regular mathematics content course for undergraduate non-science students. The approach to instruction was teaching mathematics through problem solving and included journal writing, small groups, and whole class discussions. These three metacognitive strategies helped the students to think more reflectively about what they were doing and how they were doing it, and they also provided data for the study. The paper reports on the possibilities and the limitations of these strategies for the use in the classroom.

# A TWO YEAR PROJECT FOR IMPROVING THE MATHEMATICS TEACHING FOR 11 - 13 YEAR-OLDS

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New Zealand

Many in-service professional development programs are seen as being of limited value because teachers do not have the opportunity to construct new practices in relation to the perceived shortcomings of their existing practice, and there is little opportunity for systematic reflection on the effect of new practices.

This project aims to overcome these shortcomings. It is a two-year action research program in which 18 teachers of Intermediate and first-year Secondary School classes design and carry out assessments of their own teaching. It is based on two premises: (1) that teachers need to construct their own understanding of successful teaching, under guidance, in much the same way that mathematics students need to construct understanding of mathematics, and (2) that there are metacognitive skills involved in analysing the relationship of student learning to ones' own teaching which can be developed through reflection and analysis. The educational focus of the project is to increase emphasis on students' responsibility for learning, and their flexibility in the use of mathematics. These aims are in line with the Cockcroft Report and the NCTM Standards for School Mathematics.

Results from the first year of the study (1991) show that teachers gained understanding from analyzing their practice, particularly in noting the ways in which their interaction was impeding students' understanding. Teachers' journal entries related to this include statements like: *"I noticed that whenever I talked, students' attempts at problem solving decreased."*, *"By not answering their questions I discovered many other things that they were confused about"*, and *"Once they learned to write problems for themselves, they said that they understood !! "* Tests given at the beginning and end of the year showed that while approximately 40% of students increased their understanding of a topic, 60% stayed at the same level or did more poorly on post tests.

At the beginning of 1992 teachers are undertaking a careful evaluation of students' understanding, and are already appreciating that their usual teaching is not meeting students' needs. They will undertake three or four specific projects in 1992 to improve the methodology and content of their teaching and wiill monitor these innovations to see if these enable students to increase their understanding.

# AN ASSESSMENT OF MATHEMATICS LEARNING THROUGH STUDENTS' INTRA- AND INTER- COMMUNICATION PROCESSES

Keiko, ITO-HINO
Southern Illinois University at Carbondale

*Intra-communication process means the processes of thinking which include reflection, making connections, and control of one's thought. Inter-communication process means the processes of thinking which include reflection and regulation of one's thought under the recognition of others on the one hand, and recognition of meaning, interpretation, negotiation, and elaboration on the other. By combining these two communication processes to make a context of assessment of mathematics learning, an instrument is presented with an illustration of an 8th-grader in algebra classroom.*

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# THE EFFECT OF DIFFERENT SCHOOL ENVIRONMENTS ON MATHEMATICS LEARNING ACROSS THE ELEMENTARY-SECONDARY INTERFACE

James K. Bidwell, Central Michigan University
Graham A. Jones, Illinois State University
Robin Ziukelis, Gold Coast, Australia

*This is a preliminary report of a study on 12-14 year-old students in Queensland, Australia and Michigan, USA. Australian students (n=486) were tested at the end of their last year in elementary school and then after one-third of their first secondary year (1990-1991) to measure any changes in mathematical performance (computation, concepts, and problem solving) and in mathematical perceptions and beliefs (classroom environment, self-perceptions, mathematics content). Four categories of students were measured: non-government schools vs. government schools crossed with change of campus vs. no change of campus. Attitudes means dropped significantly for the whole sample especially for the 'no change' students in both government and non-government sectors. Achievement gains were modest for the whole sample and for each of the sub-groups measured however there was a very weak gain for students changing from one non-government school to another. Further data and analyses will be available by the conference.*

# THE IMPORTANCE OF TEACHING PRACTICE IN MATHEMATICS TEACHER COURSES

Estela Kaufman Fainguelernt
Universidade Santa Úrsula

The teacher profession is going through a crisis. This affects specially the mathematics teacher, as shown by:

. the abandonnment of the teaching profission by graduated teachers through retirement or changing of activity;

. the abandonment of the teachers' course in general and of under graduation in Mathematics specifically.

We are facing the challenge to revert these trends transforming the under graduation course of mathematics teaching into a proficiency course.

In order to be a good Mathematics teacher, the professional should have:

. a goode background in the specifie contents he will teach;

. a critical view to adequate the contents of the course to the students and to the methodologies and strategies to be applied;

. a sincere commitement to the cultural reality he will face and vocation to interact with the community work.

These conditions are necessary simultaneosly and that is why our work is based on teaching practice offering student-teachers a training that permits to achieve different forms of action.

At the same time the theoretical contents of Mathematics are worked with the teachers responsible for the course, a change in classroom attitude, giving the trainees a chance to see different forms to approch the same subject.

## Results:

. Since 1990 we have had an increase in the number of students who take the teachers course.

. Many graduates from our teachers' course area earning promotions in their jobs.

. Our students have been succesful in public contests and in admission to graduate courses.

# PROB SIM AND DATASCOPE: INTERACTIVE SOFTWARE TOOLS FOR INTRODUCTORY COURSES IN PROBABILITY AND DATA ANALYSIS[1]

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**Prob Sim©** is a Macintosh application designed for teaching probability via simulations. In using the software to model some probabilistic situation: 1) a "mixer" is constructed which contains the elementary events of interest; 2) the mixer is sampled from after specifying replacement options, sample size, and number of repetitions; 3) events of interest are specified and counted; 4) specified events are counted in new random samples. The software makes step 4 especially easy. Once analyses have been conducted on one sample, the user has only to press a button to see the results of the same analyses performed on a new random sample. This features promotes the recognition and quantification of variability among results in successive random samples, and thus facilitates understanding of The Law of Large Numbers. Additional features include locating outcomes of interest, reshaping data, sampling until a specified event occurs.

**DataScope©** is an accompanying (and fully compatible) data analysis package — data is easily transferred between the two applications. This compatibility is especially useful in teaching basic principles of statistical inference via resampling. Those teaching traditional statistics courses may be disappointed with DataScope, since it includes only a limited number of statistical displays: bar graphs, box plots, scatterplots, one and two-way tables of frequencies, and tables of descriptive statistics. But the limited options make the program easy to learn, and permit more in-depth treatment of a few plot-types. The software is intended for use in courses stressing exploratory data analysis in which students work with real, and fairly large, data sets, using statistical techniques not so much to test hypotheses, but to formulate and revise them. It encourages the student to make initial judgments of relationship by visually comparing plots. A generalized "grouping" capability permits the formation of plots (and tables) grouped on different levels of a chosen variable.

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[1] The development of this software was supported by grant MDR-8954626 from the National Science Foundation. Opinions expressed here are those of the author and not necessary those of the Foundation.

# PFL-MATHEMATICS:
# AN IN-SERVICE EDUCATION UNIVERSITY COURSE FOR TEACHERS

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*PFL* is the abbreviation for the two-year university courses "Pädagogik and Fachdidaktik für LehrerInnen" (Pedagogics and Didactics for Teachers) in Austria. They are organized by the *Institute of Interdisciplinary Research and Education* (IFF) of the Austrian Universities of Innsbruck, Klagenfurt and Vienna, in cooperation with some universitary and teacher-in-service-training institutes and is supported by the Ministry for Science and Research and the Ministry for Education and Culture.

*The guiding principles of the three PFL-courses* - which are held for (mainly Austrian) secondary teachers of German, English and MATHEMATICS - can be described as follows:

- *interdisciplinary team* (e.g.: mathematics, mathematics education, pedagogics, teaching experience),
- *continuous and intensive co-operation of the staff and the participants* (each course includes: three one-week seminars, five two-day Regional Group meetings, continuous research and formation work at schools),
- *interconnection of theory and practice and of research and education,*
- *(further) education (also) for the staff members*, especially through practice-contacts for the involved scientists,
- *promoting research and formation work* - studies from teachers, especially within the framework of action research,
- *interconnection with questions of school-development* - promoting pedagogical and didactical innovations, establishing them at schools and making them visible,
- *promoting a new culture of talking about teaching and school,*
- *encouraging the participants to be (more) active in designing their further education within (and later outside) the course.*

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# THE EFFECTS OF TRAINING IN USE OF GENERATION AND MANAGEMENT STRATEGIES ON GEOMETRY PROBLEM SOLVING

Mohan Chinnappan and Michael Lawson
Flinders University of South Australia

*We report the results of two studies in which the effects of training in use of general strategies on the performance of high-achieving and low-achieving high school students in trigonometry is assessed. The studies take up two of the issues raised by John Sweller(1990),the lack of evidence of positive effects of such training, and the extent of transfer. Two forms of training derived from our earlier studies of geometry problem solving were developed. The first, Generation training emphasised analysis of the problem statement, searching of memory, and use of the diagram for cues to guide memory search. Management training directed students' attention to planning of the solution path, checking of calculations, and reviewing of the solution. No effects on training or transfer items were observed following Generation training. Management training improved the transfer performance of both high- and low-achieving students.*

# WHEN MORE IS LESS - INTERACTIVE TOOLS FOR RELATIONAL LANGUAGE

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*Abstract:* Mathematical representations such as equations can account for or unify a large class of semantic structures of arithmetic word problems, each of which can in turn be expressed linguistically in many different ways. An important task from an educational perspective is to help the elementary-level student to "see" linguistic descriptions in quantitative or mathematical terms, e.g., through progressive abstraction from a particular linguistic problem statement to an equation or other solution.

An emphasis on understanding the invariances between alternate symbolic and pictorial representation systems has led to dynamic computer environments which can extend student's ability to make connections between alternate problem representations. While the idea of dynamic linkages between representations is not new, linkages to representations which are inherently imprecise, such as natural language, have not been attempted. The interactive tools presented in this work are able to "link" with natural language representations as they are integrated with a cognitive simulation of word problem solving which includes the processes of reading and text integration.

Specifically, a *Language Link* tool enables students to manipulate icons on the computer screen and as the configuration of icons is altered the student immediately sees how this new picture alters the linguistic representation, that is, a change in the difference relation results in a new natural language sentence describing that relation.

Such "dynamic links" can encourage students to appreciate the mathematical meaning of complex linguistic constructions, e.g., the relational language of "have more than" and "have less than." A major hypothesis of the research with relational language is that "expert" problem solvers convert inconsistent relational statements into consistent relational statements, for example, converting "Jacob has 3 more cans than Kathy." to "Kathy has 3 less cans than Jacob." The implication for instruction involves the need to teach children (1) how to translate from an inconsistent relational statement to a consistent relational statement and (2) that both of these statements are simultaneously true.

# TEACHERS' QUESTIONING AND STUDENTS' RESPONSES IN CLASSROOM MATHEMATICS

**Eleanor Long**
**University of Adelaide, Australia**

The questions that teachers ask and the responses that students give are important activities in classroom learning. Questioning is a versatile and powerful means for teachers to probe student understanding and to encourage the active investigation of new ideas. Students' responses provide teachers with information that can be used to monitor and assess individual and group learning progress. Thus, effective classroom communication can be thought of as a facilitating skill that works to promote and develop students' thinking and learning.

Cognitive research shows that competence in areas of complex skill involves a critical and close relationship between reasoning processes and knowledge that is specific to the area. In this light, skilled oral communication in mathematics is communication that indicates a rich mathematical knowledge. A further suggestion is that the development of abilities such as reasoning aloud, describing and justifying one's thinking to others, and formulating questions, should go hand-in-hand with the development of knowledge in mathematics.

The study described in the poster aims to enhance communication, specifically in teachers' questioning and students' responses, in the area of fractions. From a cognitive perspective, knowledge in fractions is seen as a basis for the development of thinking and communication skills. The social aspects of knowledge development are emphasised through a classroom approach that fosters student and teacher collaboration in learning. The first stage of the project has involved the preparation of classroom materials which present fraction concepts and skills as a system of knowledge that develops meaning for individuals through the situations and conditions of use. Designed for use with 11-14 year olds, the materials feature task-sheets that pose problems in which fractions can be used in everyday life. Activities, suggestions and questions within the problem tasks are designed to encourage thinking and talking about fractions within a wide range of situations.

The intention of the study is that the materials produced will provide a key means for the promotion and development of teachers' questioning and students' responses in the learning of fractions. Initial classroom trials, currently underway, are focusing on the ways in which the materials are being used in classrooms, and the extent to which they are encouraging teachers and students to talk about fractions.

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# STUDENTS'PROBLEMS IN UNDERSTANDING THE IDEA OF CONDITIONAL PROBABILITY

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In this work the answers of students aged 16-22 years given to questions concerning the idea of conditional probability are analyzed. The arguments they used to support their answers depended on the form of the situation in which the question was set. It was confirmed also that it is more difficult to answer questions in which the conditioning event is supposed to happen later than the conditioned event, than when the conditioning event happens either simultaneously or before the conditioned event. Additionally, there was an inability to apply the formula of conditional probability because of the confusion between conjunction of events and conditioned events.

# DEVELOPMENT OF AN INSTRUMENT FOR TEACHER AND STUDENT USE IN THE MEASUREMENT OF AFFECTIVE DEVELOPMENT IN SCHOOL STUDENTS.

Ian Putt , Frederick Annesley, and John Clark, School of Education, James Cook University, 4811, Australia.

Productivity is the driving force of the 90's. We are confronted with it in our work place, at home, in industry, within government and in our schools. The central demand is always to accomplish more with less. In school, this demand has the effect of concentrating effort on cognitive processes and outcomes. But we asked the question 'what is the place of affective education in schools?'. In part the answer lies in the interrelationship between affective skills and productivity. Those persons who have developed their affective and interpersonal skills are the ones who are productive in the work place, at home, in industry and in school. The challenge then was to encourage teachers to consciously blend affective goals into their everyday teaching, and to provide them with an instrument which would reliably and validly measure the outcomes of student growth in the affective domain.

Over the four years 1987-1990 we worked with secondary teachers of English, Mathematics and Physical Education (a) to determine what affective characteristics they wanted to see developed in their students; and (b) to develop a valid and reliable instrument which teachers could use to rate students, and students could use to rate themselves on the behaviours identified in (a) above. In 1991 we tested the portability of the instrument to primary schools by repeating the same two steps above with teachers in years 5 to 7.

An analysis of secondary teachers' responses yielded two factors which were being assessed by the teachers in the three subject areas. From these factors the teachers produced a set of typical behaviours for each point on a five point scale for the affective characteristics which constituted each factor. The first factor related to classroom learning and was characterised by the following - enjoyment of learning, participation, independence, self motivation, positive attitude, and initiative. The second factor related to students' personal qualities and was characterised by the following - obedience, honesty, courtesy, responsibleness, and self discipline.

When primary (year 5-7) teachers' responses were analysed two factors also emerged and these were remarkably similar to the secondary factors. Factor 1 involved - enjoyment of learning, independence, initiative, appreciation of language, appreciation of mathematics, and curiosity. Factor 2 involved - caring for others, obedience, honesty, courtesy, and responsibleness.

In the course of establishing the final version of the instrument, teachers commented on the suitability of the instrument for use in determining the progress of their students in the affective domain, students felt empowered by being able to actively confront their affective development in school, and parents strongly supported a greater emphasis on the development of these affective characteristics in their children.

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# RESEARCH AND PSYCHOLOGICAL FACTORS INFLUENCING MATERIALS DEVELOPMENT IN MATHEMATICS: IMAGERY

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## ABSTRACT

The paper starts with 11 clarifying questions that analyse mathematical understanding. Three questions are considered in more detail to provide guidelines for materials development. In particular, the notion of a mental model is discussed within the context of learnable mathematics intelligence. How imagery may be built in to materials development to enhance accessibility is highlighted and examplars given.

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# Acquisition of meaning for pre-algebraic structures with "the Planner"

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Learning Research and Development Center
University of Pittsburgh

A computerized environment, the Planner, was constructed to be an interactive illustration of arithmetic as a language: different objects in the system embodied the different meanings of operations and arithmetic symbols. This paper describes a study with an experimental group (n=6) who learned to solve arithmetic word problems with the Planner. After seven training sessions, the group was able not only to solve difficult word problems but succeeded extremely well in a task of recalling the story problems. This result indicates that students exposed to a principled interactive illustration learn to correctly represent the algebraic structure of difficult word problems.

# CORRELATES OF DIRECT PROPORTIONAL REASONING AMONG ADOLESCENTS IN THE PHILIPPINES

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University of Minnesota

*The relationship between performance on two tests of direct proportionality (Tall-Short test of Karplus and Test of Response Types) and other measures of scholastic achievement was examined in gifted, high-achieving and low-achieving adolescents in the Philippines. Direct proportionality test scores correlated highly with achievement group, and generally improved with grade level. The distribution of response categories indicated that the patterns of reasoning were dependent on achievement level and grade level. Overall, the performance of Filipino adolescents was not fundamentally different from other racial and ethnic groups that have been studied.*

# CARDS, MIRRORS, AND HAND-HELD MODELS THAT LEAD INTO ELEMENTARY LOGIC

Shea Zellweger, Mount Union College

Learning to count takes a long time, it goes through many steps and stages, and it follows closely the development of language, sensorimotor coordination, and the ability to relate to physical objects. We take it for granted that, when we teach children to count, we will call on a long run of games and exercises. These simple beginnings, as a matter of universal importance, eventually lead to whatever we can claim for our time and culture, in regard to numbers, notation, and mathematics.

Dantzig (1954) notes that "... wherever a counting technique, worthy of the name exists at all, finger counting has been found to either precede or accompany it" (p. 9). Skemp (1982) also reminds us that " ... though the power of mathematics resides in its ideas, access to this power is largely dependent on its notation ..." (p. 204). Golden (1982) includes examples when he shows that "... the principle of using syntactically parallel notation to represent mathematically parallel concepts allows greater insight at the elementary level into the meaning of structural properties of binary functions" (p. 230). The evidence that has accumulated makes it easy for us to go along with the main line orientation contained in these statements, when we consider the standard languages of mathematics. But what about logic?

What would it be like to repeat the whole thing, at least to start moving in that direction? What would it be like to bring what we do for counting into what we might do for logic? What follows will let you know what Glenn Clark[1] and I have come up with, at least to the extent that we have made a few inroads in that direction.

All that follows will be limited to just one example. This example will center on the logic of any two things, also called the logic of two sentences, no more than that. Although it may not be evident at first, this example will gradually lead into, and then draw out, a special set of relations, namely, the relations between the two sentences (A,B), such as 'A and B,' 'A or B,' 'if A, then B,' etc. These relations are often called the sixteen binary connectives. We will also try our hand at coming up with a good notation, one that has a custom-designed syntax that runs parallel with the underlying abstract structure.

# Featured Discussion Group I

# THE THREE FACETS OF MATHEMATICS:
# THE FORMAL, THE HUMAN AND THE INSTRUMENTAL
# EDUCATIONAL IMPLICATIONS

Efraim Fischbein
Tel Aviv University
School of Education

It is surprising that so far neither the psychology of mathematics education nor the didactics of mathematics did consider the profound and complicated problems raised by the need to harmonize these three aspects as an effect of the instructional process.

Let us, first, briefly define what is meant by the respective terms:

1. Mathematics as a formal body of knowledge, that is mathematics as it is exposed in treatises and high-level textbooks. In this sense, mathematics is considered to deal only with logically governed relationships. This implies the following characteristics:

- Mathematics is an abstract domain. It does not investigate concrete, sensorial properties of real objects but only general properties considered independent of any substantial constraints.
- Mathematics is a deductive body of knowledge. Every statement is originated - through a logical, formal, deductive, consistent way - from an initial system of statements (axioms), primitive concepts and definitions. The validity of mathematical statements is accepted only if it is a consequence of previously proven statements (or accepted as axioms).

## Research-based curriculum development in high school geometry: A constructivist model[1]

W. Gary Martin
University of Hawaii

*Curriculum development in high school geometry must be thoroughly grounded on research in order to produce the kinds of dramatic changes needed. An interactive, recursive model based on constructivism is described; this model is currently being used to develop a full-year geometry course. Data collection consists of full-class and individual teaching experiments. Results are used to revise the curriculum at different levels and to produce basic knowledge about how students learn geometry.*

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## Needing Conscious Conceptions of Human Nature and Values to Inform and Develop Pedagogy

John P. Pace
Essex County College
USA

*The development of a coherent pedagogy of mathematics education is dependent upon a conscious conception of human nature and an attendant value system. One such conception of human nature is offered. Constructivism can only inform that pedagogy, never prescribe it.*

# Featured Discussion Group II

# AN ANALYSIS OF THE DEVELOPMENT OF THE NOTION OF SIMILARITY IN CONFLUENCE: MULTIPLYING STRUCTURES; SPATIAL PROPERTIES AND MECHANISMS OF LOGIC AND FORMAL FRAMEWORKS.

Lucilia Bechara Sanchez
Associação Universitária Interamericana

Advisor: Anna Maria Pessoa de Carvalho
Universidade de São Paulo - Brasil

*This paper deals with some aspects of acquiring the notion of similarity of plane figures by students between 11 and 16 years of age from three schools in São Paulo. The paper uses constructivist presuppositions and emphasizes the teaching/learning relations, development of cognitive structures and learning of specific contents. The results describe the following construction processes: a. operations with numbers, from multiplication to ratios; b. spatial relations from the syncretic view to the distinction of variables and their interdependence; c. the logic of the classifications vis-a-vis the classificatory criteria. d. the logic of the propositions from acknowledgement of the true values to the implicational relations and determination of the necessary and sufficient conditions. Given the psychopedagogical implications, the paper discusses applications aimed at optimizing the function of the school in its role of promoting learning and developing cognitive structures.*

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# APPROPRIATION AND COGNITIVE EMPOWERMENT: CULTURAL ARTIFACTS AND EDUCATIONAL PRACTICES

Maria Graciosa Veloso
**João Pedro Ponte**
*University of Lisbon*

*Appropriation of a cultural artifact or mindfact may be viewed as its progressive domain, providing a strong generative power: it becomes a thinking tool, yielding new ways to see the world and to formulate and attack problems. This paper discusses the appropriation of the calculator as a pedagogical tool by several teachers involved in a year long inservice programme. It focuses particularly on how different forms of using the calculator in the classroom can be related to different levels of appropriation.*

# EXPLORING STUDENTS' MENTAL ACTIVITY WHEN SOLVING 3-DIMENSIONAL TASKS

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Departamento de Didáctica de la Matemática. Universidad de Valencia (Spain).

## Abstract

*In this report we describe some results of an experiment aimed to analyze the mental processes and abilities of visualization used by students when they try to solve several tasks based on the manipulation of polyhedrons.*

*First, we present a task in which the students had to move several polyhedrons on the computer screen, and another task in which they had to imagine the movement of solids, and we analyze the characteristics of the different activities.*

*Afterwards, we show some students' ways of solving those tasks, and we discuss the students' behaviors in relation to their use, or lack of use, of different abilities and processes of visualization and mental representation of spatial relationships.*

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# VISUAL IMAGES, AVAILABILITY AND ANCHORING, RELATED TO THE POLYNOMIAL NUMBERS AND THE USE OF MICROCOMPUTERS

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## ABSTRACT

In this study, we focused on one hand on the role of visualizing mathematical concepts as an important facet to induce a process of forming a conjecture and developing mathematical reasoning. On a second hand, in a computational environment, we analyze some problems arisen in this context with a specific topic, the polygonal numbers.

## A VAN HIELE- BASED EXPERIMENT ON THE TEACHING OF CONGRUENCE

LILIAN NASSER
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO - BRASIL
KING'S COLLEGE LONDON

A teaching experiment has been carried out in order to investigate whether the difficulties encountered by Brazilian secondary school students in the learning of *congruence of plane shapes* can be alleviated if the instruction is based on the van Hiele model of thinking, matching the level attained by the students. In this paper, some results are presented, comparing the performances of the control and the experimental groups.

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## MODES OF USE OF THE SCALAR AND FUNCTIONAL OPERATORS WHEN SOLVING MULTIPLICATIVE PROBLEMS[1]

Mariela Orozco Hormaza
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*Multiplicative problems may include two types of relationships: scalar and functional. These may be exact or inexact. The relationship is exact if one of the two terms is the multiple of the other. A systematic study of the procedures of 50 children between 7 and 12 years of age when solving bying and selling tasks allows me to conclude -and to exemplify my conclusion- that children tend to employ the exact relationship regardless of whether it is scalar or functional. From this study it is also possible to describe a progressive construction of the scalar and functional operator and of their use.*

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# Plenary Panel

PLENARY PANEL

# VISUALIZATION AND IMAGISTIC THINKING

Tommy Dreyfus (chair), John Mason, Bernard Parzysz,
Martine Picq and Norma Presmeg

Visual, imagistic and diagrammatic elements appear to gain growing importance in learning and teaching mathematics as well as in mathematical problem solving. This is documented by a considerable number of the plenary presentations, research reports and working groups at PME-15 (Assisi, Italy; July 1991), by a recent conference on Reasoning with Diagrammatic Representations (Stanford University, Palo Alto, CA, USA; March 1992), by the volume on Visualization in Teaching and Learning Mathematics (W. Zimmermann & S. Cunningham, Eds., 1991) which has appeared in the Notes Series of the Mathematical Association of America and by numerous other events. Against this background, the panel members will address issues closely concerned with the psychological aspects of mathematical reasoning with diagrams, pictures and images. Some of these issues are

- Internal (mental) versus external (concrete) imagery
- Imagistic versus logical reasoning
- The role of visualization in abstraction
- Imagistic thinking and heuristics
- The influence of computer graphics on visual reasoning
- Methodologies for investigating imagistic thinking
- The role of age for visual thinking

After opening statements by the panelists, there will be opportunities for them to react to each other's viewpoints as well as to comments and questions from the floor.

# Plenary Addresses

# THE IMPORTANCE AND LIMITS OF EPISTEMOLOGICAL WORK IN DIDACTICS

Michèle Artigue, IUFM Reims, DIDIREM team, Université Paris 7

The student or teacher who is interested in questions on the teaching of mathematics, and who tries to establish contact with this field of research, would certainly be struck by the diversity of co-existing approaches, of the theoretical constructions elaborated, as well as by the virulence of some debates which shake up the research community.

Hence, although everyone within a group such as PME tends to claim to be a constructivist, the debates are no less lively between those who defend radical constructivism and those who defend moderate constructivism (see for instance the Proceedings of PME XI in Montreal or the Monography N°4 of JRME).

If, in methodology, the brutal opposition between partisans of quantitative methods and qualitative methods now seems outmoded, the debates are far from being finished on the nature of results in didactics and the acceptable means for their "scientific" validation, on the relevance of one approach or another.

Even outside frank differences and oppositions, the beginner would certainly get lost in the undergrowth of notions which have been introduced over the last fifteen years, to apprehend apparently close phenomena. What relationship exists between, to take but a few examples frequently encountered in the PME proceedings, "mental objects", "concept-images", "conceptions" and "intuitions", between "epistemological obstacles" and "misconceptions", between "conceptual entities", and "objects", between the processes of "encapsulation", "condensation" and "reification" ? Do we really need all these notions ?

This absence of unity in the field, this multiplicity of notions can be viewed, by referring to the theory developed by T. Kuhn, as the mark of a "scientificness" which is still developing. It is certainly so. However, I am also convinced that, up to a certain point, it is the normal mark of the partiality of any approach to the complex phenomena we are studying. The scientific advance of this field certainly goes through the search for global coherence but it must be capable of feeding off the simultaneous and partially autonomous development of these different approaches. It is this conviction which underlies the following presentation and acts as its main theme.

In this presentation, I have chosen to give a particular position to the epistemological approach to didactic phenomena which, in my life as a researcher, play a special role. The first part of the presentation will be given over to this and will present various aspects of this epistemological work. However, it must be clear that the epistemological work in question here is only a tool of the didactic work itself : although it is a useful guide, it cannot be a substitute. In the second part, limits of this work will be stressed and we will come back to a more classical cognitive approach. In this crucial domain, the need for a more coherent expression of the different theoretical frames elaborated is, I feel, obvious. I will try to clarify this demand before illustrating, from an example, the limits of cognitive work in a didactic perspective. This, finally, will lead us to the heart of didactic work : the study of the relations between teaching and learning based on the analysis of the functioning of the didactic system and more generally of didactic institutions. It is at this level that is situated, in fact, the core of much research work in didactics (mine included, despite the importance given to the epistemological approach). In my opinion, it is at this level that the search for more globally coherent organisation in the field must be situated.

## I - CONTRIBUTIONS OF EPISTEMOLOGY TO THE FIELD OF DIDACTICS

We do not mean here truly cognitive epistemology, but a more ontological reflection on the nature of mathematical concepts, the mechanisms and conditions for their development, on the characteristics of present as well as past mathematical activity, on what makes up the specific nature of one mathematical domain or another. It acts as a backdrop to the (didactic) question of knowing whether it is necessary to try to transpose, firstly, these characteristics of mathematical knowledge and activity into teaching and to the study of the conditions for viability of such transpositions, taking into account the various constraints to which teaching is subjected : cognitive constraints, constraints linked to the relationship of the different actors in the system - pupils and teachers - to mathematics and teaching, institutional constraints in particular. The fact that my research has been situated for the last ten years or so mainly on the level of Advanced Mathematical Thinking is no doubt connected to this privileged role played by the epistemological gateway into didactics : in more elementary areas, mathematical knowledge often wrongly seems more transparent.

A presentation such as this one is inevitably limited. Nevertheless, through a few examples, I would like to try to show how this type of epistemological analysis can function at a global level in the search to establish theoretical frames, and study or even question concepts elaborated elsewhere, beyond its usual role at a more local level.

### I-1 : The role of epistemology in the didactics of a specific domain :

I shall begin with this more local level of epistemological work, by considering the domain of analysis. Many other choices would, of course, have been possible but research in analysis illustrates particularly well this role of epistemological work, as well as the various forms it can take on depending on each researcher's own themata and the theoretical frames on which he bases himself. Without claiming to be exhaustive, I would like to list a number of approaches which have already given rise to substantial research.

a) An approach through epistemological obstacles which consists in identifying obstacles of this nature encountered in the historical development of concepts and in then studying, in the light of this history, the cognitive functioning of today's pupils. The notion of epistemological obstacle, introduced by G. Bachelard, was imported into the didactics of mathematics by G. Brousseau (Brousseau, 1983). An epistemological obstacle is linked to knowledge, a form or principle of knowledge which, effective and of an unrestricted field of validity, was able to establish itself and strengthen itself during the development of a notion, but which at a certain stage in this develpment becomes a blocking factor and is generally a strong, durable source of error. It continues to appear in a recurrent fashion even when one believes it has been eradicated. The hypothesis underlying this approach is firstly, that such objects exist, then that despite the differences which exist between the historical development of concepts and their learning in current school institutions, teaching cannot escape from them entirely, they are compulsory passages on the road to knowledge as they are part of it. Initially G. Brousseau exploited this notion in order to study the conceptions and persistant errors of pupils in the extension of the field of numbers to fractions and decimals. In analysis, this approach first developed around notions of limit, infinity (Cornu, 1983) (Sierpinska, 1985) and produced results supporting the hypotheses underlying this approach. Hence A. Sierpinska produced, after an in-depth historical study, a structured list of obstacles which marked out the evolution of the concept of limit. This list puts the obstacles into five categories :

- "horror infiniti" in which one finds obstacles linked to the rejection of the status of the mathematical operation for the passage to limit and those which consist in transferring automatically the methods and results of finite processes to infinite processes - the principle of permanence which is raised in 1.3 is placed here;
- obstacles linked to the concept of function on which the concept of limit depends closely;
- obstacles linked to an over-exclusively geometrical conception of limit, whereas the concept brings into play an essential interaction between geometrical and algebraic frames;
- finally, obstacles of a logic nature.

She then highlighted behaviour which can be linked to these obstacles in pupils confronted with problems concerning the definition and search for tangents. The research then widens to other notions (functions, areas....) and this evolution is accompanied by an obvious effort to target, beyond the listing of obstacles relating to a particular notion, the search for a more general process of the formation and structuring of obstacles (Sierpinska, 1988), (Schneider, 1988).

b) An approach through problems which consists in seeking in history the fields of problems which allow the organisation of teaching processes which respect the epistemology for the domain considered. M. Schneider's thesis, which has already been quoted, entitled :

"Des objets mentaux aire et volume au calcul des primitives" ("From the mental objects area and volume to the calculation of primitives") is typical of this approach even it is not limited to this. It is based on a close analysis of pupils' behaviour when faced with a field of problems, mainly constructed from authentic historical problems. This succession of problems should, without reproducing all the meanderings of history, allow "the initial intuitions of the pupils to be revealed and to allow them to develop themselves the concepts we wish to teach them". The author uses, in particular, in her research, problems which have fired controversies on the calculation of area and volume using the method of indivisibles introduced by Cavalieri in the 17th century. She demonstrates that the perception of surfaces (resp. volumes) as the piling up of segments (resp of surfaces), although not taught, is present in the mental representations of today's pupils and allows the persistant errors in the calculation of areas and volumes of solids of revolution for example to be explained. The problems chosen first serve to highlight the productive character of this perception of geometric objects and render it explicit. Next, they are used to attack the epistemological obstacle (called by the author the obstacle of heterogeneity of dimensions) which consists in the simulatneous, uncontrolled and often subconscious manipulation by the pupils, of geometrical objects of different dimensions in the calculation of areas and volumes.

c) A more systematic approach in terms of didactic transposition (Chevallard, 1985) which, beyond the epistemology of the field, aims to understand the functioning of current teaching, by studying how, historically, evolutions in mathematics and school are linked. I myself have worked in this direction in research concerning differential and integral procedures in order to try to better understand current differences between teachers' conceptions and practice in mathematics and physics (Artigue, 1986). In a later piece of research concerning the teaching of differential equations (Artigue, 1989), a similar approach helped me to understand better the way in which teaching focuses solely on the methods of algebraic resolution typical of the functioning of the field in the 18th century, and its stupefying impermeability to the epistemological evolution of the field towards qualitative geometrical and numerical approaches. Indeed, this stability results from real epistemological constraints. For example, the long domination of the algebraic frame in the historical development of the theory, the more recent appearance in the 1880's in Poincaré's work of the geometrical approach, and the difficulty of problems associated with this approach tend to oppose the early introduction of the qualitative into teaching. These epistemological constraints are coupled with cognitive constraints : the requirement of permanent mobility between frames made necessary by the qualitative study (between the geometrical frame of drawings and the algebraic one of equation) - mastering of the tools of elementary analysis necessary for qualitative proofs notably.

These two types of constraints have, in turn, been strengthened by didactic constraints : the permanent temptation of the algorithmic refuge which allows limitation to algebraic solutions - the devalued status of the graphic frame in teaching - the absence, until recently, of elementary didactic transpositions of the qualitative approach.

Technological developments today allow some of these constraints to be avoided, especially cognitive constraints, by opening up other paths to the teaching process. However, in a certain sense, the system, which has interiorised past constraints, has difficulty in envisaging its real space for freedom. From this arises the stability of a balance which is today obsolete.

In basing ourselves on this analysis, summed up briefly here, we have constructed and experimented over several years, and analysed from the point of view of viability and reproduction potential, a teaching process for students in first year of university, which better respects the field's current epistemology (Artigue, 1990).

Finally, let us underline that epistemological analysis, although it often seems rooted in history, is not necessarily subordinate to it. Research such as that of M. Legrand (Legrand et Al, 1986) and (Legrand, 1991) who, from an analysis of the current functioning of the field, highlights the rupture which exists between the functioning of mathematical thinking in algebra and that of mathematical thinking in analysis, the attempts to define the characteristics of teaching situations which allow this rupture to be pinpointed, in contrast with the practice of usual teaching, participate completely in epistemological work. We shall return to this aspect in more detail in part II.

Epistemological research, beyond these local uses, but in close interaction with them, also serves to base and question the theoretical frames we develop. I would now like to deal with this aspect through two successive examples : the status distinctions of mathematical concepts, and the theory of epistemological obstacles. On the historical level, I shall use as a basis, the history of complex numbers and their extensions, although many other choices would have been possible.

### I-2 : Epistemology and didactic concepts : the different status of mathematical concepts :

In the theory developed in her thesis, (Douady, 1984) R. Douady distinguishes between two status of mathematical concepts :

- a "tool" status, when the interest is focused on its use for solving a problem in a specific context, by somebody, at a given time,

- an "object" status, when it is considered in a cultural dimension, as a piece of knowledge, independent of any context, any person, which has a place in the body of the socially recognized scientific knowledge.

This distinction is linked to the hypothesis that, for numerous mathematical concepts, the tool function is the one which comes first and founds the construction of the object.

The history of complex numbers illustrates the epistemological relevance of this distinction. Indeed, what appeared on the mathematical scene in the 16th century, in the works of Italian algebraists, was first a tool, via the audacious extension of a technique for the solving of equations to cases in which it no longer applies but in which one knows how to prove, through geometrical and numerical reasons, the existence of roots. The extension of the technique was accompanied by the introduction of new notations (hence Bombelli introduced in his Agebra in 1572 the notations "più di meno" and "meno di meno" to represent new operations consisting in adding or subtracting the square root of a negative expression). At this period there was no question of numbers. The expressions that the signs introduced led to manipulate are empty of meaning and governed on the level of calculations by the principle of permanence stated by Leibniz. They take on the status of convenient intermediaries in calculations which, starting from real numbers must lead to real numbers (infinitesimals had a similar status in the 18th century).Slowly, however, this status evolved. Tools explicitely identified and named, imaginary quantities came out of the single frame of the solving of equations to intervene also in trigonometry. They also took on the status of autonomous variables in functional expressions, especially developments in series ; an article in Diderot and d'Alembert's Methodical Encyclopaedia is given over to them, defining pure and mixed imaginary quantities, stating on their subject, general properties of closing through usual operations and functions. A phenomena which was also undoubtedly essential in this evolution, they became the instrument for the formulation of important results : the fundamental theorem of algebra, the unification of spherical and hyperbolical trigonometries via Euler's formulae. The tool already, therefore, presents undeniable characteristics of mathematical objects, but these are purely symbolic objects, defined but not constructed and not susceptible to real interpretations. It was only in the 19th century that they acquired, undeniably, the status of mathematical object, in two distinct stages : via the geometrical interpretation proposed independently by various mathematicians (Wessel, Argand, Gauss...) then via the constructions of Hamilton and Cauchy who founded this algebraic notion algebraically and allowed it to finally come out of the frame of purely symbolic algebra.

The type of research on which this rapid presentation is based, which consists in testing our theoretical categories, illustrates, to my mind, the principle of epistemological vigilance to which the didactician must subject himself. It confirms, in this precise case, which is reassuring, that the distinction introduced by R. Douady was well-founded. At the same time it usefully reminds us of the complexity of the relations between the status of tool and object of concepts ; usefully, because experience has shown the risks of over-simplification run by any theorisation during successive transmissions and vulgarisations : the complex number object is not created suddenly by miraculous institutionalisation on the basis of "activities" in which it only had the status of tool ; before being mathematically constructed and fully legitimised, it already produces generality and in turn becomes engaged with a status of pre-object in more complex processes ; the duality of tool-object functionings is established very early, when the object is only in gestation and this duality participates in an essential way in the evolution of the status of notion ; finally, it is not the simple proof of effectiveness in the solving of problems which guarantees the obtaining of the status of object; the mechanisms of legitimation are much more subtle.

Let us also underline, even if we have not here reached this point in history, that the extension of the field of complex numbers shows that notions can appear on the mathematical scene and become objects following very different schemas to the one described above. The quaternions introduced by Hamlilton arrived fully constructed, in response to a problem of generalisation, the problem of the extension to space of the calculation on the directions permitted by the geometric interpretation of complex numbers.

The reading given here is, of course, not the only one possible. A.Sfard in (Sfard, 1991) uses the same base to found epistemologically the disticntion she operates between structural and operational conceptions of mathematical notions (conceptions viewed here also as complementary and dual) and illustrates the fact that the operational in history generally precedes the structural. The term "structural" here refers to a treatment of mathematical notions "as if they referred to some abstract objects" and the term structural to a description in terms of "processes, algorithms and actions".

These two readings are not independent and one could be tempted to assimilate "operational" and "tool", "structural" and "object". We cannot develop here a real comparative analysis, but we would like to emphasise how such an assimilation would be excessive : the tool functioning of the complex number, for example, brings into play both the operational and the structural conceptions that can be developed concerning it ; the process of "reification" which in A.Sfard allows access to a structural conception of the notion is not, either, equivalent to the "institutional legitimation" fully constituted by the object in R. Douady's sense. The distinction introduced by A.Sfard is produced by a gateway which analyses knowledge in terms of cognitive processes while that introduced by R.Douady is in terms of conceptual fields and problems with a strong reference to the mathematical institution. Both the epistemological gateways are legitimate, they do not superimpose one another, nor can one be reduced to the other.

However, it should be emphasised that each of them allows for the interrogation of teaching habits in which, frequently, the structural precedes the operational, the object precedes the tool : what can legitimize and up to what point such an epistemological inversion.

## I-3 : Epistemology and didactic concepts : rigour, control, obstacles :

In the preceding paragraph we brought up various dualities in play in mathematical concepts. We would now like to draw from these implications on control and rigour in mathematical activity. We shall base ourselves once more on the history of complex numbers but the reader will easily transpose this to other domains, especially geometry which, in cultural imagery and often even in teaching, is considered as the temple of a perfect and atemporal rationality.

Imaginary quantities, as objects in gestation and tools, are governed on the level of calculation. This governing is carried out via the principle of permanence which postulates the invariance of the rules of algebraic calculation. This same principle is then applied to the infinitesimal quantities of the beginnings of analysis. Although the principle of permanence governs calculations, the control of results is essentially pragmatic, through endorsement : no contradiction appears because of the application of the principle, one may find results accessible by other methods. There again the parallel with infinitesimal analysis is total.

This phenomenon confirms if need be that questions of rigour cannot be analysed without taking into account the levels of conceptualisation of the mathematical notions at stake, that there exist in the mathematical game subtle balances between the status of notions, the functioning permitted, the importance of the problems being sought and which one manages to "solve". The rationality of an individual, whether or not a mathematician, cannot be reduced to global cognitive characteristics.

Mathematicians, throughout history, have adapted themselves more and more to these calculations involving imaginary quantities, which, although effective, are without meaning. Hence one reaches the controversy of the logarithm of complex numbers which developed between Jean Bernouilli and Leibniz at the beginning of the 18th century, a controversy in which one sees the principle of permanence enter into conflict with itself. Indeed, among other arguments, in the name of the principle, Jean Bernouilli maintained that the logarithm of -a is necessarily equal to the logarithm of a :

$(-a)^2=a^2$ therefore $\ln(-a)^2=\ln(a)^2$ whereas $\ln(-a)^2=2.\ln(a)$ and $\ln(a)^2=2.\ln(a)$ therefore $\ln(-a)=\ln(a)$

In the name of the same principle, Leibniz maintained that if one had :

$\ln(-1)=\ln(1)$, one would have : $e^{\ln(-1)}=e^{\ln(1)}$, and therefore, as $e^{\ln(x)}=x$, $-1=1$, which is absurd.

It is important to emphasise that the solution to this controversy, brought about by Euler in 1748 occured at a purely local level and did not throw doubt on either the principle of permanence, or the use of imaginary quantities. As Euler wrote : "However, I shall demonstrate so clearly that no doubt will remain, that this doctrine is solidly established, and that all the aforementioned difficulties only originate from one single idea which is hardly correct ; in such a way that, as soon as this idea is rectified, all difficulties and contradictions, however strong they may have appeared, will vanish".

The idea which is hardly correct is in fact the following :

"One normally supposes, almost without realising it, that for each number there corresponds only one logarithm".

Euler demonstrated that for every number, real or imaginary, there corresponds in fact an infinity of logarithms and if this fact remained hidden up to then, it was only because positive real numbers, and they alone, have in this infinity a real logarithm. The principle of permanence is not rejected but in fact reestablished on a higher level.

Episodes such as this challenge the didactician. Of course, he has available to him in his theoretical arsenal notions to label the difficulty encountered. the notion of epistemological obstacle, already quoted, presents, for example, all the required characteristics : the difficulty is linked here to a principle of knowledge which proved its effectiveness in a wide field, which produces errors outside this field, persisisting even when one thought it eradicated, as the persistence of the debates long after the publication of Euler's text demonstrates. However, one problem which has still not been solved is that of didactic intervention. History shows us once more the epistemological misconceptions risked by summary "didactifications" which modelise learning through a succession of clearings of obstacles which allow one to arrive at a definitive, faultless conception. It reminds us that some epistemological obstacles which learning must face cannot be thought of in isolation, that their clearing in itself has no meaning. They repose on general principles which are the driving, fundamental principles of mathematical functioning : in this case, generalisation (via the principle of permanence), which are not to be questioned as principles but rather learnt to be controlled with vigilance. The development of such an analysis encourages us to conceive of the expert not as someone who no longer risks becoming trapped in excessive generalisation, but rather as someone who, through a certain number of local adaptations, has neutralised the errors linked to the most usual manifaestations of this excessive generalisation (in the domains he masters), and who knows, on another level of knowledge, that any new process of generalisation carries risks of excessive generalisation. He also knows, even if he usually allows himself to be carried along in research by vague principles of permanence as he allows himself to be carried along by analogy, how to change his functioning register rapidly.

Let us take the well-known example of the obstacle of the monotonous model in phenomena of convergence. Students subjected to experimental teaching processes seem to detach themselves fairly quickly from its usual manifestations and quite soon no longer allow themselves to be trapped by conjectures such as :

"If a serie is positive, convergent, of 0 limit, it is decreasing from a certain rank."

This does not prevent them from stating that a class C1 function which admits of a finite limit to infinity has a derivative which necessarily tends towards 0, which is a direct manifestation of the same obstacle, at the more hidden level of the derivative this time. As for ourselves, can we be sure, as professionals, in the grip of the action, pushed by a conjecture, that we have never included in a demonstration an argument of this type ?

Various research shows that such a view is not specific to the didactics of mathematics. In the didactics of physics, particularly, research has shown the similarity which exists between the students' way of thinking and the spontaneous reactions of teachers or professional physicists as soon as they are faced with unusual situations. However, the latter have a greater ability to change functioning register (Viennot, 1988).

This indirectly raises the question of the status of control in teaching. One continually hears teachers complain that pupils do not control the way they work and their answers. A lot of research has shown the differences which exist between mathematicians and students in solving problems in this area. What is this really about ? Where are the obstacles ? On the cognitive level ? On the didactic level ? Can this control exist publicly in the school institution? Can it be one of the stakes of learning ? It is clear that the epistemological gateway, although it leads us to these questions, is not adapted to studying them. Other, didactic, analyses in terms of didactic situations or institutions are vital. C. Margolinas' thesis provides an example of this type of work concerning validation processes (Margolinas, 1989).

## II - FROM EPISTEMOLOGY TO DIDACTICS

### II-1 : The limits of the epistemological gateway

The epistemological gateway, as with any gateway into didactics, has its limits. The history of the development of concepts, although providing a support for research in didactics, provides only a support. The epistemological obstacles identified in history only provide obstacle-candidates for the analysis of learning/teaching processes. Their power to resist appropriate didactic interventions has to be proved.

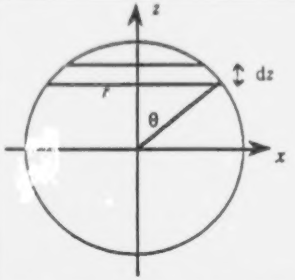
As concerns complex numbers, for example, the situation of today's pupils, for whom complex numbers, directly legitimate in the school institution, as they are taught there officially, are also endowed from the start with a punctual and vectorial geometric representation, is far from that of the 16th century Italian algebraists and even from that of their successors, even if, as is the case currently in French secondary school teaching, they are manipulated without being formally constructed and governed by a stated principle of the permanence of the rules of algebraic calculation. What obstacles put up resistance to these differences ? Which ones do not ? On another level, are the distinctions between tool and object, between structural and operational, in the form in which we made them function above, the most relevant for the study of teaching/learning relations ? I am far from being convinced of this...

In the domain of analysis quoted above, the relations established with history have proved largely fruitful. Nevertheless, they do have limits. Let us consider, for example, the algebrisation of analysis, its status in teaching, and its status in history. In teaching, it functions as a refuge in relation to unavoidable conceptual and technical difficulties :

- difficulties linked to the high level of structuration of the field's objects : since Euler, analysis has been constructed around the notion of function. When the teaching of analysis begins, the conceptualisation of this notion is frequently at a level which is at best, "operational" and the pupils' field of reference is quite limited. However, the teaching of analysis rapidly obliges one to consider functions as elements of larger categories, and to operate subtle distinctions between categories (continuity, derivability ...)
- difficulties linked to the breach pointed out above with algebraic functioning : thinking in analysis, as has written M. Legrand, first means rethinking the status of equalities and inequalities. It means first understanding that in order to prove equality between two objects: numbers, functions... one will not, in general. as in algebra, transform the information through equivalences until the equality aimed at is reached. It means understanding that such a process is frequently out of reach, and that in going the long way round to demonstrate that, for every $\varepsilon>0$, $|a-b|$ or $\|f-g\|$ is less than $\varepsilon$, the detour will be worthwhile. Next, it means understanding that in order to demonstrate that a family F1 of objects, possess a property, one will usually follow an analogous path : demonstrate the property for a simpler, more suitably chosen family F2, show that each element of the given family F1 can be approached as closely as one wants by elements of the family F2, and that the property studied resists passage to the limit. More globally, it means having understood that analysis is the domain *par excellence* of approximation and understanding the general mechanisms which underlie the conceptual and technical functioning of approximation.
- These difficulties are added to by various, more technical difficulties: difficulty of reasoning with sufficient conditions and managing the correlative succession of careful choices (losing enough to simplify but not too much to move away from the problem) basing oneself on a familiarity which can only be built up progressively, difficulties linked to the formal complexity of definitions which, in their standard version, all introduce alternance of quantifiers, difficulties of managing calculations with inequalities, absolute values...

Entrance into the field of analysis can only be considered in the long term, and even considered in this way, it is not easy to set up and manage. The problems encountered in the management of the cursus of analysis set up in 1985 in French secondary schools, as a reaction against an over-formal teaching in this area which was far from the problems at the base of this field, with the desire to promote analysis as a field of approximation, demonstrate this well.

Faced with all these difficulties, the algebrisation of analysis therefore, appears traditionally as a refuge for teachers and students. Its algorithmic strength allows one both to demand and ensure a certain success in the short term. It also allows this success to be easily evaluated on a more institutional level. Hence algebraic calculations of limits, derivatives, primitives, algebraic solving of differential equations, powerful theorems with smooth hypotheses, take precedence over questions which bring into play, in a more profound way, the concepts and techniques of analysis and, in a sense, although they complain, everyone is satisfied. Research carried out in the didactics of analysis constantly show the perverse effects of this precocious algebrisation of analysis, and we refer the reader here to the syntheses presented in "Advanced Mathematical Thinking" produced by the PME working group of the same name. These show that the students know how to calculate derivatives, primitives, limits of functions given algebraically, but they have great difficulty in reading a derivative on a graph, in explaining the meaning of the the calculations carried out, even the symbols used. Even at the end of their studies at university, most of them do not have a clear enough conception of the differences between local linearity and global linearity, of the functioning of approximation in differential and integral procedures, to be able to manage in situations of modelisation which are not strictly scholarly, in order to sort out paradoxes such as the following :



The volume of such a slice at height $z$ is assimilated to that of a right cylinder of the same thickness d$z$ and of base area: $\pi r^2$(as shown in the drawing)
So $dV = \pi r^2 dz = \pi(R^2 - z^2)dz$
and the volume of the sphere is :

$$V = \int_{-R}^{+R} \pi(R^2 - z^2)\,dz = \frac{4}{3}\pi r^3$$

If the same procedure is used to find the area of a sphere, the following expression is obtained for the area of an elementary slice of thickness d$z$ at height $z$

$$dS = 2\pi r dz = 2\pi\sqrt{R^2 - z^2}\,dz$$

and therefore the area of the sphere is given by the integral:

$$S = \int_{-R}^{+R} 2\pi\sqrt{R^2 - z^2}\,dz = \int_0^{\pi} 2\pi R^2 \sin^2\theta\, d\theta = \pi^2 R^2.$$

Could you explain why the same method leads to a correct value in the first case (volume) and to a false value in the second one (area).

However, this description of the functioning of algebrisation in teaching is far from the image history gives us : indeed, historically, this algebrisation, as it became established independently in Newton and Leibniz, provides mathematicians, through Calculus, with general, simple, effective methods to solve problems which have been worked and reworked, some since ancient times. Algebrisation is here the fertile tool of a given field of problems, for professionals familiar with techniques of approximation, who can directly realise the economy that has been made. In a sense we are at the antipodes of the usual school situation.

Once again, historical analysis provides us with grids to interrogate teaching such as the behaviour of pupils, it does not have the means to lead and legitimise, alone, the transpositional work.

### II-2 : A necessary return to the cognitive :

The cognitive gateway is therefore required. It dominates within the PME group and the wealth of its achievements is undeniable. I am certainly not the best placed to report on both the various theorisations developed and the results obtained. I would prefer, here, in relation to this gateway, to put before our community a certain number of expectations concerning the modelisation of the pupil's cognitive functioning, expectations raised by a research practice in which the cognitive approach, although still present, must exist alongside other approaches.

A modelisation will necessarily include a level of states and a dynamism which will condition the evolution of these states. Both the epistemological analyses and the research results require, it seems, supple modelisation structures which allow various phenomena to be taken into account, such as the following :

- the fact that our mechanisms for adaptation are essentially local and contextualised mechanisms,

- the fact that the coherences which are constructed are firstly local coherences which can long coexist with incoherences at a more global level,

- the fact that the construction of knowledge is just as much linked to the setting up of connections between objects, local processes, symbolic or mental representations or even the destruction or modification of unsuitable connections as to the appearance of new objects, processes or representations,

- the fact that development in one cognitive sector can modify, in later work, sectors which are not directly targeted but which are connected to the latter,

- the fact that there exist hierarchies : processes become objects or conceptual entities in order to, in turn, become caught up in processes of a superior order, but also the fact that these hierarchies are not constructed linearly but dialectically,

- the fact that the relations between operatory, technical and conceptual developments must also be envisaged dialectically,

-finally, the fact that there exist in our cognitive functioning parts which are compiled to reach wholes which can only be dissociated with difficulty.

Various conceptual tools elaborated up to now take these demands into account, even if it is not always with the necessary flexibility. Without claiming to be exhaustive, let us quote some examples : the distinction introduced by D.Tall and S.Vinner between "concept-definition" and "concept-image" linked to the notion of "potential conflict" (Tall and Vinner, 1981), the notion of "local knowledge" introduced by C.Sackur, F.Léonard (Léonard, Sackur, 1990) aim to take into account local adaptations and coherences, the notion of "interplay between settings" introduced by R.Douady, the articulations and "congruences between registers" of representations studied by R.Duval (Duval, 1989) are tools to master connections ; theorisations such as those of E.Dubinsky (Dubinsky,1991) and A.Sfard quoted above identify cognitive processes which permit a hierarchised vision of knowledge. However, for the moment, despite the efforts of some as shown in the book pulished by the AMT Woking Group from PME quoted above, these constructions remain local and their global coherence remains to be constructed.

Within what is often called the French school of the didactics of mathematics, the modelisation of the subject has mainly been organised around the notion of conception, in connection with the theory of "conceptual fields" and "schemes" set up by G.Vergnaud (Vergnaud, 1990). The notion of conception originally appeared to be guided as much by epistemological as by cognitive preoccupations. The diversity of points of view which were a priori possible on a given notion had to be pointed out (on the side of knowledge) and also, by separating concept and conception, the non-transparency of the transmission of knowledge (on the cognitive side). After that, in a lot of research, this notion took on a more strictly cognitive meaning. At present, it seems to have stabilised as an object presenting the following characteristics :

- it is an object of local nature,
- it includes various components :
    - situations favourably linked to conception : examples, typical counter examples, but also problems for which it is an effective tool (level of referents)
    - symbolic, mental representations (level of signifiers),
    - implicit or explicit invariants and processing instruments (level of signified).

The emphasis placed above on connections leads us to consider these local conceptions as elements of more global modelisations : cognitive networks with necessarily blurred outlines. However, it must also be recognised that we can only conceive of these global modelisations via what the only partial windows of our experimental research allow us to glimpse of the pupil's cognitive landscape.

Indeed, if one takes a close look at research on conceptions, it does not take as its aim the close description of such global networks. More modestly, it tries to gather together, through qualitative studies, but also according to statistical indices of implication and similarities, (Gras, 1991), behaviour observed into key categories which will be identified as conceptions and to report on their development, while trying to highlight a certain number of regularities, significant stages and reference points. This modelisation has proved to be effective over the last ten years or so, both on the level of research and the transmission and exploitation of its results. The articles published by the journal Recherches en Didactique des

Mathematiques demonstrate this clearly.

Nevertheless, I do have the impression that, in a certain number of areas which have already been explored, the way forward must now, on the cognitive level, pass by studies which go beyond the identification of major regularities, and which seek, outside the classic schemata, to better understand the diversity of the dynamic processes which underlie the evolution of conceptions. This same refining of analysis also seems to me necessary for the establishment of productive relations with the sector of artificial intelligence.

I will use one example of recent research on conceptions, that of C .Castela concerning the notion of tangent (Castela, 1992), to clarify the meaning of the questions raised in this paragraph, then to show the limits of a study which would be solely cognitive in this area. This research took as its aim the idea of clarifying the development of pupils' conceptions during secondary schooling and the effect of teaching on this development. The research was carried out through the analysis of textbooks, pupil and teacher questionaires and we will only concentrate here on one questionaire (see annex) in which various curves and straight lines are proposed to the pupils who are asked to judge the affirmation : the straight line is tangent to the curve at point A, and to justify their answers. Although this research, as far as methodology is concerned, is situated poles away from research such as that of Schoenfeld et Al, which concentrates on the microscopic study of one single student (Schoenfeld &Al., 1992), both present similarities due to the attitude they lead us to adopt on the local mechanisms of the evolution of conceptions.

The first "tangent" object encountered by the pupils in the junior cycle of secondary school is the tangent to a circle. This object is a geometric object endowed with specific properties : it does not cut through the circle, it only touches it at one point, it is perpendicular to the radius at the point of contact. All these properties are ones which, globally, bring into play the circle and the tangent and do not bring into play the idea of common direction. In order to help the pupil to become aware of the abstract status of the figure, one can even insist on the fact that although to the eye they seem to merge locally, the circle and the line only have one common point. In the same way, the tangent is linked to secants but secants of a given direction, which, when moved, help to visualise the number of intersection points : 2, 2 merged, 0. In the senior cycle of secondary school the teaching of analysis introduces other points of view on the tangent : it is a local object, with which the curve which represents the function tends to merge locally, it is also the line whose slope is given by the value of the derivative at the point considered. There is no direct relationship between these two objects and it is legitimate to wonder how the pupil's transition works, if it does work, from a "circle" conception to an "analysis" conception. It must be emphasised that in this evolution, the tangent to a circle is also reconstructed : it integrates the characteristics of locality and directionality of the general tangent, and finally becomes the prototype of the tangent to closed convex curves.

In the research mentioned, a priori conceptions were developed from different properties of the tangent to a circle, tne items in the questionaire having been chosen to allow them to be differentiated and differentiated from the tangent in the sense of analysis. 372 pupils were questioned, at different levels and from more or less scientific orientations, before the teaching of the derivative, just after, and one year later.

The answers obtained highlight local adaptation processes which are set up in the long term and do not easily allow themselves to be slotted into the usual schemata of conflicts and ruptures, or in hierarchisations of the process/object type. Before the teaching, the large majority of answers demonstrate coherent conceptions linked to the circle, which either blend all the properties of the tangent to a circle, or focus on one of these. This is translated by answers which mainly contradict the conception "analysis". After the teaching, the landscape becomes more chaotic, even though all the items, except 4 (a curve locally merged with a tangent) are grouped together globally on the side of success. Item 4, while remaining separate, only reaches 50% success in "terminale" (final year of secondary school), (even though the derivative is introduced in France from the notion of linear approximation !).

It all happens as though, progressively, while remaining an anchor point, the conception "circle" gradually gives way, through different processes : by admitting prototypical exceptions such as inflexion, by rejecting prototypical cases : angular points, by joining (for correct or incorrect reasons) the elements stemming from the analysis ("there is only one point of intersection and it is a maximum", "there is one common point and the curve approaches the straight line tangentially"), in taking on a more local character, by integrating progressively the idea of direction. This continues until, too much way have been given, the swing towards the "analysis" tangent occurs, the latter in turn becoming the dominant object in relation to which the cognitive network will become reorganised. However, even in a predominantly scientific "terminale", only 25% of the pupils presented justifications which were sufficiently homogenous to allow one to suppose that such a swing had occured (and an explicitable level had been reached), and perfectly correct answers could be accompanied by answers which were globally incoherent, step by step.

## II-3 : The need for systemic approaches in terms of didactic situations and institutions :

We have presented this research here in cognitive terms, but what it provides us with is just as much the functioning of the school institution as the functioning of the pupil, if not more. The cognitive approach encourages us to set the behaviour in a cognitive rationality but is this approach relevant ? Can we, for example, state that the research carried out tends to confirm the hypothesis of the existence of a cognitive obstacle to the setting up of the conception "analysis" because this conception resists and produces errors ? What conditions the adaptations noted here and their limits ? The pupils, the situations proposed, the demands of the system ? Are other adaptations not possible ?

The research provides us with elements for an answer on these questions. On one hand, analysis of textbooks shows that the question of the relation between the conception "circle" and the conception "analysis" is totally evacuated from teaching : either the tangent to the circle is not mentioned, or it is considered to be a transparent example. On the other hand, one of the "terminale" classes, non-scientific moreover, appears to be atypical, clearly overtaking all the other classes in its results : this was a class in which the cognitive reorganisation was not left entirely up to the pupils during the previous year.

What this research shows us, therefore, is not the way in which the pupils might construct the concept of tangent and the conceptual difficulties linked to this, but more the game the pupils play in relation to the teaching of the tangent in school and the way in which they optimise this game.

The adaptations carried out, although local and incoherent, out of step with the learning targeted, are quite adequate to allow the pupil to play his role of pupil effectively. Indeed, this role, for the marginal object that is the tangent in current teaching, consists in :

- knowing how to recognise simple cases of non-derivability : vertical tangents, angular points,
- knowing how to determine the equation of a tangent in order to satisfy given conditions,
- drawing, on a graphic representation of a function, the tangents at particular points (extrema, inflection points...)

In particular, the tangent is never really involved as a tool in the solving of problems and the hypothesis could be made that this has an influence on the cognitive adaptation process that the pupils bring into play.

It would be easy to reformulate this part of the analysis by referring explicitly to the anthropological approach recently developed by Y. Chevallard (Chevallard, 1991), which is well adapted to this institutional view. We would no longer talk in terms of conceptions, then, but more globally in terms of "relation to the object tangent". Y. Chevallard, indeed, thinks that the notion of conception has too many cognitive connotations to be an effective basis for an anthropological approach to didactics. In such an approach, we would have to distinguish different types of relations to this object tangent : the pupils' personal relations, but also the institutional relations which condition these personal relations and their evolution. It is of course, not possible here to go into the details of such an analysis, but it is clear that we will have to distinguish at least two institutions : the institutions "collège" (junior cycle of secondary school) to which one could add the first year of "lycée" for which analysis is not a teaching object, the institution "lycée", for which it is, and to clarify which relations to the object tangent both institutionalise. In other words, we are trying to clarify what "knowing what a tangent is" or "having understood the notion of tangent" could mean for the institution collège and the institution lycée. It is to this analysis that we shall relate the pupils' behaviour via what they show us of their personal relation to the object, while trying to understand up to what point this behaviour is determined by institutional constraints.

It is clear that this dimension of analysis cannot purely and simply take the place of the cognitive approach developed at the start : the pupil can no more be reduced to an institutional subject than he can be reduced to a cognitive subject. However, it does return this analysis to a more global frame and, by obliging us to introduce didactic complexity into the cognitive, gives it a new dimension. It could also oblige us to situate our interpretations in relation to others which are equally legitimate.

Beyond the example chose, crucial questions arise behind this institutional approach : what is knowledge ? If it is true that we are subjects of multiple institutions, for objects which many of them recognise, is there any coherence or compatibility between the relations with knowledge to which they tend to subject us ? These interrogations link up with those of many researchers, even if they are not expressed in exactly the same terms, especially those working with pupils from low social classes for whom cultural and social subjections are more easily a source of relations to knowledge which do not fit the expectations of the school institution (Carraher, 1988), (Perrin, 1992).

In the above, we have insisted on the fact that the cognitive functioning of the pupil whose school learning we are studying is constrained by the institution. The pupil learns by adapting but various mechanisms for adaptation coexist and are linked within the teaching process : in solving a problem set in school, the pupil brings into play his mathematical knowledge, but also uses knowledge acquired from the way in which his class and his teacher work, and in a certain way combines two types of adaptation, two types of learning. Up to what point are these adaptations compatible ? How can teaching processes which ensure this compatibility and allow it to be controlled, be set up ? The institutional approach developed here leads us naturally to these questions, and alone it does not provide the means for solving them. We have to return to a more local view of the system we are studying. To my mind, the theory of didactic situations developed by G. Brousseau (Brousseau, 1986) to which I will add here, even if it cannot be reduced to this, R. Douady's approach in terms of tool/object dialectic and interplay between settings, can be read as a theory in which these questions are central. Indeed, G. Brousseau emphasises that School is an institution which is necessarily transitory ; in these conditions its mission is to allow the pupil to construct knowledge which can exist and function outside it. In order to do this, he writes, it must offer the pupil situations such as : "the pupil is fully aware that the problem has been chosen to allow him to acquire new knowledge but he must also be aware that this knowledge is entirely justified by the internal logic of the situation and he can construct it without calling on didactic reasons".

This means that any teaching situation which claims to be such, necessarily holds didactic stakes, but also that School, in order to be "mathematically effective", must not allow itself to be trapped by these didactic stakes. It must, at times at least, make the pupil forget that it is a didactic institution. In order to take this into account on a theoretical level, G. Brousseau has introduced the notion of "a-didactic situation", the negating "a" raising the image of a situation cleared of its didactic stakes. In such a situation, the mechanisms for adaptation to the "milieu" bring into play a-didactic processes of adaptation - linked only to the relations of the pupils to the problem set - and not didactic processes of adaptation - linked to the anticipation of the teacher's expectations, to the knowledge of the habits of the class, to the didactic contract. In this theory, the "devolution" is conceived to modelise, within the global didactic situation, the process by which the teacher will negociate with the pupil the gateway into this a-didactic functioning and maintain it. Symmetrically, institutionalisation is the process by which the didactic recovers its place. This process consists in fixing the status of the contextualised knowledge constructed in the a-didactic phase, by helping in its decontextualisation and by linking some of it to the institutional knowledge aimed at by the teaching.

The work of G. Brousseau and his team, as well as that of others, provides illustrations of this theorisation applied to the teaching process in various domains, especially at the level of primary school. If we wanted to exploit this approach in relation to the tangent, we would first have to return to an epistemological interrogation, in order to define mathematically what the object tangent is, in which type of problems it can be involved at the level of secondary school teaching, what links it has with the other objects in the conceptual field of elementary analysis. From this analysis, research on situations typical of the functioning of the concept (G. Brousseau speaks of "fundamental situations") could take place which could serve as a basis for a teaching process at the level considered. The didactic shaping of each situation would take place according to the theory, trying in particular to analyse, according to the choices of variables of the situation, the possible games of the pupils and their potential meanings, then to optimise these choices in relation to the learning targeted. Finally, on the basis of this global construction called an a-priori analysis, an experiment would take place and in relation to it the behaviour observed would be analysed, and the reasons for the distorsions between what was forecast and what was acheived would be sought.

The above only represents a gateway into the theory of didactic situations. I have not tried here to give an idea of its wealth, or of its ambitions for a more global theorisation of the didactic field. To match the aims of this presentaion, I have tried, rather, to link this approach to others. However, I would like to insist on the fact that its influences are not limited to the study of processes developed following it. It has become just as much an effective instrument for the analysis of any sort of teaching situation and the concepts derived from it such as that of the didactic contract have proved their effectiveness even outside didactics of mathematics.

## III - CONCLUSION

Throughout this presentation we have tried to show how, in the didactics of mathematics, the coexistence of a variety of approaches is not simply the consequence of the immaturity of the field, that this is quite normal considering the complexity of the phenomena studied and that this can be beneficial. However, it is also obvious that these benefits are not self-evident. They depend, partly on a greater coherence of the approaches relating to the main gateways into the didactic system. They also depend on the search for links, coherences and complementarity between different approaches. Each of us favours certain gateways. We feel that even if it is possible to translate from one theoretical frame to another, these systems of translation are necessarily imperfect, that they cannot express the wealth of the respective theses central to each approach. What is central and coherent in one approach becomes a peripheral element or becomes irremediably broken up in another. Nevertheless, although we remain aware of the limits of such link-ups, we must try to develop them. It is through them that communication and a real coherence of the field of didactics can take place, a coherence which will necessarily be more global and more than just the juxtaposition of local coherences.

Through this presentation, I hope I have contributed to convincing people of the interest of such work. It has been undertaken, for example, in France, within a "Didactique" research group in the CNRS (Centre National de la Recherche Scientifique) which tries to confront, on the same corpus, analyses stemming from different approaches : analytical, situational, anthropological and psycho-social.

To conclude, I would like to emphasise that this work is certainly necessary, and it can also be fascinating.

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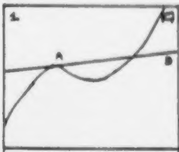
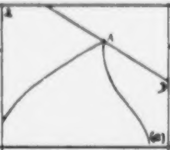
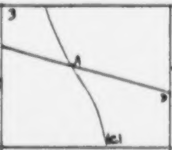
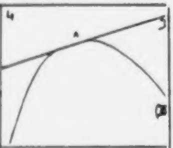
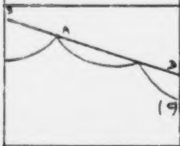
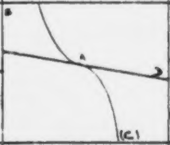
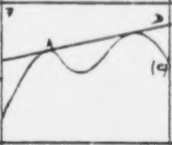
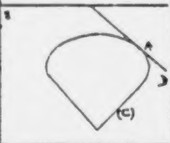
## ANNEXE :

For each of the eight following graphs :

1) Say if the line D is tangent in A to the curve (C)

(Circle the correct answer)

2) Justify your answer in the box under the graph.

| | | | |
|---|---|---|---|
|  |  |  |  |
| OUI - NON | OUI - NON | OUI - NON | OUI - NON |
| | | | |
|  |  |  |  |
| OUI - NON | OUI - NON | OUI - NON | OUI - NON |
| | | | |

# MATHEMATICS AS A FOREIGN LANGUAGE

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**Language shapes thought**

*Mathematics, although an abstract discipline, relies heavily on the use of an appropriate language. A closer examination of the way mathematicians "talk" about their theories reveals that this language is above all a written language, introducing specific symbols, which represent concepts rather than words, intermingled with bribes of natural language. We call this language* MATH. *Some of its particular characteristics are quite different from the ones linguists consider to be basic requirements for a general theory of natural languages. Among these theories, it seems that the conception of language expressed in the transformational-generative grammar (TGG) of Noam Chomsky comes closest to a suitable description of the language of mathematics. Due to the very special style of* MATH *the question of its readability is of an extreme importance in teaching.*

## 1 GENERAL

The fundamental purpose of any natural language is to serve as the expression of a set of ideas. It seems that human thinking, ifnot impossible, is at least very restricted in its way of expressing thought without the help of a language. The function of the sounds, words, sentences, which in most cases are purely conventional, is the expression of these ideas.

In understanding a language, as becomes very clear in the case of translations, one has to refer to the ideas, not to the words. Consider for example the various idioms of a particular language such as the english "to take french leave" which is translated in french by "filer à l'anglaise". Many supposedly basic characteristics of a familiar language are in fact conventional and could be dismissed, as the occurrence of grammatical gender, plurals, conjugations and declensions. These are part of the fundamental structure of many Indo-European languages but missing in e.g. japanese. Declensions are basic grammatical components in latin and russian, but have almost entirely disappeared in english; gender and conjugations are basic in french and latin but strongly reduced in english.

Considering mathematics, which consist basically of a body of interrelated concepts and ideas interacting with each others and linked by relations of various kinds, it is obvious that the core of the discipline is *meaning*, hence there is need for a language in which to express this meaning. Thinking is easier if the language is compatible with the elements and concepts of thought. It is worthwhile to raise the question of which kind the special language is which fits the formulation of the problems of mathematics and enables the human mind to perform any progress in the very special domain of formalised, abstract and deductive thinking called mathematics.

Stated differently, one has the right to ask if we are able to formulate in our common language all the questions related to a mathematical topic. The subject is not devoid of interest as one of the main problems in advanced formal logic is that if the definiability of mathematical concepts. However it is not our purpose to consider the question from the viewpoint of logic, but we'll consider rather the informal aspects of the use of the vernacular of mathematics.

Practising mathematics, soon it becomes clear that the vernacular of any natural language is not really appropriate. We need an specific way of expression, a need for a writing system which will appear to be an ideogrammatic expression of the mathematical thinking.

From the history of mathematics we know that the development of modern mathematics has been decisively stimulated by the introduction of an appropriate symbolism in the 16th and 17th century. Expressing facts about an unknown quantity became very easy representing this quantity by a symbol such as $x$ or $y$ or $t$. Another case is the tremendous progress of geometry of curves and surfaces initiated by the discovery of René Descartes substituting an analytical expression such as $ax + by + c = 0$ for the ancient concept of a straight line. The lack of an appropriate language stopped the development of Ancient Greek Geometry.

The answer to the question in the preceding paragraphs is well known; it is the language mathematicians have developed in the last three or four centuries and which enables them to express their results with the required precision. Useless to say that the position of mathematicians towards their language is pragmatic: they simply use it and pay attention to only one of its characteristics: precision. Another important characteristic: clarity, is even not always under concern. The language structure itself is usually not the object of a closer examination; current use is enough to assert its validity for their purposes.

We will use the name *mathematics* for the discipline and use MATH as the name for the language of the discipline. The name refers to the language in which mathematics is *thought* and *written*, not primarily *spoken*.

There may be reasons to consider the ability of speech as a genetic inborn ability, but the acquisition of the native language by any individual has anyway to be learned from example, from imitation of the speech of adults. Hence we consider MATH as a foreign language which has to be learned. Before a study of, say, french literature becomes possible, a study of the french language itself, in its structure and meaning, is needed. Similarly, before mathematics can be studied, the language in which it is expressed must be learned. A very brief expression of the need to be introduced into the language is found in many handbooks, where at the beginning a more or less complete list is given explaining the conventions in notation.

The STANDARDS mention the need of an appropriate knowledge of the language of mathematics (Ed. 1989, Standard 2, p.140)

> In grades 9-12, the mathematics curriculum should include the continued development of language and symbolism to communicate mathematical ideas ...

Learning a foreign language requires mastery on two domains (1) the words, grammar and syntax of the language, and (2) to get initiated to a foreign culture of which the ideas and concepts are expressed by the terms of the language. This is certainly true for MATH.

Fluency in a foreign language is achieved through the ability to think in this language, which has to be the ultimate goal of the learning. See e.g. the use of number words in a foreign language which is considered to be one of the most difficult skills to acquire in the approach towards fluency, as the french 97 *(quatre-vingt-dix-sept)* and the german 52 *(zwei-und-fünfzig)*; the japanese think in units of 10,000 and not of 1,000 as in the western languages.

Luckily, in practise MATH is taught though examples, using language and content simultaneously. Teaching of the language goes together with the teaching of mathematical thinking itself. This is a most favorable situation; it is difficult to imagine how a succesful learning of mathematics could be achieved through the preliminary teaching of first an extensive set of symbols, followed only when the first objective is realized, by the exposition of the theoretical frame which uses the symbols. Yet, the recent endeavour to organise the teaching curriculum in the formalistic way starting with axioms and definitions followed by the formalised deduction of all dependent theorems with proofs seems to have been very close to the former procedure. Formalised/axiomatic teaching of mathematics in high school may be considered equivalent to the learning of a foreign language not through a manual for beginners but through a dictionnary.

## 2 CHARACTERISTICS

Among characteristics of the language MATH which distinguishes it from natural languages the following are worth while to mention.

**A WRITTEN LANGUAGE.** To many of us MATH becomes unintelligible after a few minutes if a blackboard is not available. There seems to be very few exceptions, it is said that Euler was almost blind at the end of his life and dictated his mathematical thinking to his daughter.

The empirical necessity to *read* mathematics, as opposed to the alternative of *listening* to mathematics, forces us to consider the MATH as a written language. As opposed to many languages which are spoken but have no written tradition, MATH is a written language with a rather weak spoken tradition. The pronunciation accompanying the characters of MATH is largely unimportant. One could sustain the conjecture that MATH is a language without sounds. Its usefulness is almost completely concentrated in its written form, not in the oral form. In that respect it is very similar to the reading of an ancient, extinct language as Ancient Egyptian, Accadian, Hettite. Reading is uncertain in the sense that the sounds are no longer well known and that a page of text is not read at an eyeglance but rather has to be *decoded*. Of course, in conversational interactions mathematicians use sounds but the pronunciation is of little importance. Look at the symbol "5" which sounds as *cinq* or *five* or *pyat'* or *go* but is understood as soon as the symbol 5 is written on paper or on the blackboard. In reading a natural language the prevailing direction of understanding is directed from the written text towards the sounds. On the contrary, in MATH the direction of understanding is from sounds to ideograms. From the teaching of mathematics for beginners to the exposition of new results to research mathematicians the teacher/speaker invariably accompanies his talking by writing on the blackboard, or using an equivalent device such as transparencies o a computer screen, in any case: a written document.

**USE OF IDEOGRAMS.** Mathematical literacy depends on recognition of patterns: the symbols of MATH are ideograms, it is an ideographic language. Look at the meaning of something as

$$\sum_{n=0}^{n=+\infty} \qquad \text{or} \qquad \lim_{x\to+\infty} f.$$

This writing system is similar to the one of Chinese, Japanese and the hieroglyphs of Ancient Egyptian. Graphs and geometrical figures are part of MATH as well, as they are needed for understanding. But these are impossible to pronounce, hence may be called ideograms. The ideograms of MATH provide e.g. a convenient way, depending on the context, to introduce the infinite into the language, as with ... (decimal notation of numbers); $\pm\infty$ (limit of functions); $\lim_{n\to\infty}$ (convergent sequences); $\Sigma$ (convergent series).

Mathematics relies heavily on the formation of concepts which are abstractions from facts referring to real situations; the ideograms serve as devices for the communication of concepts. It is easy to talk about facts but concepts have to be constructed and are not that easily communicated, even not if all the underlying facts are communicated. Talking about a concept becomes easier through the construction of the corresponding ideogram. The most common example maybe are the numerals. The development of number is a long evolution exemplifying a sequence of increasing levels of complexity: (1) numerals used to count real objects (as in the sumerian civilisation) (2) numerals as ideograms (3) characters from the alphabet used as representations of numbers (4) the concept of sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbf{Q}, \mathbb{R} \ldots$

**A CLASSIFICATION.** A consequence of the widespread use of ideograms is that MATH may be classified from a structural viewpoint as an *agglutinative* and *polysynthetic* language. From that viewpoint MATH is totally different from our familiar Indo-European languages which have an analytic structure.

Agglutinative languages such as Hungarian and Turkish have the characteristic of building an extension of a single word concept by joining suffixes to that word. E.g. from the hungarian *ház* (house) the forms *házam* (my house), *házaim* (my houses), *házaimban* (in my houses) are derived. These constructions are comparable with

$$\oint_C I\overline{dl} \times \overline{B} \qquad \text{or} \qquad \frac{\partial^3(g \circ f)}{\partial z \partial y \partial x}$$

In MATH a symbol is often extendable and acquires a derived meaning if we adjoin affixes to it: pre- and suffixes, even infixes. The most common affixes are subscripts and superscripts, others have the meaning of exponentiation as in $x^n$. (The affix may even be "void" as in $x^1$.)

In polysynthetic languages, as many of the North American Indian languages are, several among the seperate components of a sentence have no meaning of their own but acquire a meaning through the juxtaposition with the other elements of the sentence. In these language a sentence is a whole which becomes ununderstandable if broken up in smaller pieces. E.g. in Oneida (an Iroquois language) *g-nagla-sl-i-zak-s* means "I am looking for the village". To say that MATH has the same polysynthetic character means that a sentence in MATH becomes understandable only when the very last symbol is written. An equation in not solvable (i.e. not understandable) if the sentence $5x^3 + 2x^2 - 10x - 2 = 0$ is not written as a whole and looked upon as a whole. The sentence stripped of its last element: $5x^3 + 2x^2 - 10x - 2 =$ has no meaning in mathematics.

Substituting an analytical conception for the polysynthetic ideogram of the antederivative of the exponential function, a student wrote

$$\int e^x dx = e^{\int x dx} = e^{\frac{1}{2}x^2}.$$

The function symbol $e^x$ is interpreted in an analytical way, not as an inseparable part of the integral.

**SIMPLICITY and COMPACTNESS.** Professional users of MATH show a constant tendency to keep their language as simple and concise as possible. There is no presumption to develop a literature or poetry in MATH. An often repeated advice for authors of mathematical papers sounds "*Don't tie yourself into verbal knots.*" For example, the use of negative sentences entails usually a more complicated grammatical structure which may be considered a decline of readability. Classroom practice taught me the reading of an inequality $x \leq y$ as *"x is smaller than or equal to y"* or, for short *"x is smaller than y"* is more easily assimilated by students than the equivalent alternative *"x is not bigger than y"*. Moreover in the former there is clearly a conceptual link with the symbolic expression $x \leq y$. In more intricate situations the danger of blocking the understanding of the reader increases, as in expressing the property $x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$ as *"f is a non-decreasing function"* instead of *"f is an increasing function"*.

A more advanced example, about normal subgroups: the sentence *"we know a normal subgroup N remains invariant under an inner automorphism of the group G"* seems esoteric to a beginning student of Group Theory. Why not tell him/her: "$\forall g \in G,\ g^{-1}Ng = N$" ? This is an example of the fact that mathematical symbolism might be simpler and easier to understand than a verbal expression, because the verbal expression is too much context-dependent.

Besides simple, MATH is compact hence many mathematical texts are terse. This may be variable from expository to research papers, the former are expected to dwell more at lenght on details than the latter. Anyway, it is not exceptional that many steps are omitted in mathematical discourse. This is extremely important in writing proofs.

A consequence of the compactness is that correct reading in MATH implies the regeneration of missing steps. In this respect MATH is quite different from an algorithmic language, where no missing steps are regenerated, the most extreme case is a computer program. Regeneration of missing links in a deductive inference implies understanding of conventions such as the use of parentheses, the symbol $\sqrt{x}$ which refers to the positive root, . . . and awareness of hidden hypotheses as in the equation $\sqrt{x-1} = x$ ($x$ real) which supposes $x \geq 1$.

The equality symbol "=" is a typical example of hidden meaning: compare the expressions $(x+1)^2 = x^2 + 2x + 1$ and $(x-1)^2 = 2x + 3$. Examining its use the following cases may be listed, in each case the meaning of the symbol is different, and there are probably other cases.

1. A *elementary equality*, as in $1 + 2 = 3$ or in $f(1) = 2$ where $f$ is a given function.
2. An *equation*, as in $x^2 + 3x - 8 = 0$.
3. An *identity*, as in $(x-2)^2 = x^2 - 4x + 4$.

4. A *definition*, as in defining the *tangens* function if the *sinus* and *cosinus* functions are given

$$\tan x = \frac{\sin x}{\cos x}.$$

5. A *substitution rule* which confers a meaning to, say, a variable as in "*let* $x = 4$" or to a function symbol, as in "*let* $f(x) = e^x$". This case is related to the previous one, but is not quite the same.

6. A *selection rule, or constraint or hypothesis* as, working with functions of several variables, in "*let* $x = y$".

**THE SPELLING AND THE LEXICON OF** MATH. The choice of a particular ideogram for the representation of a mathematical concept is a priori free, although some constraints exist: up to some degree consistency, clarity and tradition should be respected. Hence it can be said, roughly, that MATH is spelling-free. Consistency is required but one should be aware of the divergence due to the occurence of local definitions which restrict the meaning of a symbol to only one paragraph or chapter. In local definitions MATH is variable in meaning: in the same paper a symbol as $x$ or $f$ may have different meanings from one paragraph to the other. Hence, in speaking/writing the MATH language a switch may easily occur, sometimes the symbols maintain their meaning, sometimes there is a change in the meaning of the ideograms.

MATH may even be character-independent as well: the following expressions have the same meaning

$$\forall x \in \mathbb{R},\ x^2 \geq 0 \qquad \text{and} \qquad \forall t \in \mathbb{R},\ t^2 \geq 0.$$

The use of dummy variables, as the indices in summation formulas or the variable of a definite integral, is another example where the mathematical concept is not dependent on the particular symbol used as an index or as the variable of a function. More examples of the character-independence of MATH are: the symbol $x, t, \ldots$ which represents the variable in an identity or the unkown quantity in an equation. In the latter case $x$ is in fact another notation for a numeral: as in $2x - 4 = 0$ where $x$ stands for 2 and in $x^2 - 9 = 0$ where $x$ stands for 3 or -3 or, more exactly, $x$ stands for a member of the set $\{3, -3\}$.

MATH uses symbols which have a rather faint existence as a concept and which could be discarded from the language. The only purpose of an ideogram such as $\pm\infty$ is its use as a shorthand, as in

$$(a, +\infty) = \{x \in I\!R \parallel a < x\}$$

or in $\lim_a f = +\infty$. The definition of such a limit doesn't refer to "infinity" in any way or another; in the definition

$$\lim_{x \to +\infty} f = -\infty \iff \forall N < 0,\ \exists\, M > 0 \ \text{such that}\ \forall x,\ x > M \implies f(x) < N$$

there is no need to mention the words and the symbols for infinity.

An important problem in the study of any language and a major concern in the conception of language of Chomsky is to draw up an inventory of the lexicon. In contrast, through the role of definitions, MATH has potentially a completely defined lexicon. A string of symbols, such as in $\sqrt[3]{z}$ ($z$ complex), $0^0$, $f(x)^{g(x)}$, doesn't have a meaning of its own in MATH but needs a definition. A definition is the introduction into the lexicon of a new item, entirely dependent on and completely determined by the preceding terms in the lexicon.

**LINEARITY.** MATH is *linear* as all other languages, but deviations from linearity are considerably more frequent in MATH than in a natural language. Some instances of reading mathematics which don't respect the usual *left-to-right* rule are

- *Up-down*: a fraction $\frac{2}{3}$
- *Down-up*: the second term in adding fractions $\frac{2}{3} + \frac{5}{7}$
- *Diagonal*: a term of a subsequence of a sequence $x_{n_i}$
- *Backwards*: composition of functions $(h \circ g \circ f)$ and partial derivatives of higher order $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$
- *Circular*: Matrices, in the order $x_{11} \to x_{1n} \to x_{nn} \to x_{n1} \to x_{ij}$
- *Planar*: Graphs and geometrical figures in the plane
- *Spatial*: Geometrical figures in space (use of perspective)
- And what about reading the Monge projections of the old Descriptive Geometry ?

**CONNECTIVES.** MATH has a peculiar category of words, such as *let, thus, so, hence, obviously, ....* Their purpose is to link the deductive development together. They are prepositions ...in natural languages but are in fact an extended category of CONNECTIVES in the sense of formal logic.

**INFORMAL LOGIC.** A language appeals to some informal logical principles, which are well known to its speakers. But in MATH some are quite different from what is usual in natural languages and may create an obstacle to understanding. This is the case with the logical connective *or* ($\vee$); its meaning doesn't coincide with the conjunction *or*, the latter being often used in an exclusive sense. Of particularly importance is the True/False dichotomy of the natural language because in mathematics it is very often superseded by a three valued logic True/False/Undefined (as in the case of the limit of a function).

## 3 STRUCTURE

Among present day descriptions of the structure of natural languages many characteristics of the transformational-generative grammar (TGG) of Noam Chomsky (1957) seem to be appropriate for a linguistic analysis of MATH.

The older theory of the School of Bloomfield remains too much at the surface of the linguistic phenomena to be appropriate for a study of the language of mathematics. As Bloomfield associates the description of language structure mainly with a study of the observable linguistic phenomena, i.e. with what can be done with the utterances of the language, applying his theory to the language of mathematics would restrict us far too much to considering what can be done with the theorems and formulas under consideration. In other words, it would force us to pay almost exclusively attention to applications of mathematics. And how useful applied mathematics may be, a broader field of investigations is needed if we want to understand how MATH is conceived and how it is structured and works.

We consider Chomsky's theory as the closest approximation of the structure of the language used to express mathematical thought.

We see MATH as a language consisting of two main components: a Formal Deductive System (FDS) together with a Model/Interpretation, both embedded in between two informal layers of natural language: the hypo-language below the FDS and the meta-language above it.

The concept and role of the term "meta-language" is well known and needs no clarification; the term "hypo-language" refers to the informal language which is needed to verbalize the most elementary ideas about what will be the subject of investigation in mathematics and which has to come at the very beginning of the learning or research process. Without comprehension of the primary sounds of the hypo-language any understanding would be totally impossible. E.g. the axiomatic definition of a Group beginning with the sentence *We call "Group" any set of objects, called the elements of the Group, which satisfies the following requirements* ... would be ununderstandable if the reader doesn't have a previous knowledge of what is meant by the verb *to call* or the adjective *any* etc. Suppose a non-native speaker of english understands "any" as "two": he will be looking for two sets of objects, running eventually into confusion. Moreover, there exists a permanent feedback from the meta- to the hypo-language such that the latter may be changed or extended by the results of the FDS and the lifting of these results into the meta-language. An example is the expression "*clopen set*" in topology which is not in the Oxford dictionnary. One may consider with some right this term as part of the meta-language, colloquial usage however reveals this term is rather part of the hypo-language, due to feedback at some moment in the development of the theory. If MATH is considered to be a language we are forced to raise the question of what kind is the Grammar of MATH ? What is the Syntax of MATH ? A quotation from an advisory text about the writing of mathematics sounds "*Rightly or wrongly, mathematics is regarded as having an existence independent of the words used to describe it.*" This statement is based on the fact that in order to write in the appropriate way about a mathematical concept you should first internalize it and make it your own before restating it. And if you have internalized the concept, you can recreate it, no doubt with different words.

Phonology (the sounds of the language) is totally unimportant, mathematicians use in fact the hypo-language to pronounce the ideograms of MATH. Grammar (how the words are formed and transformed, morphology) is probably not much more important. Ex. is plural important? The statement *"A and B are sets"* says nothing in particular (number of elements, etc.) about $A$ and $B$. What counts is the syntax: the agglutination of the ideograms, and the syntax of MATH is closely related to *meaning*, hence to the semantics of the language. The

TGG is primarily a *syntactical* description of a language and considers a language as a system of rules; in that respect it agrees with the fact that syntax seems to be the most important in MATH.

Let's look at some distinguished characteristics of language, according to TGG, and establish the link with MATH. However, it is important to keep in mind that the TGG gives a description of how the language is structured and operates. It is not a theory of how language concepts are generated in the mind; the latter is an other issue which is part of psychology, not of language theory.

**CREATIVITY.** Language is a creative activity: speakers produce new sentences over and over again. As the construction of a meaningful sentence is an activity of the mind, it seems acceptable to pretend that even a previous sentence, if repeated, is recreated anew by the mind and is not, in most cases, a simple repetititon of sounds, as a parrot would do. This is exactly what we expect from an individual thinking and speaking/writing mathematically. Teaching has not been successful if a student is unable to recreate its content, doing nothing more but repeating mindlessly a theorem or a formula. Creativity as a continuous injection of meaning is a basic requirement for valuable performances in mathematics. As an example of the openess of the MATH language to the creation of new items, think of the many symbols devised, more or less in that order, to express for different purposes, the idea of the derivative in the language: $f'$, $f$, $\frac{dy}{dx}$, $\frac{df}{dx}$, $Df$.

**UNLIMITED PERFORMANCE.** In Chomsky's conception a language is capable of producing an unlimited number of sentence constructions from a limited number of basic sounds (phonemes). This follows from the possibility that for any given sentence, the sentence may be extended by the adjunction of descriptors (e.g. adjectives) or even complete subordinated sentences. From a rather restricted set of generative rules the production of an infinite number of statements is possible in a natural language. MATH has the same ability: any formula can be extended to a longer one; a statement can be made more specific (hence extended) introducing some additional hypotheses; a concept may gain an extended meaning in lifting it into a more general context. Ex. from the middle of the 19th century and even more through the introduction of Measure Theory in the early 20th century, the concept of integration went through a long series of refinements.

In an actually spoken language there is not a corpus of material, as the available set of sentences may always be extended. The opposite situation occurs in the study of a dead language: the available set of texts may be too restricted to allow the decypherment of the (unknown) language and extention is impossible as there are no longer native speakers. This is the case with e.g. Etruscan. In MATH the latter situation doesn't occur; as a corrolary it makes the study of MATH a versatile enterprise, as at any moment an unexpected but mathematically sound structure may be met which forces the investigator into new conjectures about the structure of MATH.

**EXPLICITY.** A sentence in MATH is explicit, it means what has been written, not the concept its author had in mind. This characteristic expresses an important difference with natural language. A natural language is not always explicit. If someone says "it has been raining today" then the intention of the message coincides probably (unless he is a liar) with the sentence. But if someone says "it has been raining cats and dogs today" it doesn't mean that cats and dogs have been falling from the sky, nor was it the speakers' intention to pretend that this remarkable effect happened in reality. A sentence in a natural language expresses the concept, not exactly what is said or written.

In MATH the meaning of a sentence is determined by the sentence itself, as it is written. It expresses the concept if and only if the sentence is structured in the appropriate way. The derivative of a product $fg$ of two functions is $f'g + fg'$; if someone writes $f'g - fg'$ the results are wrong and there is no restructuration of the mistake, as in "the man *see* the dog" which is correctly understood, because the reader is clever enough to remove the mistake. Famous examples are the false proofs of Euclids' parallel axiom and of the four color problem.

Amusing (but unrealistic) would be the case of an equation $2ax - 6 = 0$ with the solution $x = 3$ and the remark that the $a$ doesn't count, the $a$ is there to make it difficult. Similar situations occur in the spelling of english and french.

In practice there is often a slight difference of meaning between a sentence in MATH and its intended meaning; if the gap is too large an error occurs.

**RECURSIVITY.** In Chomsky's theory the term recursivity has a broader meaning than it has in mathematics, it is the name used to express the possibility that a sentence is integrated into another sentence as in "John noticed that the man saw the dog". MATH has the same power: a formula may be incorporated into an other formula, a concept may be part of a more elaborated concept, as the function concept may be incorporated into the distributive rule $f \cdot (g + h) = (f \cdot g) + (f \cdot h)$. MATH is even more powerful in the application of the recursivity character. The same object may come back in the same context. It is not exceptional that the value of a function $f(x)$ enters the same function again: $f(f(x))$, . (This is what is usually called recursion in mathematics, in a strict sense). This ability is not usual in natural language: it is not common to say "he saw that he saw the dog". An other example is: in $(a + b) \cdot c = a \cdot c + b \cdot c$ we reintroduce $a + b$ as the value for $c$: $(a + b) \cdot (a + b) = \dots$

**GRAMMATICAL CONSTRUCTIONS.** TGG faces the problem of deciding which hypothetical constructions are grammatically acceptable and which are not. Ex. the sentence "the man saw the dog" is grammatically sound, but a construction as "Man the dog saw not" is not acceptable, and there is even a category of statements which seems to be hard to classify in one of both classes.

MATH produces statements which are acceptable and others which are not, but also several statements of a doubtful status which occur rather often and, more important, often doesn't disturb understanding, such as the use of $\frac{dy}{dx}$ for a derivative in stead of $\frac{df}{dx}$. This confirms the viewpoint that grammar is not that important in MATH but that meaning prevails and may save an incorrect statement from being ununderstandable. Here we find, again, support for the conjecture of the prominence of syntax over grammar.

## 4 READABILITY

The readability of a text written in a natural language has been the subject of much research, but it seems that not enough attention has been paid to the problem of reading the language of mathematics. A possible definition may be

> The property of a text which secures the correct transfer of the ideas of the author to the mind of the reader is called the *readability* of the text.

Because MATH is almost exclusively a written language the problem of its readability is even more decisive as regards understanding of the meaning of its ideograms. Reading a text there is more often than not a loss of communication. The author wants to express some ideas which are only partially assimilated by the reader. Readability may be mesured in percentage of the volume of the writers' ideas which are effectively transferred to the reader. We don't consider here the additional images which might be evoked in the readers' mind and which may be out of the realm of what the author intended to say. This supplement could better be called the *creativity* of the reader.

In the teaching of mathematics the readability problem becomes even more important because of the extensive use of a handbook which is often the decisive element of **what** shall be taught (the curriculum) and **how** it shall be taught (the teaching strategy) . The readabiblity of the text of a handbook establishes the link between the reader/student and the content of the teaching (the mathematical concepts). The communicative role and didactical value of a handbook are only secured as far as the text is readable. A relevant research would be to see in how far students really read a handbook in a coherent way or if they consider it only as stock of formulas and exercises. More intriguing is even the question if they read sometimes some pages which have not been the subject of teaching in the classroom.

The level of readability is the result of a balance between two components: (i) the characteristics of the language MATH, as it is written in the handbook, and (ii) the reading abilities of the students.

As for (i), some characteristics of a readable mathematical text may be listed here:

1. The text is clear and precise.
2. The text appeals to the readers' attention.
3. The major part of what is said is understandable or, the context allows the reader to construct a correct concept image of new ideas.

   This requirement is often a stumbling-block: we all know, reading mathematical papers, the trouble resulting from the occurrence of expressions as *it is easy to check that ...*,

> *obviously ..., after a short calculation the reader will come to the following result ...* and even worse *the proof/details are left to the reader ...*, which in fact are used as connectives but often the reader is left behind baffled.
>
> Another case are the pages of homework or the answers to a final given by some students where you are facing a page with a list of formulas and equations and not one single word of comment on what the students is doing (or wants to do) : all connectives are dismissed.

The question in how far a given text matches the above requirements brings us to the hard task of devising a method to measure the readability of a mathematical text. It seems that very little has been done in that respect. Several methods have been developed to measure readability in the case of literary texts, but these are not ready for a straightforward application to the MATH language. It seems that no progress has been realized beyond the application of the "subjective" method which is based on the judgment of experts in the field (researchers, mathematics teachers, trained readers).

As for (ii), this component requires an investigation of the reading processes of the students. Because mathematics requires a high level of precision it will often be impossible to reduce much of the sophistication of the presentation, using fewer symbols, etc. Hence the reading ability of the students has to be stimulated. There may be inner obstacles to this objective due to a lack of preliminary knowledge and outer obstacles which are of a psychological or of a linguistic nature. Part of the second obstacle is the particular case of students who are not native speakers. It is clear that, if understanding of the vernacular commonly used in the classroom is not sufficiently developed, understanding may be blocked. But is seems that this problem is situated out of the realm of what we are discussing here and possibly the only answer is to advise those students to get a better understanding of the local language (the hypo-language).

Among Reading/Thinking processes involved (in that order) in the assimilation of a mathematical text the following are often mentioned.

1. Observation and detection of the symbols and figures.

   E.g. classroom experience tells that representing the derivative as $f'$ with an accent is not always appropriate as the accent may become unobservable or even disappears in writing. A symbol as $Df$ or the more elaborate $\frac{df}{dx}$ is more appropriate in that respect.

2. Attribution of a meaning to these symbols and figures, which requires mastery of a more or less extended part of the techncal vocabulary of mathematics. Most disturbing is the use of terms which have in the natural language a meaning unrelated to the meaning in mathematics. Ex. the use of the term *open set* in topology which often entails the conclusion that a set which is not open has to be closed, because a door or a window which is not open is said to be closed.

3. Analysis of the relationships between the symbols (as in a formula), between groups of symbols (as in understanding the proof of a theorem), between whole paragraphs expressing a consolidated idea (as in understanding the purpose of a part of the theory).

4. Construction of the reader's own concept image of a theoretical concept; if the text is well readable the concept image should be close what the author wants to forward.

Word problems might be an exceptional but important and frequent case; transforming a verbal problem into mathematical formalism, the nature and succession of the processes might be quite different. The teacher is facing here the difficulties inherent in the translation from a natural language into MATH. Does the conversion from the verbally expressed problem towards MATH operate in a formal or informal style, or in both, at the same time ? The most realistic answer is probably to say that writing goes from informal to formal, gradually from the natural language to MATH, taking profit of the support offered by the use of both the hypo- and the meta-language.

# ON DEVELOPING A UNIFIED MODEL FOR THE PSYCHOLOGY OF MATHEMATICAL LEARNING AND PROBLEM SOLVING

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*Theoretical descriptions of mathematical learning and problem solving often highlight just one or a few key constructs associated with identifiable patterns in behavior. This paper explores what is involved in developing a broader, unified psychological model--complex enough to account for diverse empirical results that have been obtained (i.e., scientifically adequate), and at the same time parsimonious enough to be accessible and useful in the practice of mathematics education. Some components of such a model are proposed, including familiar constructs as well as less familiar ones related to the internal representation of mathematical concepts and strategies.*

## Introduction: The Need for and Value of a Unified Model

As we proceed with increasingly sophisticated studies of learning and problem solving in mathematics, the need grows for theoretical bases to guide our empirical and exploratory work.

The past quarter century has seen mathematics education theory evolve considerably. One strand of thought moved from behaviorism and rule learning into neobehaviorism, including the study of strategies and strategy scoring systems (patterns in behavior), analyses of formal mathematical problem structures and their interactions with strategies, and algorithmic learning and debugging. A second theme was the explicit study of problem-solving heuristics, which further enlarged researchers' vision of the complex thinking processes involved in "doing mathematics". The emerging discipline of cognitive science brought to the psychology of mathematics education ideas from artificial intelligence research and heuristic programming--together with new tools, such as computer simulations of learning and problem solving, that helped foster greater precision in modeling specific cognitive processes. Meanwhile the genetic epistemology of the Geneva school successfully challenged the highly behaviorist orientation that had prevailed in most psychology departments in the United States, leading to an international emphasis on epistemological and structuralist analyses of children's arithmetical behavior, and to the considerable attention given to developmental stages in mathematical learning.

The fields of psychology, formal linguistics, semantics and semiotics, together with the study of mathematical structures, motivated a great deal of work on representation and symbol systems in the psychology of mathematics education. Visualization, spatial and kinesthetic representation, image schemata, and imagistic representation in general (a theme of the present PME meeting) came to be recognized as critical to understanding how mathematical concepts are meaningfully understood; while another construct, that of learners' and problem solvers' metacognitive processes (self-monitoring, reflection on strategies, etc.), also came to occupy center stage. Various kinds of constructivist perspectives posed alternate interpretations of mathematical knowledge and meaning; and changing perspectives on mathematics itself--on "what mathematics is"--greatly influenced the thinking in our field. Social anthropology contributed renewed attention to the cultural contexts within which mathematical meanings are constructed and mathematical problems are solved; while research on attitudes, belief systems, and affect provided convincing evidence of the critical role these constructs, too, must play in any adequate theoretical model of mathematical learning and problem solving.

In addition to these broad considerations, focused studies of conceptual understanding in *particular content domains* of mathematics led to new constructs that are crucial to theory. Included here are descriptions of specific cognitive structures or schemata: analyses of additive and multiplicative structures; descriptions of semantic structures of "story problem" tasks that influence behavior; schemata underlying rational number learning, proportionality, algebraic and probabilistic thinking, the function concept, and so forth; variations on the "van Hiele levels" model in geometry; as well as the identification of certain cognitive obstacles and misconceptions that occur widely in relation to specific mathematical concepts.

The diverse concepts, structures, and/or processes that are conjectured by researchers to occur inside students' heads, cannot of course be seen directly. They are, rather, *inferred* from the observation, classification, and measurement of various kinds of behavior. Such observations now include not only measures of the "products" of learning or problem-solving success (such as scores on standardized tests, patterns of correct and incorrect answers on structured sets of problems, subproblem scores, solution times, numbers of steps to solution, or paths through previously-defined, external problem representations or state-spaces), but also measures of "processes" (such as strategy usage, application of particular planning and heuristic processes, spatial visualization, verbalization, metacognitive awareness, and occurrence of affective states). Accordingly, the methods of inquiry in mathematical learning and problem-solving research have also widened greatly. They range from statistical studies of variables influencing various outcome measures, to detailed analyses of video- and audio-taped individual clinical interviews, classroom episodes, and small-group problem solving, to cross-cultural comparisons. Qualitative data drawn from concurrent "thinking aloud" protocols, retrospective verbal descriptions, responses to structured questions in clinical interviews, uses of external notations and representations, problem solvers' eye, hand and body motions, etc., now augment earlier, quantitative measures.

All these developments give us a rich pool of theoretical constructs and investigative methods from which to draw. Such brief mention cannot possibly do the ideas justice, or credit properly those who developed them; however, the incomplete "supplemental bibliography" following this paper highlights a selection of relevant books and articles, not cited specifically, which influenced my own thinking considerably. References therein may permit the reader to trace the history at least partially; I would apologize in advance for the many omissions.

In surveying these ideas, I would like to mention what I believe are two obstacles to theoretical progress. The first is that we in the psychology of mathematics education often do not learn sufficiently from each other, build sufficiently on each other's perspectives, or give each other's work sufficient acknowledgment. Mathematics education research has in many respects been *ahead* of other disciplines in recognizing the importance of, and developing, key psychological concepts--but there is an unfortunate tendency to disregard each other, and to attribute these achievements to researchers in other disciplines who later rediscovered and articulated similar ideas. To cite an example, the books by Lakoff (1987) and Johnson (1987) have been put forth as providing a new linguistic and philosophical theory incorporating imagistic thinking (image schemata), with the suggestion that we who are doing research in mathematical learning and problem solving would do well to adopt it. Dörfler (1991) introduced one such discussion thus:

> "I want to stress that the theoretical approach presented here points to a rather neglected direction but is needed to understand various cognitive and linguistic phenomena.
>
> "I will take as a starting point a theory developed by Johnson and Lakoff. They use the term *image schema* to denote a (hypothesized) cognitive mechanism which regulates the use of words (and other linguistic units). [It] is a schematic structure which in a highly stylized form depicts or exhibits the main features and relationships of situations and processes to which potentially the word refers. . . ."
>
> (Dörfler, 1991, pp. 18-19)

Now Lakoff and Johnson do make many fascinating points about language and metaphor, categorization, and the bodily basis of meaning; and I do not mean to detract from their valuable contributions. Nevertheless, I would hold that researchers in psychology, mathematics, and the psychology of mathematics education--including many within PME--have long been developing the theoretical underpinnings of visual/spatial, kinesthetic, and other forms of imagistic thought!

Imagery and kinesthetic representation were already tacitly embodied in the *practices* exemplified by Montessori's approach to mathematics teaching, Bruner's emphasis on "enactive" representation, and Robert Davis' early advocacy of guided discovery learning. Visualization was also seen as fundamental by many great mathematicians; e.g. Poincaré, and later Polya, helped to make this explicit. From the theoretical and empirical studies of Piaget, Inhelder, Vygotsky, Ausubel, and others, we can trace a set of ideas underlying "image schemata" that were developed further by many in psychology and mathematics education research. In the psychological literature, Paivio (1978, 1983) argued for a "dual code" model for cognitive representation, including "verbal" and "imaginal" systems; Kosslyn (1980) reviewed the debate about imagery, and persuasively rebutted earlier arguments against the validity of internal representation based on it (see below); Bischof (1987) provided an interesting overview of research on visual representation. In the psychology of mathematics education, the papers in the edited volumes by Skemp (1982) on mathematical symbolism, and by Janvier (1987) on representation, are noteworthy; Scholz (1987) addresses intuition and pictorial encoding in relation to strategic decisions in probabilistic contexts; and the contributions of Mason (1980, 1987) about symbolism and metaphor are highly relevant (see also Mason and Pimm, 1984). The series of papers by Bergeron and Herscovics, with their stress on qualitative and operational physical understandings underlying children's conceptual development, contributed further to the theory from an epistemological perspective (e.g. Herscovics and Bergeron, 1983, 1984, 1988; Herscovics, 1989). The model I proposed for the structure of mathematical problem-solving competency (Goldin, 1982, 1983) and subsequently elaborated (Goldin, 1987, 1988a,b), incorporated imagistic representation as absolutely fundamental; and as the current chair of the PME Working Group on Representations, I have observed how thoroughly the emphasis on imagery, imagistic representation, and image schemata has developed within PME. Thus, to attribute these constructs to recent work in linguistics and philosophy is not only historically incorrect, but downplays the prior advances of many in our own field who have argued for their central importance--often, in opposition to others who took approaches based exclusively on propositional, syntactic, or algorithmic reasoning processes. The struggle and the very real progress of those working in our own discipline are relegated, perhaps inadvertently, to a derivative role instead of an innovative one. And in our slowness to understand thoroughly and digest each other's theoretical ideas, we retard our overall research progress.

A second obstacle to progress has been the tendency of some of our main theoretical frameworks, whether originating within or outside the psychology of mathematics education, to exclude *a priori* the most useful constructs of other frameworks. This has occurred repeatedly, despite the fact that those studying mathematical learning and problem solving want to understand and explain similar observable phenomena. It is almost as if the sociology of our discipline is such that to gain adherents and wide attention, a theory or methodology must be *exclusive* rather than *inclusive*, and dismiss as illegitimate the conceptual entities of "rival" perspectives. Several examples come to mind. For a long time behaviorists and many neobehaviorists rejected on first principles discussions of internal mental states and mental representations, accepting at best constructs built from "internal responses" to stimulus situations. Later, some cognitive theorists maintained on *a priori* grounds of parsimony that all cognitive encoding must be represented propositionally, rejecting the very possibility of internal imagistic representation. Some computer-oriented cognitive scientists rejected mechanisms that were difficult to simulate mechanically, maintaining in effect that constructs are impermissibly vague unless they are programmed; they consequently accorded an unwarranted, broad validity as descriptions of human thought processes to readily-programmed constructs (e.g., search algorithms in problem solving). Radical constructivists, again on *a priori* grounds, have excluded the possibility of talking about external mathematical structures, apart from individual knowers and problem solvers, and have tended to dismiss population studies or investigative methods based on controlled experimentation. In contrast, some of those tied to quantitative methods have rejected qualitative and exploratory investigations as unworthy. The methodology of meta-analysis often leads to the exclusion of all but the most readily-quantified outcome measures; for a recent exchange of views, see Hembree (1992a,b) and Goldin (1992).

I would maintain that the history of progress in the natural sciences should teach us it is rarely fruitful to deny on first principles the admissibility of one or another kind of construct. Despite the many achievements of all the theories and methods mentioned, each has built in unnecessary limitations. Here I want to argue for the value of developing an adequate, *unified* theoretical foundation; one that can accommodate the most helpful and applicable constructs from a variety of frameworks.

There is a need to be able to use not just one idea in isolation--be it rule learning, algorithms, strategies, image schemata, visualization, heuristics, metacognition, metaphor, construction of meaning, affect, belief systems, or any other--but *ideas in combination with each other*, to build a realistic, structurally adequate, sufficiently detailed theoretical model. Examples of this need abound. Yuille (1983), for instance, discusses work on "mental rotations" (Shepard and Metzler, 1971; Shepard, 1978), in which subjects viewing pairs of diagrams are asked to determine if they are the same or different. In general it is found that the time required to identify two diagrams correctly as "the same" is a constant plus a term directly proportional to their pictured relative angle of rotation; it is concluded that this reflects the time needed to rotate a mental image holistically. Other researchers form similar conclusions about the comparison of image size, judgments of distance, etc. However, Yuille challenges such conclusions, reporting on a series of experiments in which the effects of figural complexity are investigated and taking the position that findings such as these can be regarded as consequences of the manipulation of task demands: when more complex images are involved, the data no longer fit a process of *holistic rotation*, but are more consistent with *feature comparison*. In turn, I would maintain that feature comparison inevitably leads to the need to discuss not only the conceptual origins of the "features" depicted, but also problem solving and problem-solving heuristics, such as "trial and error" with monitoring of the outcomes of trials. In short, the adequate analysis of seemingly straightforward empirical findings on visual imagery requires a theoretical base that is sufficiently inclusive: i.e., a synthesis of many perspectives.

The following development borrows heavily from the ideas of many people, from behaviorists to radical constructivists (though I am neither), to try to take a step toward achieving such a synthesis (see also Goldin and Herscovics, 1991a,b). I will try to identify and describe very briefly some of the components that a unified theoretical model should include. The goal is to construct a framework sufficient to allow the description of conceptual systems (as systems of competence) at a given time, the characterization of how competencies change and grow over time, and the study of how external learning environments can foster their development. The framework should *admit*, rather than exclude, the main constructs I have just surveyed.

## Task Variables, External Representations, and Representational Systems

The first ingredient in the framework should be the description of task environments external to learners and problem solvers.

One strand of research in mathematical problem solving views tasks as *instruments* that can be designed to elicit various kinds of problem-solving behavior; thus allowing the planned observation and measurement of specific outcomes. A system of classification to organize the kinds of characteristics of tasks (i.e., task variables) that can influence problem-solving behavior, outlined by Kulm (1984), formed the basis for the systematic investigation and discussion of task variable effects (Goldin and McClintock, 1984). The main categories are *syntax* variables, describing the words and symbols in the problem statement, their arrangement and the grammatical relationships among them; *content* and *context* variables, describing respectively the mathematical and non-mathematical semantic information in the problem statement; *structure* variables, describing mathematical properties of formal problem representations; and *heuristic behavior* variables, describing heuristic characteristics intrinsic to tasks associated with, or eliciting particular heuristic processes and strategies in solvers.

"Representational systems" or "representational modes", which include spoken symbols, written symbols, static figural models or pictures, manipulative models, and real world situations, are discussed by Lesh (1981) and Lesh, Landau, and Hamilton (1983) in analyzing applied problem solving. These, too, are seen as *external* to the learner or problem solver, who may (among other things) translate among distinct modes. External representations can be highly structured, and the analysis of task structure as external to the individual provides important insights into problem solving behavior (e.g. Jeeves and Greer, 1983; Goldin, 1984a). As computer technology has advanced to permit the intentional structuring of highly complex mathematical environments ("microworlds"), the analysis of external representational systems becomes still more important, and we must consider not just *task* structure but *representational* structure if we wish to understand how human cognitions change while interacting with such environments (Kaput, 1991).

It may be helpful, before proceeding further, to compare the terminology used here with that of others. My use of the term *representational system* bears some resemblance to what Kaput (1987, p. 162) calls a *symbol scheme*. A major difference appears to be that symbol schemes are taken to be "in principle machine compilable", while I tend to place great emphasis on the essential presence of *ambiguities* in representational systems. What Kaput (1985, 1987), following Palmer (1977), calls a "representation" or "representation system" corresponds most closely to a *relationship of symbolization* between two representational systems as I use the term. Von Glasersfeld (1987) contrasts the German words *Darstellung* and *Vorstellung*, both of which can be translated as "representation". The former corresponds most closely to what I am here calling "external" representation; the latter, to what I call "internal" representation (see below).

Let me now elaborate on the technical interpretation of this term. A representational system consists first of primitive *characters* or *signs* (two words I use interchangeably), embodied in some fashion. These can be discrete entities drawn from a fairly well-defined set, such as spoken words, letters in an alphabet, punctuation marks, numerals, or arithmetic operation symbols. They can be also be less well-defined entities, such as physical objects and their attributes. They can be abstract mathematical or physical entities such as numbers, vectors, velocities, or forces. An important feature of these characters, when they are regarded as the elementary entities within a representational system, is that we do not yet ascribe to them further meaning or interpretation. Taking the words in a (spoken) natural language to be the elementary characters, embodied orally, we would not yet interpret the words through their meanings, provide their spellings, specify their grammatical parts of speech, etc. Now, in addition to the elementary signs, a representational system includes rules for combining the signs into *permitted configurations*. Thus letters of the alphabet, as characters, may be combined in certain sequences to spell English words, while other sequences do not form words. Likewise words-labeled-as-parts-of-speech (taken as a collection of primitive signs) may be joined to form sentences; single-digit numerals may be combined according to place value rules to yield multidigit numerals; numerals and operation signs may form mathematical commands or mathematical equations; and so forth.

Typically a representational system has still more structure than this. There may for instance be rules for *moving* from one permitted configuration in the system to another, or from sets of configurations to new sets, establishing a kind of network structure. In formal logic, the sentential calculus may thus be regarded as a representational system; we are provided with elementary characters, with rules that specify the allowed configurations of these characters (called "well-formed formulas", or wff's), and with inferencing rules for moving from previously established wff's to new ones by proving theorems. Representational systems may also include other kinds of higher-level structure: e.g. configurations of configurations; relations such as partial or total orderings on the family of configurations; mathematical operations of various kinds; rules for valuing configurations (e.g., the assignment of truth values to logical propositions); and so forth.

The rules for forming configurations from characters, and the relationships among configurations established by higher-level structures, are one way of giving *meaning* to the characters and configurations in a representational system, which might be called a purely *syntactic* or *structural* interpretation of meaning. But the main reason for calling these systems *representational* systems is because characters, configurations, or structures in one such system can *stand for, represent*, or *symbolize* those in another. This *symbolic relationship* then provides a *semantic* interpretation of meaning. Thus, I am viewing a representational system as having *intrinsic* structural relationships (i.e., within itself), and *extrinsic* symbolic relationships (with other systems of representation). For instance numerals and arithmetic symbols (signs in a system of numeration) not only have syntactic relationships with each other, but also can *stand for* other things--e.g., action sequences corresponding to the counting of sets of objects, the joining of two nonoverlapping sets, etc.

Representational systems are *conventional* constructs. The decision as to where one representational system leaves off and another begins, or whether to view various additional structures as intrinsic to a given representational system or as arising from a symbolic relationship between two systems, is an arbitrary one motivated by convenience and simplicity of description. Furthermore, there is the inescapable fact that for most systems, there are exceptions to virtually any proposed set of syntactic and semantic rules. To try to enumerate all the exceptions to a set of rules makes the structure of a theory complicated and unwieldy, to the point of sacrificing usefulness--and new contexts can always be suggested which engender new exceptions. Consequently, I would take the view that with certain exceptions *ambiguity* is a necessary feature in the concept of a representational system. The initial family of signs may be well-defined (as in the case of "letters of the Roman alphabet" or "Arabic numerals"), close to well-defined (as in the case of "English words"), or highly ambiguous (as in the case of "real-life objects"). Similarly rules for forming configurations, and any higher-level structures in the representational system, may be ambiguously specified. Finally, there may be ambiguity in the symbolic relationship between two representational systems.

In practice, most ambiguities in representational systems are resolved straightforwardly; and we say that this is done from the *context* in which the ambiguous sign, configuration, or symbolic relationship occurs. For example, homonyms are distinguished easily in spoken language--the phrase "to tell a tale" is normally sufficient to ensure that "tale" is not misconstrued as "tail"--but this requires *semantic interpretation* of the spoken words. "Context" in this sense refers to that which is *not* part of the representational system under discussion. The ambiguity *within* the system of words requires going *outside* it to a different system, one that permits words to be interpreted semantically. Thus ambiguities in one system are resolved by means of *unambiguous* features of another system that stands in a symbolic relationship to the first. This principle can apply to ambiguity in the signs, rules for forming configurations, or higher-level structures of a representational system. To resolve ambiguity in the symbolic relationship between two representational systems, contextual information can sometimes be incorporated by going to a *third* system, to which each of the first two bear a symbolic relationship, or where the symbolic relationship between the first two is itself represented.

When we discuss external representations, it is usually possible to point explicitly to the signs, to construct configurations explicitly, and then to discuss various kinds of structures in a more-or-less concrete way. But to understand learning and problem solving in mathematics, it is necessary also to provide a model for how and why individuals interact as they do with their external task environments. Schemes for classifying and studying the effects of external variables do not tell us *how* task variables affect problem-solving outcomes in the ways observed, how external representational modes are mediated when translation or other processes take place, or how the structures of external representations influence learning and subsequent behavior. In fact, internal constructs are, to some extent, already *implicit* in the descriptions of external variables. The model one adopts to describe learners' or problem solvers' internal processes, cognitive and affective, inevitably and profoundly influences the external characteristics of task environments and learning contexts to which one attends. It is extremely important to make such influences explicit.

## Internal Systems of Cognitive and Affective Representation

Actions on external task environments (e.g., steps within external representational systems, translations from one representational system to another, or construction of entirely new representations) are viewed here as mediated by *internal* cognitive representational systems, which may (or may not) bear some structural resemblance to external systems. The type of unified framework I envision includes internal representational systems that can stand in symbolic relationship not only to external systems, but to each other. Here I shall discuss briefly some ideas in connection with such internal representational systems, and how their structure can in principle be inferred from observations of mathematical behavior.

### *Competencies*

Cognitive representational systems are intended to be descriptive of *competencies*, rather than of behaviors directly. The distinction between directly observable behavior (i. e., "performance") and competence, an operational construct, is fundamental to the approach I take. Some may regard the following definition as unsatisfying, but I think it is the best one can do in characterizing the knowledge of human beings: *human competence refers to the ability to perform a task some of the time, under conditions which are incompletely specified.* Competencies are viewed as *theoretical constructs*, more stable than observations of behavioral success, that account for the steps taken during problem solving, but necessarily embody some *context-dependence* (at least implicitly). Thus from the outset our approach requires attention to factors that are not subject to purely mechanistic description. Ultimately, of course, the presence of competencies, or structures of competencies, is inferred from the observation and interpretation of behavior. Now *learning*--whether interpreted generally, or in the specific domains of mathematics or science--can be defined operationally as *the acquisition of competencies;* thus learning, like competence, will depend not just on directly observable behaviors, but on constructs *inferred* from observations.

To give one brief illustration, a standard arithmetic algorithm (for, say, subtraction of multi-digit numbers) can be viewed as a system of structured competencies (to perform the steps, use the decision procedures, etc.). A student who correctly solves three or four exercises using such an algorithm might be validly inferred to have that structure of competencies--even if on another occasion, the same student is distracted and makes errors, or loses interest and does not complete a problem set. We cannot *define* competence unreservedly as successful performance at a particular level, or with a particular probability. Nevertheless we can *infer* it under conditions that may depend on context, culture, social expectations, and other factors. To stress this point: competence is always inferred, never observed directly. Consequently, there are inevitable *ambiguities in the characterization of the competency* whose existence is inferred. One of these ambiguities is in the specification of the *domain* of problem situations or configurations in which the problem solver is competent to take particular steps, use particular rules, etc. A second kind of ambiguity in the specification of a competency is in the *outcome* of taking the step or using the rule. When there is a unique, well-defined, observable behavior that the competency is interpreted as generating, there is no difficulty. But the outcome may entail one of a *set* of possible behaviors (e.g., choosing a number to make a trial), or the outcome might be a mental construct or constructs, a configuration or set of configurations in a cognitive representation that is impossible to observe directly. Then the competency will be specificiable only approximately, or within some range of possibility.

Now one can move to a discussion of *competency structures*, especially contingency structures (in which one competency can depend on others): subcompetencies that constitute a given competency, competency to access other competencies, and so forth. Competencies can be simultaneously metacognitive and cognitive--e.g., one could speak of a student's competency to *decide* to employ a trial-and-error strategy in a given context. In general I would like to interpret competencies as describing individual capabilities for moves among configurations in internal, cognitive representational systems, including the construction of new configurations (and even new systems).

### *Systems of internal representation*

For some time I have sought to develop a model for the structure of problem-solving competence based on five types or categories of internal representational systems: (1) *verbal/ syntactic* systems, (2) *imagistic* systems, (3) *formal notational* systems, (4) a system of *planning, monitoring, and executive control*, and (5) a system of *affective* representation. Some cognitive psychologists have proposed models for cognition based on one such system (propositional encoding), or two (propositional encoding and visual imagery, so-called "dual code" models), or even three (including an emotional code). In my view the five types of internal representation described here are *all* to be regarded as psychologically "fundamental," in the sense that not only do they occur universally among mathematical and scientific problem solvers, but with the possible exception of formal notational systems, all also occur universally in human beings. *None* of them are taken as biologically "fundamental" in the sense of actually *being* the structures which make up the human brain and nervous system. They are only *descriptive* of those structures. The structures themselves I would regard as most likely encoded at a neural level; description at that level would be of enormous scientific interest, but might or might not be useful to us as psychologists or educators. Further, I would argue that the decision to regard a proposed system of cognitive representation as a single system or as more than one distinct system is *arbitrary* (that is, a matter of convention). In modeling problem-solving competence through representational systems, greater parsimony is not achieved by assuming fewer systems.

There again appear to be inherent limitations to the precision with which such systems can be characterized--some essential ambiguity is present, even when the competencies described are fairly well-defined. Like competencies, internal representational systems are context-dependent constructs, not observed directly but *inferred*. They are conjectured structures, intended to account for observable behavior. Here I shall discuss each type of representational system briefly, and comment on some processes for moving from one to another.

### *Verbal/syntactic systems*

A verbal/syntactic system of representation describes the individual's capabilities for processing natural language, on the level of *words, phrases, and sentences* (only). *Input* channels for such a system include hearing and reading; *output* channels include speaking and writing. The competencies within this system may include verbal "dictionary" information such as common definitions and verbal descriptions; word-word associations such as synonyms, recalled related phrases, and antonyms; word categorical relationships; and the parsing of sentences based on grammar and syntax information. A bilingual person may be usefully regarded as possessing two distinct verbal/syntactic systems. Verbal/syntactic configurations can *correspond* to--that is, be semantically descriptive of--configurations in other systems: imagistic, formal, heuristic, or affective configurations are all described in words. There are also *self-referential* competencies in natural language; as words and sentences can be used to describe words and sentences.

### *Imagistic systems*

Several different non-verbal, non-notational cognitive systems are included under the general heading of imagistic systems. The most important of these for the psychology of mathematics education seem to be *visual/spatial, auditory/rhythmic*, and *tactile/kinesthetic* systems of representation. The term "imagistic" as I use it is not intended to be restricted to visual imagery, such as might be represented or processed within just the visual/spatial system. It is intended to have the broader connotations of the word "imagination". Imagistic systems incorporate non-verbal, internal configurations at the level of objects, attributes, relations, and transformations, encoding what might loosely be called "semantic" information. Thus their inclusion in a unified model allows for the description of mechanisms whereby semantic structures influence mathematical problem solving. Competencies in accessing and processing such nonverbal configurations are *needed* for the meaningful or insightful interpretation of verbal problem statements. Within imagistic systems dwell also students' non-verbal, non-quantitative conceptions and misconceptions. Modeling the interesting theoretical construct of *phenomenological primitives* ("p-prims") due to diSessa (1983), descriptive in a sense of the individual's most basic "intuition" about a real-life phenomenon, also belongs in the domain of imagistic cognitive representation. Paivio (1978, 1983), Kosslyn (1980), and others argue for at least two fundamentally distinct means of internal cognitive representation, one of these being visual imagery. They thus challenge some advocates of computational models based exclusively on propositional encoding, for whom a single system is more parsimonious:

> "*Imagery is not an inherently flawed concept.* . . . The essential claim that imagery is a distinct representation system, utilizing data-structures of a special quasi- pictorial format, is neither internally inconsistent, incoherent, nor paradoxical.
>
> . . .
>
> ". . . Although some sort of propositional system probably could be formulated to account for data on imagery, it is not clear that this is desirable. A special imagery representation system could have evolved, and positing one may be empirically fruitful. If this turns out to be true, the wisdom of developing a separate imagery theory will have been vindicated.
>
> "*Images are not pictures.* . . . the claim that images are like percepts that arise from memory fails to pick out the properties of images that distinguish them from distinctly nonimagistic representations. We need an independent characterization of what it means for a representation--arising from long-term memory *or* the senses--to be 'quasi-pictorial.' . . . Clearly, theories that posit that people can use quasi-pictorial mental representations will be very different from ones that do not."
>
> Kosslyn (1980, pp. 27-28)

Kosslyn argues for "quasi-pictorial" representation as a means of mental representation fundamentally distinct from propositional representation. He then develops in impressive detail a model for such representation. His work is persuasive concerning the ability of this kind of theory to account for much of the experimental psychological evidence. With Kosslyn, then, we reject the arguments that visual/spatial systems of cognitive representation are *a priori* inadmissible; the competencies he calls quasi-pictorial correspond roughly to what I am here labeling visual/spatial representation. But in mathematical as well as non-mathematical domains, there are imagistic competencies encoded in other than visual/spatial ways.

When children clap and sing and tap their feet, when they learn to count in rhythm, when they accent the numbers as they make use of counting-on strategies in solving early addition problems, we see evidence of auditory/rhythmic encoding--applied to mathematics--that is different from the aural processing of spoken words or symbols. I would want to treat this as a distinct system of imagistic representation.

The tactile/kinesthetic system pertains to internally represented, imagined *physical actions* by the individual or on the individual, as distinct from imagined sights or spatial transformations. In a sense, this is an internal version of what Bruner called *enactive* representation. It is straightforward to identify mathematical concepts with which such a cognitive system assists in conceptual understanding. Level curves in mathematics, satisfying $f(x,y) = c$, where $f$ is a function of two real variables and $c$ is a constant, can be interpreted kinesthetically by the imagined action of walking at a fixed elevation on a hillside; the gradient of the function then gives the direction of steepest ascent. "Turtle geometry" in the LOGO computer language requires the interpretation of instructions in a way that makes reference to the turtle's frame of reference, rather than a fixed background coordinate system; the child can encode this kinesthetically by imagining himself or herself to be the turtle. A tactile/kinesthetic system should probably be taken to include internal configurations representing imagined physical actions of the following kinds: (1) imagined actions performed by the individual, such as reaching out with the hand or hands, touching and feeling shapes (flat, round, pointed, etc.) and sensations (warm, cold, fuzzy, etc.), walking or acting with the body, estimating distances between the hands or between parts of the body kinesthetically, pushing and feeling pressure with hand(s) or body, and so forth; (2) imagined actions by the environment *on* the individual--by another person or object, or a field of force (gravity, etc.), resulting in touching, feeling, or kinesthetic estimation; (3) imagined actions by or on the person *in place of* something external in the environment, such as turning one's own body to correspond to the rotation of a figure, imagining one's own hands pressing together to represent the forces exerted by two bodies on each other, or imagining one's arms angling outward to represent an angle; (4) imagined physical actions by or on the person, utilizing the individual's constructed "self- other" correspondence to another person.

Input channels for imagistic systems of representation take the form of *interpreted sensory experiences*. Output channels include interpreted actions--creating, or acting on, external enactive, figural, or manipulative representations.

To comprehend words and sentences includes the competencies of accessing internal imagistic configurations corresponding appropriately to verbal/syntactic configurations--roughly speaking, to *imagine* the situations described by the words. To comprehend formal mathematical symbols also includes (fairly sophisticated) competencies of accessing or constructing corresponding imagistic configurations. In addition, the characters and configurations in a verbal/syntactic system, or a formal notational system, can *themselves* function as "objects", and be manipulated imagistically! In algebra, for instance, a student learns to "move the $x$ over to the other side of the $=$ sign, and put a $-$ sign in front of it", as if the $x$ were an object. Words-as-objects, sentences-as-objects, or symbols-as-objects can also be *described* (as when we discuss in words, for example, the inclusion of particular words in a sentence). Thus a great deal of *level crossing* occurs in the systems of representation in this model, through their mutual- and self-reference.

### *Formal notational systems*

The conventional, formal notations of mathematics are highly structured, symbolic systems--numeration systems, arithmetic algorithms, rational number and algebraic notations, rules for symbol manipulation, etc. Their structure as external representations for mathematical and logical problem solving is readily explored. Correspondingly, internal systems of competencies associated with the construction and manipulation of such formal notational configurations are taken here as a separate type of internal, cognitive representational system. The fact that formal notations can be subjected to detailed structural analysis gives us an advantage in research--it is possible to model these competencies utilizing such constructs as algorithms and debugging, tactics, and strategies. While it is clear that formal notational systems are essential to analyze mathematical problem solving effectively, their necessity is not so evident in other, less formal problem-solving domains (such as non-quantitative, real-life situations). Incorporation of this type of system is a consequence of looking specifically at problem solving in mathematics.

Some cognitive processing can be seen as occurring *within* a system of formal notational (e.g., the execution of an algorithm); but much of what we would call *meaningful understanding* in mathematics has to do with relationships that formal configurations of symbols have to other kinds of internal representation. An understanding of the meaning of a mathematical notation, or of why an algorithm works, involves being able to talk *about* the notation or procedure, not merely to take steps within it. It also involves correspondences between symbols, imagistic configurations, and words. A highly competent problem solver can not only *translate* a verbal problem statement directly into formal symbols, but can also *imagine* the situation that the statement describes (i.e., construct appropriate internal, imagistic configurations), *formulate* notational description of the situation imagined (i.e., move from the imagistic to the formal), and conversely, *interpret* formal descriptions imagistically. The use of imagistic competencies often remains *tacit* when verbal problems are solved--unless, through some *monitoring* of words and symbols, imagistic descriptions are evoked.

*Planning, monitoring, and executive control*

A cognitive representational system that includes strategic planning, monitoring, and decision-making (executive control) can be regarded as *directing* the problem solving process. While time and space do not permit an adequate discussion here of the complexities of this representational system, it should be clear that without it, some of the most important aspects of the theory of mathematics education cannot be adequately discussed. This system includes competencies for (1) keeping track of the state of affairs in the other systems, and in itself; (2) deciding the steps to be taken, or moves to be made, within all of the internal representational systems, including itself; and (3) *modifying* the other systems--deciding to improve the formal notational conventions, invent new words, and so forth. Thus there are many important senses in which this system stands in a *metacognitive* relationship to the others. My view, however, is that one cannot firmly maintain the distinction between cognitive and metacognitive processes; instead, the concepts of *mutual- and self-reference* among the systems of internal representation are vital. Each of the systems can represent information *about* the others as well as *within* the others; each can also represent information about itself.

The *heuristic process* is in my view the most useful organizational unit, and culminating construct, in a system of planning, monitoring, and executive control--"trial and error", "think of a simpler problem", "explore special cases", "draw a diagram", etc. These processes, however, are not single, unitary skills or techniques; each is itself a complicated structure of subcompetencies. In earlier work (Goldin and Germain, 1983; Goldin, 1984b), I proposed four dimensions of analysis for examining and comparing heuritic processes: (1) *advance planning reasons* for using a particular process, (2) *domain-specific methods* of applying a process, (3) domains and levels where a process *can* be applied, and (4) prescriptive criteria for suggesting *that* a particular process be applied. The complexity of this representational system is partly due to the fact that heuristic processes can *access each other, and act on each other,* in the course of their use; the diversity and complexity of the heuristic processes that require consideration is also a consequence of the choice of mathematics as a learning and problem-solving domain to model (as distinct from every-day learning and problem solving). Based solely on the problem-solving cognitions involved in domains less highly-structured than mathematics, such as planning a shopping trip or searching the television channels for the best show, one might have thought that a single, controlling process such as means-ends analysis might suffice; it is clear from looking at mathematics that it cannot.

*Affective representation*

A fifth type of internal, representational system, a system of affective representation, is needed not only to model learning and problem solving effectively, but to discuss educational goals that maximize *enjoyment and positive self-concept* as well as cognitive competencies. Zajonc (1980) advocated enlarging the then-current "dual-code", verbal/imaginal models, to "triple-code" models, in which emotion or affect is included as an independent system, and Rogers (1983) discusses some general ways in which what he calls "the Emotional code" can interacts cognition. Affective representation refers here not merely to relatively stable, attitudinal constructs in relation to mathematics (what I have called *global* affect)--but to the *changing states of feeling* that problem solvers experience, and *utilize*, during problem solving (*local* affect): e.g., curiosity, puzzlement, bewilderment, frustration, anxiety, fear, despair, encouragement, pleasure, elation, and satisfaction. The term "utilize" is critical. It suggests that we need to consider *affective competencies*, and regard the affect appropriate for problem solving as something that can be learned, and is susceptible to being taught (Goldin, 1988b; McLeod and Adams, 1989; DeBellis and Goldin, 1991).

Related competencies include processes which "translate" from imagistic and heuristic configurations to affective states, reflecting the fact that capable problem solvers make use of affect to interpret and evaluate their problem-solving progress. The interaction between affect and heuristics is subtle and pervasive. Affective configurations can not only influence, but override executive decisions; sometimes with consequences that are valuable, at other times with unfortunate results. In view of the strong evidence that negative emotion is widespread in relation to mathematics, considerable further attention must be devoted to the psychology of developing effective affect in students.

## Key Constructs Expressible in a Unified Model

I have sketched some of the ingredients of a model, one that admits both external and internal representational systems of various kinds. Let us next observe the extent to which such a model can serve as a unifying framework. I shall mention how a few major, useful ideas from different theories of mathematics learning and problem solving can fit together in this framework.

*Adequacy of the systems of external and internal representation*

Ideas that pertain to mathematical structure as external to or independent of the individual, including problem complexity, isomorphic and homomorphic mathematical tasks, strategy scores as patterns in external structured environments, and the design of artificially-structured computer microworlds, are expressible fairly directly in terms of external representational systems. The major types of internal systems proposed, allowing for the presence of ambiguity, would appear sufficient to express and analyze much existing theory. By making direct reference to the five specific kinds of internal representation suggested, and exploring the structures of the correspoding *systems*, we can readily incorporate theory relating (respectively) to the following: (1) verbal processing and the effects of problem syntax; (2) empirical evidence on visual imagery, image schemata, the "physical tier" of conceptual development, and the effects of problem semantic structure variables in mathematics; (3) the roles of notational and algorithmic complexity, consequences of the manipulation of task structure variables, and the "logico-mathematical" tier of conceptual development; (4) the use and development of heuristic processes and executive decisions as these govern learning and problem solving; and (5) the fundamental importance of affect. Somewhat less directly, we can also incorporate *interactions* between the internal systems--how they act on and affect each other, how they symbolize each other, and how they interact with and symbolize configurations in external representational systems.

But we can also do a lot more. Let us consider a few additional, key theoretical ideas.

### *Learning as a constructive process*

Both moderate and radical constructivist researchers argue that knowledge is *constructed* by each individual, rather than *transmitted* or *communicated* (Ausubel, 1963; Cobb and Steffe, 1983; von Glasersfeld, 1987, 1991). Thus the constructivist perspective on learning suggests that internal representations and systems of representation do not appear full-blown, but are built up over time. Is each system established on its own, on an *ad hoc* basis, or are there overall laws governing the development of representations, processes that govern all or most of the kinds of systems considered? The metaphor of construction can be intepreted in greater detail by examining the development of these systems as fundamental entities.

Ausubel's work is especially interesting in this regard because he was an advocate of "reception" and a critic of "discovery" learning. A key technique according to him was the use of *advanced organizers*, which would serve to provide students with prior frameworks--scaffolding, as it were, or a "template"--upon which pieces of new knowledge might be "fit". Thus new cognitive structures could be built up through efficient reception learning. Indeed, *pure* discovery learning can be set in opposition to *pure* reception learning, with the former subject to criticism for its inefficiency and the latter for its rote, uncreative nature; but guided discovery learning has much in common with reception learning when the latter is accompanied by advanced organizers. In employing guided discovery, the teacher must rely on some assumptions about students' pre-existing cognitions in shaping meaningful hints or suggestions; these prior structures are in effect serving as templates for the construction of new knowledge. For meaningful reception learning, the advanced organizer (provided externally, by the teacher) functions as the template. But empirical studies of the effectiveness of advanced organizers have difficulties: e.g., how are we to know when something is serving as an advanced organizer, and when it is simply a review of prerequisite knowledge? This function is not based on surface characteristics of the learning environment, but requires for its definition a *model of what is happening internally*, which the analysis of cognitive representational systems can provide.

To the extent that a theory governing the construction of internal systems can be proposed, we have the beginning of an understanding of learning that goes beyond addressing the acquisition of discrete skills. One element of such a theory involves stages in their development. Three main stages have been proposed: (1) an *inventive-semiotic* stage, (2) a period of *structural development* , and (3) an *autonomous* stage.

During the inventive-semiotic stage (Piaget, 1969, p. 31), new signs are created or learned; and, most important, these are taken from the outset to *symbolize* aspects of a *previously established* system of representation. This is the act called *semiotic.* The prior system serves as a *semantic domain* for the new symbols. Sometimes the characters are treated during this stage not just as symbolizing, but as *being* the aspects of the prior system that they represent. For example, when exponentiation is first introduced, the notation $3^4$ may be taken to *be* the act of calculating the product 3 x 3 x 3 x 3. When words are first learned, children (or adults) sometimes *identify* words with the objects that they label. With this inventive-semiotic stage in mind, we might say that a presentation in a learner's environment serves as an advanced organizer for the construction of new knowledge when it is an *externalization of certain features* (signs, configurations, higher structures) of a prior cognitive system (already developed), that can serve as a semantic domain. The degree to which this can serve as an operational definition of advanced organizers depends on our ability to characterize the student's existing systems of cognitive representation through the assessment of competencies.

During the structural development stage of a new representational system, configurations are built up from characters, and intrinsic syntactical rules for the new system are constructed, *with the construction driven primarily by structural features of the previously established system*. The possibility of *alternate* symbolic relationships for the new system is generally unrecognized at this stage, as the structure of the new system is seen as a *necessary* consequence of the meanings of the new signs and configurations in the semantic domain of the previous system. Gradually, the new characters and configurations cease to be discrete entities unrelated to each other, and form instead a structural whole. Redundancies are built in, so that relationships from the earlier system are reinforced through the developing intrinsic structure of the new one.

Finally we enter an autonomous stage, during which the new cognitive system of representation *separates* from the old. The new system can now stand in symbolic relationships with systems *other* than the original template, and the characters and configurations in the new system can have meanings other than those initially assigned to them! As these new possibilities for the new system are realized, the transfer of competencies to new domains becomes possible.

The development of representational systems through such stages requires interaction with representational structures in the individual's environment, ranging from spoken language to mathematical experiences. Tracing this development permits the unification of a considerable degree of cognitive theory in relation to the learning of mathematics.

### *Domain-specific mathematics learning, schemata, belief systems, and misconceptions*

Cognitive representation always takes places in a specific, contextual domain of knowledge. In my view, conceptual understanding in a particular content domain *always* involves not just one, but many types of representational systems. I tend to resist the idea that individual learners can be classified as "verbal", "visual", etc. based on preferential encoding in just one system. Rather, I would attribute to individuals a preferential use of certain *heuristic processes*, which then permit easier access to, or more rapid development of, one system of representation over another.

The notion of a *schema* (e.g., for mathematical proportionality) is neither purely verbal, purely notational, purely imagistic, nor purely heuristic. Rather, it involves highly structured relationships among competencies represented in all these ways--including semantic structures, verbal structures, notational structures, etc. Thus schemata are structured collections of domain-specific competencies, organized into sets of related configurations (and information-processing action-sequences), in *different* representational systems. The redundancy that this multiple-encoding provides can account not only for the persistence of schemata in long-term memory once they are constructed, but also for the ability of individuals to *reconstruct* concepts when they seem to have been temporarily forgotten.

Like schemata, belief systems are broad constructs cutting across systems of representation. For example the widespread and pervasive belief that "mathematics is the obtaining of correct answers by following established procedures" pertains quite obviously to the individual's planning and executive decision-making in mathematics. But it also is likely to involve a considerable affective component; e.g. a sense of security with familiar procedures, satisfaction in following precise instructions, or anxiety when a rule has not been provided. Such a belief system no doubt also entails some fairly elaborate schemata describing what the acceptable "established procedures" are, so that formal notations are rapidly invoked.

Schoenfeld (1985) sees students' adherence to certain mathematical misconceptions as a part of their belief systems; thus he stresses, at least implicitly, the affective as well as the experiential factors that strengthen adherence to such beliefs in the face of contrary learning. And the characterization by diSessa (1983) of the cognitions involved in such beliefs as resting on phenomenological primitives, "simple knowledge structures that are monolithic in the sense that they are evoked as a whole and their meanings, when evoked, are relatively independent of context", is strongly suggestive of internal imagistic representation. The elaboration and multiple encoding of belief systems can in part account for how surprisingly difficult it is to modify them through verbal discussion or formal instruction.

### *Developmental stages, imagistic representation, and related ideas*

In an earlier paper (Goldin, 1988a) I suggested that developmental precursors of imagistic systems of representation, a *brain system* and a constructed *sensory interpreter*, could be posited. Construction of the latter would correspond roughly to Piaget's *sensorimotor* stage of development; while construction of imagistic systems (on the template of interpreted sensory experience) occurs during the *preoperational* stage. Such broad correspondences between the construction of representational systems and developmental stages in children need further exploration. I would further propose that imagistic systems of representation (providing semantic content and structure), together with a brain system (providing "deep" linguistic/ syntactic structure), are the twin templates on which the verbal/syntactic system is built.

As a final observation in this section, I would note that the role of metaphor in understanding language and in reasoning can be explored by interpreting metaphors as maps, expressed in words, from one coherent imagistic representation to another.

I hope these observations, sketchy as they are, provide some indication that a model based on internal and external systems of representation is capable of incorporating the most important constructs from a wide variety of theoretical perspectives.

## Some Implications for Mathematics Education in Practice

A model has consequences not just for theory, but for the practice of mathematics education. Most fundamentally, the present model suggests that our over-arching goal should not be to convey specific mathematical content to students, or even to teach them specific problem-solving processes; rather it should be to foster in students the *construction of powerful, internal systems of representation* of the kinds discussed.

Much of school mathematics is still devoted explicitly to the manipulation of formal notational systems, despite considerable effort to change classroom practice to emphasize problem solving strategies, visualization, pattern recognition, and other more conceptually-oriented techniques. This comment applies to all subject areas, with the possible exception of high school geometry. In the earliest grades, children learn to write and manipulate numerals and operation signs; they continue with standard algorithms for addition and subtraction, multiplication and division--of whole numbers, fractions, and decimals. New formal characters, such as $ and % are introduced for special purposes. In high school, students are introduced to writing and manipulating algebraic expressions, graphing equations using Cartesian coordinates, proving trigonometric identities, etc.; and in college, they learn to manipulate formal expressions for functions, derivatives, definite and indefinite integrals, partial derivatives, and so on. Even the teaching of estimation is often limited to an algorithmic process for "rounding", followed by use of an arithmetic operation on the rounded quantities. Problem solving may be limited to artificial story problems, where the goal is merely to *translate* from the words and sentences of the problem to previously learned formal notations (and then to solve the problem in the mathematical notation, using a standard algorithm).

If, on the other hand, our goal is to develop internal representational systems of many kinds, we must provide an equivalent level of attention to imagistic representation, planning and executive control, and affective representation. Developing power in these representational systems must become our explicit objective, and methods of assessment must be developed which can evaluate our success. Furthermore, if we take seriously the construction of representational systems according to the stages discussed, we must consciously foster these processes in students: inventive-semiotic acts, where children create new signs and assign them meaning, must themselves be seen as instructional goals; and the students should then be exploring the logico-mathematical consequences of their inventions--in effect, learning to build their own (external and internal) mathematical representations.

## Conclusion

The main strands of theory in mathematics education, as they have been developed during the past twenty-five or more years, are on the whole compatible, not mutually contradictory. They fit together into a unified whole, if we are willing to develop the necessary theoretical underpinnings (and sort through each other's terminology), instead of rejecting each others' constructs on *a priori* grounds. Hopefully the framework for a model described here represents at least a step toward developing the necessary components of such a unified, inclusive system. It has consequences both for research, and for the practice of mathematics education.

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# ILLUMINATIONS AND REFLECTIONS — TEACHERS, METHODOLOGIES AND MATHEMATICS

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In his presidential address to the 14th annual meeting of PME in Mexico 1990 Balacheff called for a step beyond a psychological approach "in order to understand the nature of the complexity of mathematics learning in a didactical context" (Balacheff, 1990, p.2). He suggested that we must find ways to integrate a social dimension into our methodologies basing his argument on two premises. First, "students have to learn mathematics as social knowledge; they are not free to choose the meaning they construct" and second, "mathematics ... requires the confrontation of the student's cognitive model with that of other students, of the teacher, in the context of a given mathematical activity" (Balacheff, 1990, p.2).

## Trends within PME

Balacheff's remarks reflect a changing paradigm of work within the community of PME. There is a clear shift in methodology: from experiments identifying childrens' errors and their possible 'remediation' by 'treatments', to case studies within a more holistic framework; from theoretical perspectives which are limited to the student/mathematics diad, to those which encompass this diad within social settings while incorporating a more conscious problematising of the nature of mathematics itself. I aim to suggest in this paper that, whilst these changes have brought a richer more authentic thrust to our work, they have also led to some confusion in focus for investigations within the framework of psychology of mathematics education, together with some concern over the validity and generalisability of our research findings.

There are two trends evident in the work of PME: a quantitative increase in research incorporating the teacher as an integral — and crucial — facet of learning mathematics, and a series of qualitative shifts as to how the teacher and the teacher's role are conceptualised. I will attempt to trace these trends within the proceedings of PME over the last twelve years.

## Recognising the Teacher

Of the 45 papers included in the published proceedings of the third PME conference in 1979, all but three focused on student understanding of mathematical concepts. Some papers centred on the diagnostic, identifying the kinds of difficulties students encountered, the strategies used to solve particular sets of problems, and the types of errors that arose. Others considered development in mathematical thought either from the standpoint of identifying barriers to further development, or in terms of advocating aids for promoting understanding. If the teacher was mentioned at all, s/he was discussed purely as a facilitator — to dispense facts and information; to identify errors or misunderstandings; to provide materials or strategies to overcome misconceptions; or to promote further mathematical development. Within this body of papers, Skemp's paper (1979) proved to be an exception, describing the dynamic development of the mental processes of the learner as "necessarily related to the goals of the learner; and if a teacher is involved, *to those of the teacher also*" (p.197, my emphasis). Skemp also pointed to the possible 'mismatch' between the learning goals of student and teacher.

In 1980, the majority of papers again concentrated on student ability, student understanding and representation of specific mathematical concepts and the 'attributes', 'attitudes' and 'processes' of students in the mathematics classroom. As at the 1979 conference, the teacher's role was described in terms of promoting student progress, either on the basis of identified driving or inhibiting characteristics of the students themselves, or by adopting theoretically-derived methods of instruction. However, the teachers themselves featured a little more prominently, at least in terms of introducing ideas or methods to 'connect' with students and overcome problems in the classroom (for example Herscovics and Bergeron, 1980; Bergeron and Herscovics, 1980).

In contrast with the dominant theme of most papers in 1979 and 1980, which may broadly be classified as "what makes learning difficult?", Shroyer (p.331) asked "What makes teaching difficult?". Referring to the work of researchers who have shown an interest in classifying and quantifying teacher thought (for example, Peterson and Clark, 1978), Shroyer herself analysed 'critical moments' whilst teaching mathematics and proposed that differences between teachers reflected differences in "teaching styles, instructional goals and expectations" (1980, p.336). Bishop (1980) also argued that, just as children develop their own problem-solving strategies, so do teachers develop their own personal techniques for dealing with classroom situations. He had "no doubt that the teacher was the key person in mathematics education, and research which ignored this fact stood a good chance itself of being ignored" (p.343).

Within the context of the present review, the 1981 conference papers were notable only in terms of a marked absence of any substantial new discussion of teacher beliefs and, in 1982, despite an increase in the number of PME papers, there was no corresponding increase in papers devoted to the teacher's role in the classroom. However, between 1982 and 1984 a new strand emerged which recognised the influence of teachers' expectations and attributional interpretations of students' mathematics learning and ability. (for example, Hoyles and Bishop, 1982; Zehavi and Bruckheimer, 1983; and Romberg, 1984). Romberg began to identify a more clearly defined link between beliefs and practice. Taking as a conceptual framework the "Model of Pedagogy" (Romberg et al, 1979), he proposed that teacher beliefs about schooling, learning and mathematics are mediated by the specific content to be taught, but also reflected in instructional plans, teacher's actions and students' behaviour and performance. He argued that the character of plans could be inferred from the allocation of time and the emphasis given to various content, and observed that teacher beliefs relating to student ability seemed to determine the different types of tasks students were set in respect of content and difficulty. Thus 'low' ability students were given 'drill and practice' tasks, high ability students 'explorations'.

In 1985/86 little new can be reported, and indeed the 10th anniversary conference in 1986 appeared to relegate the teacher once again to the position of a passive conveyor of facts and information, albeit using a variety of diagnostic methods and teaching tools. However, a growing appreciation of the differentiated nature of teacher beliefs can be discerned. Brown (1986) suggested that a conception of mathematics teaching consisted of three components: beliefs about mathematics; appropriate goals and tasks for the mathematics classroom; and the relative responsibilities of teacher and students regarding motivation, discipline and evaluation. Although drawing conclusions from an analysis of only a single teacher, Brown highlighted the possible conflict between a teacher's own conception of mathematics and his/her perception of students' needs and interests, to the point where classroom actions might become inconsistent with the teacher's own expressed beliefs about mathematics teaching. This is the first mention of the influence of context on beliefs and the potential mismatch between beliefs and beliefs-in-practice — here framed to suggest that teachers have 'true' beliefs which may not be enacted in practice.

In 1987, Oprea and Stonewater took up this issue of inconsistency, noting that the relationship between belief systems and instructional practice was far from simple. They argued for a distinction to be made between how teachers thought about teaching mathematics and how they perceived mathematical content. More generally, Cooney and Grouws (1987) identified a number of research issues and themes for the future: "Studies of teacher knowledge and teacher beliefs and especially how they moderate teaching behaviour and student learning are needed" (Vol. III, p.423).

In 1988, a rather more theoretical analysis was undertaken by Ernest (1988) who, drawing on previous research (Begle, 1979; Bishop and Nickson, 1983) detailed the complex nature of the claimed relationship between teacher attitudes towards mathematics and effectiveness of teaching suggesting that the picture was complicated by the multi-dimensional nature of attitudes towards mathematics. Nonetheless, Ernest concluded after work with student primary school teachers that it was attitude towards teaching the subject rather than towards the subject per se which was the most important factor in 'determining' teaching style.

In 1990, a number of papers from a variety of perspectives continued to point to the influence of the teachers' beliefs, cognitions, and competence, on behaviour in the classroom and assessment practices or how the latter threw light on the former (Dougherty; Flener; Grouws et al; Schifter, (all 1990)). This trend continued into the following year, where once again beliefs were reported as influencing classroom practice, (Jaworski; Even and Markovits;

Jurdak, (all 1991)) or framing teachers' expectations of students and their treatment of errors (Boufi and Kafoussi, 1991).

The issue of complexity was taken up by Underhill (1990) who concluded that mathematical conceptions were 'watered down' as one moved from the mathematics specialists at school division level through the principal to the classroom teacher. He suggested that this 'web of beliefs' influenced the actions and statements of novice teachers. This takes me on to a consideration of factors influencing teacher beliefs and how beliefs might be changed.

## Changing the Teacher

Alongside the trend towards 'researching the teacher', a developing interest can be discerned in PME concerning the teacher's role in mediating curriculum change. 1987 marked the first time interest in teacher beliefs in relation to curriculum innovation was specifically noted, when teachers were viewed as potential obstacles to innovation, as 'something' to take into account and to be changed. For example, Oprea and Stonewater (1987) cited Carpenter et al. (1986) who reported that teachers' beliefs affected how they perceived in-service training and new curricula, and therefore influenced the relation of 'implementation' to the intentions of the original developers. The same year saw a series of papers collected under the heading 'In-service Teacher Training'. These tended to argue that it made sense to explore the belief systems of teachers *before* attempting to introduce changes (Dionne; Waxman and Zelman; Jaworski and Gates; Shaugnessy; Kuendiger, (all 1987)). Again there appears to be an implicit view that teachers have something called 'beliefs' in some decontextualised sense which need to be accessed and changed.

In the following year, several papers continued to present evidence concerning how teachers' beliefs mediated their behaviour and the ways curricular innovation was taken up (Klein and Habermann; Jaworski; Simon, (all 1988)). Positive change was reported by Carpenter and Fennema (1989) who described how curriculum support in the form of research-based knowledge about children's thinking and problem solving was identifiable in teachers' decision-making and instructional techniques, which in turn affected students' learning. These findings contrasted with those of Clark and Peterson (1986) who had contended that teachers did not tend to base instructional decisions on any assessment of children's knowledge.

In a retrospective study, Nolder (1990) examined some of the consequences of curriculum changes on teachers' belief systems and identified some factors influencing decisions about the innovative practice: anticipatory anxiety (e.g. that parents might oppose the new methods and that experimentation might lead to deterioration in examination results); mismatch between teacher's residual ideologies and ideas underpinning curriculum innovations; and concern about the time needed to 'implement' new approaches. She also recognised that the social context of teachers' work imposed limitations on classroom practice and curriculum innovation. This paper illustrates the growing appreciation within PME of the teacher's reality and its structuring of the teaching/learning process.

Finally, technology reappeared in a discussion of beliefs (after a gap of 11 years) with Ponte (1990) and Noss, Hoyles and Sutherland (1990) examining conceptions and attitudes of teachers enrolled on computer-based inservice programmes. This discussion was continued by Noss and Hoyles in 1991, who adopted an approach which will be elaborated later in this paper.

## Contextualising the Teacher

In the late eighties, papers in PME began to exhibit a shift in research attention away from subject matter structures, teacher beliefs and student achievement, towards a perspective of social interaction and construction, mainly through the report of the Research Agenda Conference on Effective Teaching (Cooney and Grouws, 1987). Bauersfeld, in particular, argued that a study of the development of mathematics within social interaction in classrooms showed 'degeneration' into linear accumulation of tried and tested routines: "The mathematical logic of an ideal teaching-learning process ... becomes replaced by the social logic ..." (Bauersfeld, 1988, p. 38). He suggested that the reality of teachers, students and researchers could be seen as a product of constructions by each of the parties involved. Taking an even broader view (perhaps leaving the realm of psychology?), Stigler and Perry (1987) considered teacher beliefs and attitudes in the context of cross-cultural studies of mathematics teaching and learning, claiming that what happened in the classroom was a reflection not only of the culture of the classroom but also of the wider society.

This particular perspective seems little developed within PME although in the introduction to the proceedings of PME 15, the organisers commented: "More than in the past the general interest of the different activities scheduled in the conference seems focussed not only on students' behaviour, but on the figure of the teacher and on the context and social factors intervening in the teaching learning processes" (Furinghetti, 1991, p.iii). This trend is discernable in the organisation of the PME conference itself, where some of the discussion groups of previous conferences had, by 1991, developed into full blown working groups, including: Social Psychology of Mathematics Education; the Psychology of In-service Education of Mathematics Teachers; Research on the Psychology of Mathematics Teacher Development.

Additionally in 1991, some attention was paid to teacher beliefs and attitudes within a cultural context with an emphasis not so much on identifying existing attitudes, but on the effects of courses aimed at changing attitudes. Bishop and Pompeu Jr (1991), after introducing an ethnomathematical initiative in Brazil, questioned whether a simple shift from one approach to another was sufficient to guarantee better results in teaching. Moreira (1991), in a comparison of attitudes to mathematics and mathematics teaching in English and Portuguese teachers, found quite different attitude profiles in the two countries attributing these to different educational systems and to different social contexts within which the schools operated. These papers point to the difficulty of identifying generalisable patterns within a paradigm which incorporates a social and cultural dimension, particularly in view of the different theoretical frameworks adopted by the researchers.

## Researching the Teacher

Evident from the review of the research above is the bewildering range of perspectives adopted. The studies cited are characterised by the range of theoretical perspectives guiding the research, by the variety of investigative tools used, and by the diversity of methodologies employed. Lerman (1992a) has argued that one of the reasons that research on teaching has grown in significance lies in the fact that qualitative research methods have become established over time and correspondingly gained wider acceptance.

Within the PME community, the issue of methodology was given serious consideration for the first time when McLeod (1987) emphasised the need to refine frameworks and develop new methodologies[1]. However, there exists a dilemma between, on the one hand, the desire to provide creative methodologies to explore and access teacher beliefs (as advocated, for example, by Shaugnessy, 1987) and, on the other, the problem of interpreting multiple methodology research findings to compare results and identify emergent patterns.

Perhaps the concern over methodology was a catalyst for the new innovation in 1990: the plenary symposium, considering the responsibilities of the PME research community? Unfortunately, since the proceedings are produced prior to the conference, there is no indication of whether any attention was paid to the topic of teacher beliefs and attitudes — and although I was there I just cannot remember! It is therefore unclear as to whether progress was made but nevertheless apparent that continued debate is needed.

## Summary

An overview of papers presented at PME conferences over the years indicates that the teacher has attracted increasing attention. There has been a shift in focus from an explicit recognition of the 'formatting' role of teacher interventions to a consideration of teachers' underlying attitudes and beliefs. Different trends under this umbrella can be discerned: studies of the nature of beliefs, and studies seeking to identify their influence on classroom activity and teaching/learning behaviours. Alongside these trends, a growing interest is displayed in the part played by social and cultural factors and how these frame attitudes at least in the process of classroom practice. Many papers also admit that much remains to be done in formulating appropriate methodologies and theories.

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[1] This is in fact the third 'new' trend initiated in 1987. Is it at all relevant that the 11th PME conference was characterised by the largest number of scientific presentations and widest geographical distribution of contributors in its history, with conference reports presented in three volumes, each about the size of the single volumes containing the entire presentations of previous years?

Thus, in 1992 'teacher research' is a major focus of effort in mathematics education but one that can still be regarded as emergent: there is no consistent approach or straightforward way to understand and interpret either empirical findings or theoretical approach. A range of methodologies are used but few comparisons made between them; there is little evidence of general patterns. Each research seems almost to "start again" with a new theory, a new method of data collection. However, my review has thrown up some issues which will serve to organise my paper: the need to disaggregate teacher attitudes taking into account attitudes to teaching/learning, to mathematics and views of students; the need to analyse the 'framing' of context on teacher attitudes and take on board the possibility of multiple, even contradictory, belief systems; and finally, the need to investigate the interaction of teacher beliefs and curriculum innovation.

The changing focus of research evident in PME can also be discerned outside the community of mathematics educators as well as within it: for example, from the perspective of educational psychology, Berliner has asserted:

> "Studies of the individual learner, a dominant paradigm in our field for decades, may no longer be the appropriate conceptual framework for understanding many kinds of learning. The educational psychologist of the 21st century needs training in the concepts and methods needed to study groups, to discriminate among and describe environments, and to think systematically or holistically rather than analytically or simplistically about the nature of causality in the learning process (Salomen, in press)" (Berliner, 1991, p.150).

An attempt to identify appropriate 'concepts and methods' in this new paradigm leads me to my next concern; the identification of a theoretical framework.

## A Search for a Theoretical Framework

In trying to set out a theoretical framework to research teaching and teachers of mathematics, I look first outside mathematics education to search for similar trends and to provoke me to question what I might have previously taken for granted.

### Some lessons from outside Mathematics Education

A look at research in reading suggests many parallels with mathematics education: there have been similar shifts in emphasis over the last decade. Early reading research tended to assume a sequence of skill acquisition based on the notion of developmental stages. More recently, emphasis has been on reading as a constructive process whereby readers interpret text according to their own existing understandings. Additionally, the mediating role of the reading teacher is now more clearly in focus with a relationship hypothesized between expressed teacher beliefs and theoretical understandings on classroom practice (see for example, Roehler et al, 1988). On the methodological front, Richardson et al, (1991) reviewed reading research exploring the relationship between theoretical orientation towards reading (phonics, skills, whole language) and teacher behaviours. They suggested that the contradictory results reported could be interpreted as stemming from problems of measurement. The authors argued that since there was no universal agreement on theoretical orientation or approach to teaching reading, variations in teacher beliefs as well as practice must be expected. In summary, reading research seems to mirror the move in mathematics away from process-product research to constructivism — with all the concomitant methodological advantages and problems .

Moving beyond psychology and specific subject teaching, we find that in his keynote address to the 1985 conference of the International Study Association on Teaching Thinking, Clark reflected on the development of researchers' conceptions of teachers and teaching, the available conceptual frameworks and methodological issues (see Halkes, 1986). Clark argued that the teacher had previously been viewed as a physician-like clinical decision-maker but now tended to be conceptualized as a professional sense-making constructivist, developing and testing personal theories of the world; a change which required new language to mirror and express life in classrooms as seen by teachers and students. The need for precise terminology was illustrated in the use of terms such as 'teachers' cognitions', 'constructs', 'subjective imperatives' or 'practical knowledge' which, though often used interchangeably, originated from different frameworks. Commenting on the conference as a whole, Halkes also noted that, with the growing attention to the individual teacher's perspective and thought processes, a range of non-traditional research methods had been adopted, the diversity of which inevitably leading to problems of comparability and generalisability.

Clandinin and Connelly (1987), commenting on recent reviews of teacher thinking, theories, beliefs and knowledge claimed that "what is especially interesting about these studies is that....they purport to study 'the personal', that is the what, why and wherefore of individual pedagogical action." (p. 487). However the authors questioned what was exactly meant by 'the personal' and its relationship to pedagogical action.

Clandinin and Connelly's paper attracted considerable interest from other researchers in the field, as evidenced by the publication of four response papers[2]. I briefly pick out some arguments of three of them. Halkes (1988) differentiated between the 'particular' perspective exploring idiosyncrasies of an individual teacher (that of Clandinin and Connelly), and a 'general' perspective asking how different ways of teaching might explain teaching or classroom practices. He also pointed to the need for a third perspective linking teacher behaviour and thinking with the task of teaching as 'explicit intentional behaviour'. He argued that such a view would require an analysis independent of the individual teacher and the demands of both general and specific teaching tasks. Roehler et al. (1988), whilst accepting that the Clandinin and Connelly review provided a useful basis for examining teachers' personal knowledge, argued that the personal knowledge which most influenced teachers' instructional practice was not beliefs or implicit theories, but rather the structures which organized instructional knowledge. They suggested that these knowledge structures were fluid and evolved as the teacher integrated knowledge from new experiences into existing schema, in contrast to beliefs and theories which were usually presented as static and eternal truths. Roehler et al. also pointed to the difference between the associations of beliefs and knowledge structures to practice — the former influencing what teachers said *outside* the classroom, whilst the latter structuring their responses *in* classrooms. Olson (1988) similarly argued that making sense of teaching meant interpreting what teachers did and what rules they were following — an approach recognizing the teachers' culture which parallels the work of myself and Noss (see below). The contrast of the general and the particular, the focus on methodological issues and the growing awareness of the social constraints of the classroom mirror our own concerns in mathematics education to which I will now return.

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[2] Initiating a debate in a similar way amongst researchers in our community on the pages of our journals would, in my view, be a major step forward in our endeavours.

## Beliefs and Beliefs-in-practice

Thompson (1992), in a comprehensive review of this area noted that, since 1980, a flurry of studies in mathematics education have focused on teachers' beliefs about mathematics and mathematics teaching and learning. Drawing on her own work (Thompson, 1984), she identified a consistency in views of mathematics amongst junior high school teachers, which, she argued, suggested a well-integrated system. She also noted a strong, though subtle, relationship between these views and classroom practice — although recognising that this could not be characterised as one of cause-effect. Other work has identified mismatches between beliefs and practice, the espoused-enacted distinction (for example, Ernest, 1989) and experimental research within French didactics has also thrown light on this issue.

The problem for French researchers as set out by Arsac, Balacheff and Mante (1991) is 'one of the reproducibility of didactical situations' in the researching of which a growing recognition of the role of the teacher has been identified. In their first study, it became clear that, in practice, the teachers needed 'closure' and the production of an acceptable mathematical solution — even though this was not part of the plan agreed with the researchers. In a second experiment, the teacher was given an even more precise formulation of what she was allowed to do and say[3], but once again clear 'gaps' between the intended scenario and its implementation were discerned, gaps which were interpreted as stemming from the teacher's personal relationship to mathematics and her ideas about teaching and learning mathematics.

The inconsistency of teachers' beliefs and their practice is also manifested in the findings of Sosniak, Ethington and Varelas (1991) who used the data from the Second International Mathematics Study to explore teacher beliefs. They found that:

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[3] Since this teacher, unlike the former one, had not been a member of the research team

> ... eighth-grade mathematics teachers in the US apparently teach their subject matter *without* a theoretically coherent point of view. They hold positions about the aims of instruction in mathematics, the role of the teacher, the nature of learning, and the nature of the subject matter itself which would seem to be logically incompatible. (p.127, emphasis in text).

Sosniak et al. recognised that their findings might be artifacts of the research instruments they employed: questionnaires in contrast to Thompson's in-depth case study approach. This point is explored further by Glidden (1991) who expressed concern over the lack of concrete referents in the questionnaire items and the absence of any distinction between short and long term goals. Lerman (1992b) suggested that similar 'inconsistencies' might merely be in the eye of the beholder, arguing that theory and practice should be "seamless rather than separate" (p.3). I tend towards one interpretation suggested by the authors themselves since it rejects (albeit implicitly) any intimation that 'teachers are not up to standard'. They posited the notion of 'distance' of the variables in their study from the activity of teaching — views of mathematics at one extreme and of student activity at the other and suggested that:

> We have a set of findings regarding teachers' curricular orientations which shift systematically from 'progressive' to 'traditional' as the teachers move from considering the issue most distant from schooling and classroom instruction to the issue most central to schooling and classroom instruction (Sosniak et al., p.129).

From this perspective, inconsistency is merely a clear and accurate representation by teachers of the different concerns brought to bear in the act of teaching in schools and that of discussing mathematics education. Brown and Cooney (1991) subscribed to this more interactive interpretation of teachers' actions, building on the work of Schön (1983), which recognises the limitation of technical rationality in professional action: "(the teacher's) inquiry is not limited to a deliberation about means which depends on a prior agreement about ends. He does not keep means and ends separate, but defines them interactively as he frames a problematic situation" (Schön, 1983, p.68). Brown and Cooney also pointed to the duality of practice referring to the work of Shulman (1978) "that when prescribed practice is perceived by teachers to be inconsistent with their existing theories (implicit perhaps) of teaching, that practice is discarded" (p.113) — that is, when it is a question of cognitively-guided instruction or relinquishing classroom control, the latter must take precedence as teachers need to be able to predict and control classroom events. Decision-making ('reframing' as Schön called it) occurs as reaction to the data of the classroom. As Desforges and Cockburn (1987) have pointed out with reference to primary classrooms:

> We have worked with many teachers who were well informed on all these matters [children's mathematical thinking, clear objectives, attractive teaching materials — CH] and yet who routinely failed to meet their own aspirations

> which they shared with mathematical educators. We do not believe that this failure has been due to idle or uninformed teaching. On the contrary, we suspect that the failure lies in the unwillingness of the mathematics education establishment to take seriously the complexities of the teacher's job. (Desforges and Cockburn, 1987, p.1-2)

It seems that research must accept that actions and beliefs are shaped by the conditions of classrooms. But, if we go too far along this road, there is a danger of seeing the teacher as 'determined' by the constraints of the role, an issue I will return to below.

## Towards more specificity

In attempting to unravel general strands within and outside mathematics education, we run the risk of losing the self-evident diversity amongst teachers and classrooms — across different age ranges, and within different cultures. In fact I have been struck by the question as to how far the change in paradigm identified so far is in fact a 'western' phenomenon? Is 'the teacher' a well-defined 'object' of study? Is it the case that the teacher's culture is homogeneous, across different countries, and phases? Just limiting myself to the last distinction, there is in fact evidence pointing to a different stance taken towards investigations with primary teachers in comparison with those of secondary or high school teachers. The majority of studies on the former group tend to assume that understanding of mathematics, or at least self-image regarding competence in the subject, are the most influential factors in determining the functioning of the teacher. They therefore tend to concentrate on misconceptions in the teacher's knowledge base or, less frequently, explore teachers' beliefs regarding their perceived lack of competence. In contrast, the majority of studies of teachers of older students are more interested in defining the knowledge and/or beliefs held by teachers regarding mathematics and mathematics teaching and learning.

Outside the realm of mathematics, similar differences are identifiable. For example a study of pre-service elementary and secondary science teachers, indicated basic differences between the belief structures and concerns of the two groups (Cronin-Jones and Shaw, 1992) with the former apparently more 'simplistic' than the latter. Examples of areas of focus unique to the secondary group included concern for the subject and for its assessment.

There is also diversity amongst students, in terms of their perceived mathematical competence (as previously noted) but also in terms of their social position. As Secada (1991) has argued:

> Work on teaching, while attending to teacher beliefs, knowledge, and behaviours in terms of content knowledge, needs to expand to include teacher beliefs, knowledge, and behaviours as a *function of the sorts of students who are in their classrooms.* (p.46-7) [my emphasis].

A failure to acknowledge this diversity in attempts at simplification, inevitably rules out consideration of emotional, cultural and social issues, leaving a blandness which does not resonate with life in real classrooms with real teachers and real kids. There is diversity too in mathematics itself. Distinctions have been made between school and academic mathematics, relational and instrumental mathematics[4]. It might be sensible to disaggregate mathematics still further and in different ways in order to map out a more complete picture of teacher beliefs. We might usefully classify mathematics according to different sign systems, different modes of expression leading us to distinguish 'textbook' mathematics, oral mathematics, Logomathematics etc.[5]

To undertake research which takes account of the diversity amongst teachers, students and mathematics raises enormous methodological problems and I am in danger of moving beyond the realms of mathematics and psychology. Our approach has been to study the teaching/learning setting in the context of using computers, an approach which throws into relief teachers' beliefs in their interaction with the curriculum innovation and classroom culture.

## Innovation and beliefs

In the discussion above, I have in fact touched upon the debate in 1986 between Brophy (1986a,b) and Confrey (1986)[6] concerning the ways in which teachers are viewed in the enterprise of teaching: to 'meet objectives' or 'to construct meanings'. These different views, I contend, are also reflected in the different interpretations of the frequently reported mismatch between the intention and implementation of a curriculum innovation.

## Perspectives on teachers and change

One perspective sees teachers as potential 'impediments' to curriculum innovation who do not or cannot take on board all the objectives of the reform. For example Peterson 1991, states:

> One problem is that teachers often seen the 'surface level' features of the reforms being advocated, such as the changes in instructional practices, without seeing the assumptions and theoretical frames of the persons who have constructed the hoped-for changes in instructional practices including the researcher, the reformer, the textbook writer, or the expert teacher. (Peterson, 1991, p.3).

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[4] Thompson (1992) returned to Skemp's (1978) distinction, noticing that he had suggested that these were effectively two different subjects being taught under the same banner of mathematics.

[5] This suggestion presuppose that representation is an essential part of a mathematical concept following Vergnaud (1982), Carraher, Schliemann and Carraher (1988). Extendng these notions from our experiences with Logo, we have coined the term 'situated abstraction', a mathematical generalisation articulated in the language of a particular setting (see Hoyles and Noss, 1992, and Hoyles, in press).

[6] The debate is presented in greater detail in Noss & Hoyles (1992, in press).

Similarly, in relation to a discussion of didactic engineering, Artigue and Perrin-Glorian (1991) suggest that: "the teachers, faced with the products of didactic engineering, will attempt, consciously or not, to reduce the distance between what is proposed and their usual way of functioning" (p.17). It is evident from these studies that a 'treatment' approach which fails to take account of the teacher and the teachers' work situation as mediators of any innovation is not only doomed to failure but theoretically inadequate.

Olson (1985) proposes an alternative stance towards curriculum change, viewing change as being brought about by teachers as they reflect on and become more aware of their own practices. From this reflexive perspective, teachers' behaviour is viewed as fundamentally sensible involving the constant resolution of multiple interacting dilemmas and demands as part of their professional life. He rejects viewing teachers as being controlled either by the plans of the system or the constraints of the classroom.

Innovation not only reveals differences amongst the designers of the innovation in terms of their views of teachers but also, because it inevitably perturbs the dynamics of a classroom, makes more apparent the mathematical beliefs and understandings of teachers and students. We have studied this phenomenon in the context of introducing computers into mathematics classrooms. Even if computers are simply 'just another innovation', they would adequately serve my purpose as a means to study teacher beliefs. However I claim more, and argue that computers can clarify and amplify the operation and influence of the social norms of the classroom and the belief systems within it. Seeing how computers are incorporated into practice throws all the issues surrounding beliefs into particularly sharp relief.

We have long maintained that introducing computers into the teaching/learning process in mathematics whilst changing the setting also provides a window[7], a magnifying glass even, on the interaction processes, the mathematical conceptions of the students and the effects of teacher intervention. Others have noticed this tendency: "Computers can make the things we already do badly in schools even worse, just as they can make many things we already do well infinitely better" (Smith, 1986, p.206). Their exaggerating effect is apparent within kinds of scenarios beyond the scope of this paper, for example, class and gender differentiation within hierarchies of 'competence'. As far as teacher beliefs are concerned, I contend it is no coincidence that the first papers I found in the PME proceedings which faced up to the influence of teachers' beliefs, arose in settings where technology was being introduced (JBH du Boulay, 1979; Hutton, 1979). I will elaborate on this claim in the next sections.

## The computer as a window on learning and a mirror on intervention strategies

I first became aware of the power of the metaphor 'computer as a magnifying glass' whilst reflecting on data from the Logo Maths Project (Hoyles and Sutherland, 1989). In this project we set out to investigate the ways Logo could be used as an aid to students' thinking and learning in mathematics at the secondary age level. Our study pointed to the collaborative efforts of the students in pursuit of long term projects negotiated together. We talked of the effects on attitude to and motivation towards learning mathematics and the shift towards student decision-making. However, unanticipated issues of dominance, particularly in relation to boys and girls, could not be ignored and we became increasingly aware of the influence of the medium on mathematical expression and the fragmented and situated nature of students' mathematical understandings.

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[7] This idea was first introduced by Weir (1986).

More relevant here is that although we set out to intervene 'lightly', that is in the context of the students' own work — to suggest ideas to explore or to point to 'interesting' mathematical extensions — on analysis of the transcripts, we were surprised by the significant structuring role our 'subtle' interventions had on student progress and the direction of their work. In the areas we had emphasised, the students made consistent and excellent progress, whilst in others, development was haphazard. In retrospect, the theories of Vygotsky provide a coherent framework for interpreting these findings within the realm of psychology. Initially we had taken a Piagetian approach, expecting that students would construct mathematical knowledge through interaction within our microworlds. We hypothesised that they would build their ideas through interaction and reflection on the results of their actions — a reflective process facilitated by the feedback provided by the computer. However, we came to appreciate how mathematical knowledge emerged through social interaction — with the teacher and other students offering 'scaffolding' within the zone of proximal development (Vygotsky, 1978)[8].

This interpretation allows us to map out a child's dynamic developmental state and his/her future intellectual growth. It also brings pedagogic intervention to centre stage — as a mediator between the child and his/her experience. Inevitably, the nature and intentions of this intervention, either in general or in particular (to take Clandinin and Connelly's distinction) become legitimate objects of study.

Returning to the Logo Maths project, it is interesting for me now to reflect on our role as participant observers. We were the teachers, organising and guiding the work, so, given the previous analysis, it is interesting to speculate how far our findings were contingent on *our* beliefs about mathematics and mathematics learning. In fact, the metaphor of the computer not only as a window but also as a mirror on beliefs is compelling here. Our interventions and the students' responses bear witness to our commitment to giving students autonomy over their learning and knowledge-building processes — as such we were constructivist, though without the label! But we were not laissez-faire and very evidently had a strong mathematical agenda which, as the research unfolded, became increasingly articulated as we watched the effects of our suggestions in the mirror of the computer screen.

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[8] 'Scaffolding' is a metaphor first coined by Wood et al. (1976) to describe support provided by teacher or parent to a child which is 'just enough' for their progression in an activity. I develop this notion in the context of mathematical education in Hoyles (1991).

We have argued that groupwork can play an important intermediary role in the transition from closely -supported work with the computer (as in the Logo Maths Project) to the 'real' world of classrooms. We have pointed to "the pivotal importance of small-group work in serving as a bridge between pupil's own meanings and mathematical meaning" (Hoyles and Noss, 1992, p.54). However, in mathematics education, there is much rhetoric (at least in the US and UK) that groupwork is 'good' but rather little analysis of the conditions which optimise the potential of groupwork in mathematics or indeed elaboration of appropriate methodologies to research such a question. Again in retrospect, I see our research on groupwork with computers as highlighting some crucial aspects of learning mathematics in groups. The major conclusion of our study (Hoyles, Healy, and Pozzi, 1992) was that the social norms and expectations of the student group could *not* be ignored in any analysis of group interactions: a necessary condition for the success of any group setting was that students were sufficiently mature to manage themselves and their resources and were unimpeded by interpersonal antagonism. Computer-use highlighted the tension between group outcome (a computer product) and learning (reflection on process), brought to the surface interpersonal conflicts, emphasised the need to negotiate and share, and drew attention to the absence of connection between students' constructive activity on the computer, their expression orally and on paper.

It might be odd to suggest that a study like this one, where we did not intervene, could have anything to say about the teacher's role and the teacher's beliefs. But in a similar way to that described in the context of the Logo Maths Project, our beliefs were in fact all too apparent — not least in our choice of computer software: the computer was used within microworlds and employed by students as an expressive medium to explore mathematics; we do not use 'instructional' software or courseware to deliver pre-specified curriculum objectives, which, in our view, shifts decision-making power away from teachers and students[9]. Our convictions concerning learning mathematics are also clear: the importance of 'bumping up' against different perspectives and solution strategies to develop mathematical knowledge together with an understanding of its limitations; the view that learning is not sequential but a function of many encounters; a concern for student decision-making and the provision of opportunities for them to resolve their own dilemmas. What was clear to us though during the research was how often we *longed* to intervene as we observed the group interactions — but if we resisted this temptation the children could frequently sort things out for themselves together *given time*[10].

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[9] We reject the position of 'the neutrality of the tool' and the idea that computers merely implement the curriculum (for further discussion of microworlds and different visions of computer use, see Bullin, 1991 and Hoyles,in press).

[10] We are certain though that interventions of the following characters could have assisted progress or at least rendered it more efficient: conceptual — to notice something, recall a point made earlier by a child;

Once again the presence of the computer with which the children could construct, negotiate, debug and validate their mathematical ideas set us free to reflect upon our beliefs as they were acted out in the small group situation. But what about beliefs-in-practice in the 'real' world of classroom life? To consider this, I first need to give a brief overview of computer use in schools.

## The computer as a window on teacher beliefs and a mirror on innovation

It is clear from a review of the available literature (mainly in the U.S. and the U.K.), that the impact of computers on school life does not match the early claims of the 'computer enthusiasts'. As Becker put it: "There were 'dreams' about computer using students ... dreams of voice-communicating, intelligent human tutors, dreams of realistic scientific simulations, dreams of young adolescent problem solvers adept at general-purpose programming languages — but alongside these dreams was the truth that computers played a minimal role in real schools" (Becker, 1982, p.6). In the same vein, Becker later argued: "As we enter the 1990's, it is important to understand how much of that early limited reality still remains and to understand how much of the idea of transforming teaching and learning through computers remains plausible. We need to be aware of the 'old habits' and 'conventional beliefs' that are common among practising educators and the 'institutional constraints' that impede even the best of intentions to improve schooling through technology" (Becker, 1991, p.6). Once again we have the juxtaposition of beliefs and classroom cultures.

Thus the 'average' picture painted is one of stability of attitude to computers with possible incremental rather than major change. Absent from such a review are the, admittedly rare, but imaginative usages which have been researched by Sheingold and Hadley (1990). In their study of 'accomplished teachers' — those reputed to be especially adept at incorporating the computer into their classroom practice — the majority had been teaching for over 13 years, worked in schools more generously endowed with hardware and had easy access to technical and educational assistance. Within this rather special sample, teachers reported that using computers had radically changed their conceptions and methods in ways that could be interpreted as 'constructivist' — becoming more student-centred in terms of expectation and direction of learning. Prerequisities to reaching such a position were identified as resources, time and perseverance — factors necessary to survive the phase of 'disruption' and then to maintain a willingness to reflect upon the new classroom dynamic.

We came to rather similar conclusions after the Microworlds project (Sutherland, Hoyles and Noss, 1991) where we had set up a course of in-service education for 'expert' secondary

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practical/managerial — to overcome any disruptions due to dominance, and status hierarchies; syntactic — to assist with the formal structures of the computer language.

mathematics teachers. In addition to the time and support required to integrate computers into practice, we also stressed the need for the teachers to experience the power of the medium for expressing *their own* mathematical ideas. It is important to underline here that we recognised from the outset that any study involving innovation with mathematics teachers, must of necessity look at teachers' beliefs as well as their practices. Additionally, we aimed to map the interrelationships between beliefs and practices. We were interested in how the interactions of the teachers with the computer activities on our course, and the ways they incorporated them into their practice, provided a window onto their views and beliefs about mathematics teaching — but in a dynamic way whereby beliefs and course activities were both in the process of change.

I thus extend the idea of the computer as a window and a mirror to this new setting — a window on our teachers'/students' beliefs and a mirror on our own. We set out with a strong commitment to conduct a course focussed on mathematics education and not on technology but it was not until the project was underway that what we meant by mathematics education became clearer to us and to our students — particularly with respect to the importance we accorded to groupwork, social interaction, problem solving and 'learner control'.

There is still a gap between these general considerations and their practical realisation. We collected interview, observational and descriptive data throughout the course (30 days spread over one year) and wrote case studies of each teacher-participant. Analysis of these revealed a phenomenon of *projection* — the tendency for teachers to attribute their own feelings to their students. Once noticed this serves as a very useful and transparent window on beliefs! The next stage of analysis revealed our notions of teachers and teaching in a rather more clear and precise way. We constructed *caricatures* of the course participants to provide a synthesis of views, attitudes and practices of a cluster of case studies. The caricatures attempted to draw attention to teacher characteristics and behaviours which *we* deemed crucial by exaggeration of some facets and omission of others[11]. Thus they reflected our ideas about categories by which to gauge mathematics teaching and teachers —they mirrored our beliefs as well as reflecting teacher beliefs in so far as they resonated with any individual viewpoint.

The five caricatures which emerged from our study were: Mary, the frustrated idealist; Rowena, the confident investigator; Denis, the controlling pragmatist; Fiona, the anxious traditionalist and Bob, the curriculum deliverer (Noss, Sutherland and Hoyles, 1991; Noss and Hoyles, 1992). Each illustrated very different ways of integrating computers into their practice and different foci upon which they reflected during this process. What is very evident is that any curriculum change is complex and subtle. If it is be anything more than a 'technical fix', it must interact with the very essence of teaching and learning mathematics. Thus, Polin (1991) suggested: "we need to instill a different vision of teacher development in our impatient policy-makers and in our harried teachers, a vision that acknowledges the many years of practice it takes to acquire and integrate a new way of teaching" (Polin, 1991, p.7).

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[11] The caricatures consisted of attitudes and behaviours within a set of dimensions which coalesced as the data analysis proceeded, and which when put together created a recognisable 'person'.

Change must occur from within and cannot be imposed from above — a constructivist message about learning 'transposed' to a new arena? I suggest that we might approach innovation from a different perspective: not trying to change beliefs in order to have the 'right' effect but rather as a means to throw light on beliefs, beliefs-in-practice, and on the innovation itself. But I go further, and problematise the distinction between beliefs and beliefs-*in-practice* and posit that all beliefs are *situated*. This framework not only influences our approach to innovation but presents a new way in which to interpret the inconsistencies in research findings on teacher beliefs reviewed earlier.

## Situated Beliefs

The caricatures described in the previous section were derived from rich and diverse data collected over a period of a year. Each caricature had a 'personality', 'beliefs about teaching and mathematics teaching', and consistent, well-established classroom practices with which we (and teachers) could identify. Each caricature held a myriad of beliefs, each constructed within the practice in which they were realised (in this case the classroom, the computer activities, the interviews). I propose to view all beliefs as situated — dialectical constructions, products of activity, context and culture. By analogy with situated cognition and situated knowledge (see, for example, Seely Brown et al, 1989), the notion of situated beliefs challenges the separation of what is believed from how beliefs emerge: 'situations might be said to co-produce *beliefs* through activities' (Seely Brown et al, 1989, p.32, replacing 'knowledge' in the original text with 'beliefs'). Once the embedded nature of beliefs is recognised, it is self evident that any individual can hold multiple (even contradictory) beliefs and 'mismatch', 'transfer' and 'inconsistency' are irrelevant considerations and replaced by notions of constraint and scaffolding within settings.

To develop further this notion of situated beliefs, I revisited some research I carried out in the early 1980's, 'the Mathematics Teaching Project' (Hoyles et al, 1984). We had set out to identify the characteristics of 'good practice' in secondary school mathematics classrooms and tried to capture the reality of a good mathematics teacher from the point of view of the teacher (through repertory grids and interviews), the students (by questionnaire and interviews) and a classroom observer. Returning to this project now, I am struck by how ambitious we were! Additionally, it seems to me in retrospect, that our design suggested (albeit implicitly) that teacher beliefs were something which could be accessed outside the classroom and then observed as applied to the classroom. We found it difficult to make links between the teacher/student/classroom perspectives or to identify the interplay of beliefs and practice. This could, of course, be because of our inadequacies, or the inadequacies of the data-gathering techniques and conceptual tools available at the time. I feel now that we were chasing a mirage:

given the research paradigm in which we were working, we simply had not appreciated how beliefs of teachers and students could be constructed differently in different settings.

In Hoyles et al (1985) we reported a snapshot of a mathematics teacher, Ms X. At the beginning of the paper we described some contextual features of the setting; for example, the class comprised a high ability group of 15 year old girls from a comprehensive school. What is noteworthy from a perspective a decade later, is our failure to see these features as major structuring influences on the vignette. With the benefit of hindsight, the following questions come to mind: How far was Ms X's mathematical perspective constructed by the 'high ability' of the group? How far was her emphasis on effort related to her sex and the sex of the students? Was her particular blend of exposition/interaction partly a function of the age and specialism of the students? I would of course now answer these questions in the affirmative!

On re-reading the classroom snapshot, I am also exercised by the remarkable harmony between what the teacher 'delivered' and 'what the students wanted'. Such harmony exemplifies *a* good practice — a different blend of variables in the setting could produce alternative pictures of success. The notion of situated beliefs allows us to cope with such diversity. This does not mean that we are constrained to consider only individual differences at the level of the teacher; rather we need to seek crucial variables and general patterns at the level of setting.

## Concluding Remarks

At the beginning of this paper, I set out my agenda to map out the next step forward from that proposed by Balacheff. I am now wondering if my argument has not led me in another direction! In their quest for scientific status, the French didacticians have pushed us towards a more precise and careful analysis of the processes of interaction between student/teacher and mathematical knowledge. The influence of this research in the U.K. is very evident, although in our group at least, we were always very clear that any microworlds we 'produced' were not 'scripts', not innovations to be 'implemented' — we expected, and found, their take up to be very different in different classrooms. By extending the idea of situatedness to beliefs we not only point to the enormity of trying to capture the complexities of the classroom, but also call into question the possibility of 'reducing' teaching to a science. Teaching is a 'human' activity which involves the feelings and beliefs of the participants, each of whom have a personal and cultural history colouring their actions — or, in line with the spirit of this paper, does this interpretation merely reflect my beliefs and my Englishness?

Woven throughout this presentation has been the issue of methodology. I have pointed to the multitude of approaches adopted within and outside PME and to the diversity of research tools borrowed from psychology, sociology, etc.[12]. Eisenhart, in her plenary address to PME-

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[12] To list just a few we can find: Vygotsky; metacognition; constructivism; humanism; personal construct theory; attitude theory; psychoanalytic theory; social/cultural perspectives; using a bewildering array of tools

NA (1991), strongly argued the need for studies to be grounded within some framework, be it theoretical, practical or conceptual. There is still some way to go in this direction and research tools need to be reconstructed within the practice of the psychology of mathematics education. Within research incorporating the teacher, case studies are increasingly the dominant methodology. There is a danger however, that by recognising the influence of setting in which teachers' beliefs are operationalised, we descend into a path of relativism and, moreover, lose our psychology/mathematics focus — it is difficult to trace development; mathematics itself becomes merely a 'placeholder'. Somehow, case studies need to illuminate the general through the particular, whilst keeping intact the complexity of the situation.

It is also the case that case studies are hard to write and harder still to read — their very richness mitigates against their communicability. Different approaches to this issue have been found to be effective: critical incidents, evocative episodes, gambits[13] and caricatures to name but a few. Can we find ways to critique and build upon other case studies, keeping track of failures as well as successes? How do we judge case studies? Do we have the criteria available to falsify any interpretations? When we write up case studies, is it possible to develop different stories (even unpalatable ones!) from the same data? Perhaps, having largely rejected experimental work and quantitative methods, we can now review them from a new perspective in order to provide complementary or contradictory insights? Perhaps we need more explicit humility, recognising the limitations of our work and acknowledging the tensions within it?

I assert that these tensions reflect those in schools: between acquisition and construction; collaboration and schooling; the aspirations of mathematical experts and the constraining factors of the classroom. As I write this, I note a further tension: the move to situatedness and a holistic, constructivist approach in research, and a parallel development (at least in the U.K.) towards curriculum and assessment mechanisms based on mastery, fragmentation and the separation of outcomes from experiences.

In this paper, I have traced the evolution of research on teacher beliefs and their interaction with mathematics learning which threw up evidence of inconsistencies between beliefs and beliefs-in practice. I argued that this mismatch was thrown into relief when teachers were faced with an innovation, particularly when the innovation involved computers — a point brought home by the adoption of the metaphor of the computer as a window and a mirror on beliefs. The contention that teachers reconstruct their beliefs whilst interacting with an innovation, led me to propose the notion of situated beliefs — all beliefs are, to a certain extent, constructed in settings. Finally, I have addressed the issue of methodology and the need for more coherence, development and attention to be paid to the centrality of psychology and mathematics. I end with

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such as: repetory grids; observational techniques; gambits; analysis of metaphors,videotapes of teaching scenarios, student errors, narratives; attitude scales; belief scales.

[13] These methods have been adopted in many of the researchs cited in this text with the exception of gambits,which are discussed by Mason and Davis, 1987 and Pimm, 1987.

an anecdote — a critical incident which highlights some of the points raised and the tensions presented:

> I recently heard that a researcher[14] in Logo went to his university computer room and wanted to load up Logo from the network. He found that Logo was no longer there and when confronting the manager received this reply: "Ah, but we have *Dazzledraw* [a flashy paint program] ... I just assumed that now you have *Dazzledraw* you will not need Logo".

I will leave you with some questions: What does this short vignette tell us about the beliefs of the manager — regarding education? student learning? mathematics? and his colleagues in teacher training? Perhaps most significantly, if we reflect on our reaction to this vignette, what does it tell us about our own beliefs?

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[14] Doug Clements, personal communication

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